Under the assumption that y_t is stationary with autocovariances

$$\gamma_k = E(y_t - \mu)(y_{t-k} - \mu)$$

and spectral density

$$f(\omega) = \frac{1}{2\pi}\left(\gamma_0 + 2\sum_{k=1}^{\infty}\gamma_k \cos(\omega k)\right),$$

$$\begin{aligned}
V &= nE(\bar{y} - \mu)^2 = nE\left(\frac{1}{n}\sum_{t=1}^{n}(y_t - \mu)\right)^2 = \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{T} E(y_t - \mu)(y_s - \mu) \\
&= \frac{1}{n}\sum_{t=1}^{n} E(y_t - \mu)^2 + \frac{2}{n}\sum_{t=2}^{n} E(y_t - \mu)(y_{t-1} - \mu) + \cdots + \frac{2}{n}E(y_n - \mu)(y_1 - \mu) \\
&= \gamma_0 + 2\sum_{k=1}^{n-1}\frac{n-k}{n}\gamma_k \\
&= \gamma_0 + 2\sum_{k=1}^{n}\left(1 - \frac{k}{n}\right)\gamma_k \;\rightarrow\; \gamma_0 + 2\sum_{k=1}^{\infty}\gamma_k = 2\pi f(0).
\end{aligned}$$

Sample autocovariances:

$$\hat{\gamma}_k = \frac{1}{n}\sum_{t=k+1}^{n}(y_t - \bar{y})(y_{t-k} - \bar{y})$$

Weighted covariance estimator of f:

$$\hat{f}(\omega) = \frac{1}{2\pi}\left(\hat{\gamma}_0 + 2\sum_{k=1}^{q}(1 - \frac{k}{q+1})\hat{\gamma}_k \cos(\omega k)\right) \quad \text{(Bartlett estimator: triangular window)}$$

$Cov(y_t, y_s) = 0 \;\; if \;\; |t - s| > q$:

$$V = \frac{1}{n}\sum_{t=1}^{n} E(y_t - \mu)^2 + \frac{2}{n}\sum_{t=2}^{n} E(y_t - \mu)(y_{t-1} - \mu) + \cdots + \frac{2}{n}\sum_{t=q+1}^{n} E(y_t - \mu)(y_{t-q} - \mu)$$

$$\begin{aligned}
\tilde{V} &= \frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 + \frac{2}{n}\sum_{t=2}^{n}(y_t - \bar{y})(y_{t-1} - \bar{y}) + \cdots + \frac{2}{n}\sum_{t=q+1}^{n}(y_t - \bar{y})(y_{t-q} - \bar{y}) \\
&= \hat{\gamma}_0 + 2\sum_{k=1}^{q}\hat{\gamma}_k
\end{aligned}$$

$$\begin{aligned}
\hat{V} &= \hat{\gamma}_0 + 2\sum_{k=1}^{q}(1 - \frac{k}{q+1})\hat{\gamma}_k \qquad \text{(Newey-West estimator)} \\
&= 2\pi \hat{f}(0)
\end{aligned}$$

Remark: $\frac{1}{n}\sum_{t=k+1}^{n}(y_t - \bar{y})(y_{t-k} - \bar{y})$ may converge in probability to $\frac{1}{n}\sum_{t=k+1}^{n} E(y_t - \mu)(y_{t-k} - \mu)$ even if the latter depends on t. The Newey-West estimator is therefore regarded as robust both to autocorrelation and heteroscedasticity.

Given a sample $p_{10}, \ldots, p_{1n}, \ldots, p_{m0}, \ldots, p_{mn}$ of stock prices observed at the n+1=79 time points 9:30 a.m., 9:35 a.m., ..., 4:00 p.m. on m successive trading days, we can simply use the square root of the mean squared (logarithmic) close-to-close return

$$\frac{1}{m-1}\sum_{j=2}^{m} R_j^2 = \frac{1}{m-1}\sum_{j=2}^{m}\left(\log\left(p_{jn}\right) - \log\left(p_{(j-1)n}\right)\right)^2$$

as a measure of volatility or, more sophisticatedly, the square root of the sum of the mean squared overnight return

$$\frac{1}{m-1}\sum_{j=2}^{m} Q_j^2 = \frac{1}{m-1}\sum_{j=2}^{m}\left(\log\left(p_{j0}\right) - \log\left(p_{(j-1)n}\right)\right)^2$$

and the mean realized variance

$$\frac{1}{m}\sum_{j=1}^{m}\sum_{t=1}^{n} r_{jt}^2 = \frac{1}{m}\sum_{j=1}^{m}\sum_{t=1}^{n}\left(\log\left(p_{jt}\right) - \log\left(p_{j(t-1)}\right)\right)^2.$$

In either case, it is assumed that the returns have mean zero and are serially uncorrelated.

Still assuming that the mean vanishes and observing that

$$E\left(\sum_{t=1}^{n} r_{jt}\right)^2 = n\left(nE\left(\bar{r}_j - 0\right)^2\right),$$

we could try to replace the realized variance

$$\sum_{t=1}^{n} r_{jt}^2 = n\hat{\gamma}_0(j)$$

by n times the Newey-West estimator

$$n\hat{V}_j = n\left(\hat{\gamma}_0(j) + 2\sum_{k=1}^{q}\left(1 - \frac{k}{q+1}\right)\hat{\gamma}_k(j)\right)$$

in order to obtain a more robust measure of volatility (e.g., with q=1).

We might also want to exclude the volatile overnight return and compensate for it by scaling up the realized volatility. However, the practice of using a fixed scaling factor (e.g., 1.18) is dangerous because the magnitude of the overnight return changes over time and differs from one stock to another.