EXPONENTIAL SMOOTHING

The Holt-Winters algorithm

A general, possibly nonlinear trend $\mu_t = EY_t$ may be forecasted recursively with the **Holt-Winters algorithm**.

Estimated level at time 2: $\hat{a}_2 = Y_2$

Estimated slope at time 2: $\hat{b}_2 = Y_2 - Y_1$

Forecasted trend at time 3: $\hat{\mu}_3 = \hat{a}_2 + \hat{b}_2$

Estimated level at time 3: $\hat{a}_3 = \alpha Y_3 + (1-\alpha)\hat{\mu}_3$

Estimated slope at time 3: $\hat{b}_3 = \beta(\hat{a}_3 - \hat{a}_2) + (1-\beta)\hat{b}_2$

Forecasted trend at time 4: $\hat{\mu}_4 = \hat{a}_3 + \hat{b}_3$

$$\vdots$$

Estimated level at time n: $\hat{a}_n = \alpha Y_n + (1-\alpha)\hat{\mu}_n$

Estimated slope at time n: $\hat{b}_n = \beta(\hat{a}_n - \hat{a}_{n-1}) + (1-\beta)\hat{b}_{n-1}$

Forecasted trend at time n+1: $\hat{\mu}_{n+1} = \hat{a}_n + \hat{b}_n$

Forecasted trend at time n+2: $\hat{\mu}_{n+2} = \hat{a}_n + 2\hat{b}_n$

Forecasted trend at time n+3: $\hat{\mu}_{n+3} = \hat{a}_n + 3\hat{b}_n$

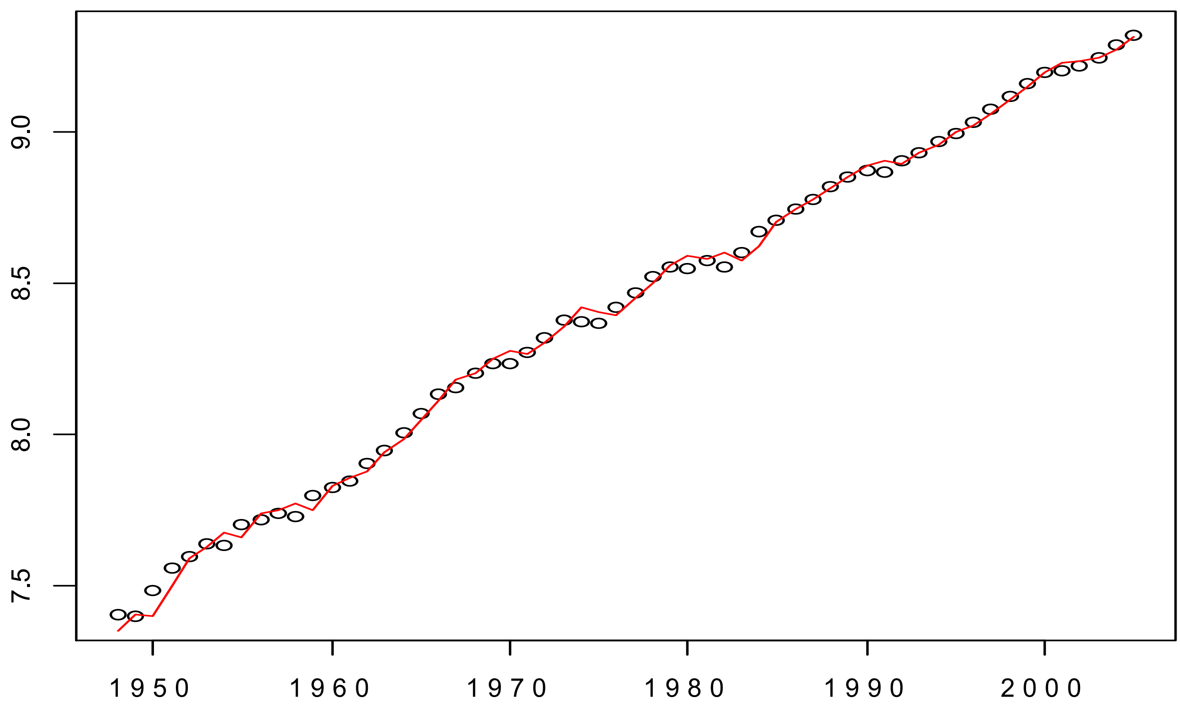
$$\vdots$$

The smoothing parameters α and β lie between 0 and 1. The smaller the values of α and β, the smoother is the result.

Exercise: Forecast the trend of the log GDP with the Holt-Winters algorithm using the parameters α and β determined by R to minimize the sum of squares of the squared one-step prediction errors.

```
HW <- HoltWinters(GDP.lts,gamma=FALSE)
          # gamma=FALSE: no seasonality
c(HW$alpha,HW$beta) # parameters selected by R
alpha      beta
1.0000000 0.2264229

t3 <- 1948:(1946+N-1) # no forecasts for first two years
plot(t3,GDP.lts[c(-1,-2)],type="p",xlab=" ",ylab=" ")
lines(t3,HW$fitted[,1],col="red")       # forecasts
```



Exponential smoothing

If we ignore the slope, the Holst-Winter recursions reduce to

$$\hat{\mu}_t = \hat{a}_{t-1}$$
$$\hat{a}_t = \alpha Y_t + (1-\alpha)\hat{\mu}_t = \alpha Y_t + (1-\alpha)\hat{a}_{t-1}.$$

Regarding the estimated level $\hat{a}_t$ at time t as an estimate $\tilde{\mu}_t$ of the trend at time t, we may write

$$\tilde{\mu}_t = \alpha Y_t + (1-\alpha)\tilde{\mu}_{t-1}.$$

An obvious choice for an initial condition is

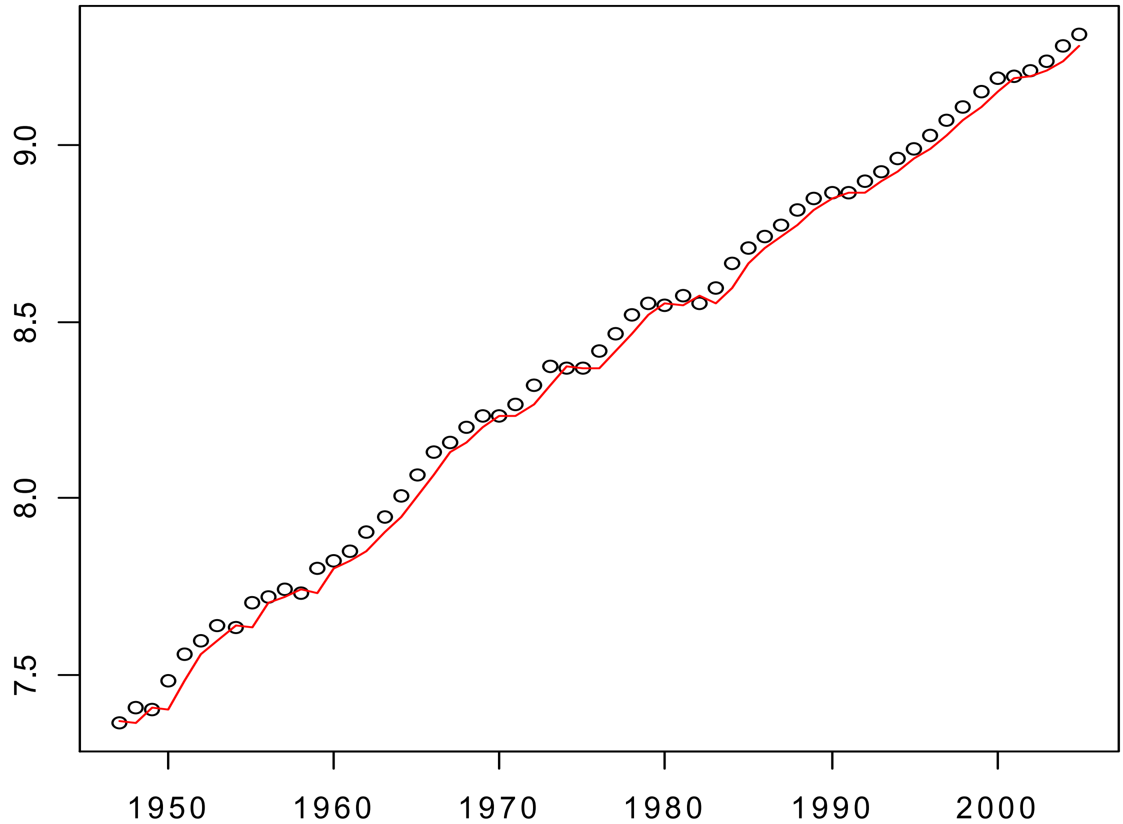
$$\tilde{\mu}_1 = Y_1.$$

This technique of computing trend forecasts $\hat{\mu}_t$ or trend estimates $\tilde{\mu}_t$ is referred to as **exponential smoothing**, because

$$\begin{aligned}
\hat{a}_t &= \alpha Y_t + (1-\alpha)\hat{a}_{t-1} \\
&= \alpha Y_t + (1-\alpha)(\alpha Y_{t-1} + (1-\alpha)\hat{a}_{t-2}) \\
&= \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + (1-\alpha)^2\hat{a}_{t-2} \\
&= \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + (1-\alpha)^2(\alpha Y_{t-2} + (1-\alpha)\hat{a}_{t-3}) \\
&\ \ \vdots \\
&= \alpha(Y_t + (1-\alpha)Y_{t-1} + \ldots + (1-\alpha)^k Y_{t-k}) + (1-\alpha)^{k+1}\hat{a}_{t-(k+1)}.
\end{aligned}$$

Exercise: Forecast the trend of the log GDP by exponential smoothing.

```
ES <- HoltWinters(GDP.lts,beta=F,gamma=F)
t2 <- 1947:(1946+N-1) # no forecast for first year
plot(t2,GDP.lts[-1],type="p",xlab=" ",ylab=" ")
lines(t2,ES$fitted[,1],col="red")
```



Because of the omission of the slope, the forecasts are, of course, biased downwards.

The forecasts obtained by exponential smoothing are biased downwards in case of an uptrend and upwards in case of a downtrend. Exponential smoothing might therefore be used to identify the direction of the trend.

<u>Exercise:</u> Forecast the trend of the monthly log S&P 500.

```
t <- 1950+(0:(N-1))/12 # y=log(S&P500) has length N
t2 <- t[-1] # no forecast for first year
plot(t2,y[-1],type="l",xlab=" ",ylab=" ")
ES <- HoltWinters(y,alpha=0.1,beta=F,gamma=F)
a0.1 <- ES$fitted[,1]; lines(t2,a0.1,col="red")
```