THE WOLD DECOMPOSITION

<u>Definitions:</u> (i) A family $(X_t)_{t\in T}$ of random variables $X_t:\Omega\to\mathbb{R}$ defined on a probability space $(\Omega,\mathcal{A},P)$ is called a **stochastic process**.

(ii) A stochastic process $(X_t)_{t\in T}$ is called a **discrete-time process** if its index set T is either a finite or countably infinite subset of $\mathbb{R}$ (e.g., if $T=\mathbb{Z}$).

(iii) A discrete-time process $(X_t)_{t\in\mathbb{Z}}$ is called **strictly stationary** if for all $k\in\mathbb{N}^*$ and $t_1,\ldots,t_k,h\in\mathbb{Z}$ the joint distributions of $X_{t_1},\ldots,X_{t_k}$ and $X_{t_1+h},\ldots,X_{t_k+h}$ are the same.

<u>Remarks:</u> (i) If the distribution fuctions of X_t and X_{t+h} are the same, then

$$EX_t\in\mathbb{R}\Rightarrow EX_t=EX_{t+h}.$$

(ii) If the joint distribution fuctions of X_s,X_t and X_{s+h},X_{t+h} are the same, then

$$EX_sX_t\in\mathbb{R}\Rightarrow EX_sX_t=EX_{s+h}X_{t+h}.$$

<u>Definition:</u> A discrete-time process $(X_t)_{t\in\mathbb{Z}}$ is called **weakly stationary** if for all $s,t,h\in\mathbb{Z}$

(i) $EX_t^2\in\mathbb{R}$,

(ii) $EX_t=EX_{t+h}$,

(iii) $EX_sX_t=EX_{s+h}X_{t+h}$.

Exercise: Let $(X_t)_{t\in\mathbb{Z}}$ be a weakly stationary process. Show that for all $s,t,h\in\mathbb{Z}$

$$Cov(X_s,X_t)= Cov(X_{s+h},X_{t+h}).$$

Definition: The **autocovariance function** and the **autocorrelation function** of a weakly stationary process $(X_t)_{t\in\mathbb{Z}}$ are defined by

$$\gamma(h)=Cov(X_0,X_{-h}),\ h\in\mathbb{Z},$$

and

$$\rho(h)=\frac{\gamma(h)}{\gamma(0)},\ h\in\mathbb{Z},$$

respectively.

$\gamma(h)$ is called the **autocovariance at lag h** and $\rho(h)$ is called the **autocorrelation at lag h**.

Exercise: Show that γ is an even function of h.

Solution:
$$\begin{aligned}\gamma(-h) &= Cov(X_0,X_h)\\ &= Cov(X_{0-h},X_{h-h})\\ &= Cov(X_{-h},X_0)\\ &= Cov(X_0,X_{-h})\\ &= \gamma(h)\end{aligned}$$

Definition: A weakly stationary process is called **white noise** if $\gamma(h)=0$ for all $h\neq 0$.

Remark: Since the concept of orthogonal projections is useful for the discussion of the linear prediction of weakly stationary processes, we will henceforth regard the random variables X_t, $t\in\mathbb{Z}$ of a weakly stationary process as elements of the Hilbert space L^2 if occasion demands.

Definitions: (i) The **innovations process** $(U_t)_{t\in\mathbb{Z}}$ of a weakly stationary process $(X_t)_{t\in\mathbb{Z}}$ is defined by

$$U_t=X_t-P_{H_{t-1}}(X_t),$$

where $P_{H_{t-1}}$ is the projection mapping of the Hilbert space L^2 onto the closed subspace $H_{t-1}=\overline{\text{span}}(X_{t-1},X_{t-2},\ldots)$.

(ii) $(X_t)_{t\in\mathbb{Z}}$ is said to be **deterministic** if $X_t\in H_{t-1}$ for all $t\in\mathbb{Z}$.

(iii) A not deterministic process is called **non-deterministic**.

(iv) $(X_t)_{t\in\mathbb{Z}}$ is said to be **purely indeterministic** or **regular** if

$$\bigcap_{t\in\mathbb{Z}} H_t=\{0\}.$$

Exercise: Show that the innovations U_t, $t\in\mathbb{Z}$, of a zero-mean weakly stationary process $(X_t)_{t\in\mathbb{Z}}$ are uncorrelated.

Solution:

$$s<t \Rightarrow U_s=X_s-P_{H_{s-1}}(X_s)\in H_s\subseteq H_{t-1},\ U_t=X_t-P_{H_{t-1}}(X_t)\in H_{t-1}^{\perp}$$

$$\Rightarrow U_s\perp U_t$$

$$\Rightarrow EU_sU_t=0$$

The Wold Decomposition: If a zero-mean weakly stationary process $(X_t)_{t\in\mathbb{Z}}$ is non-deterministic, it can be expressed as

$$X_t = U_t + \sum_{k=1}^{\infty} \psi_k U_{t-k} + V_t,$$

where

(i) $(U_t)_{t\in\mathbb{Z}}$ is white noise,

(ii) $(V_t)_{t\in\mathbb{Z}}$ is deterministic,

(iii) $EU_sV_t = 0 \ \forall s,t\in\mathbb{Z}$,

(iv) $\sum_{k=1}^{\infty} \psi_k^2 < \infty$,

(v) $U_t \in H_t \ \forall t\in\mathbb{Z}$,

(vi) $V_t \in \bigcap_{t\in\mathbb{Z}} H_t \ \forall t\in\mathbb{Z}$.

All quantities occurring in this decomposition are uniquely determined:

$$U_t = X_t - P_{H_{t-1}}(X_t) \ \forall t\in\mathbb{Z} \text{ (U_t is the innovation at time t)},$$

$$\psi_k = \frac{1}{\sigma^2} \mathrm{Cov}(X_t, U_{t-k}) \ \forall k\in\mathbb{N}^*,$$

$$V_t = X_t - U_t - \sum_{k=1}^{\infty} \psi_k U_{t-k} \ \ \forall t\in\mathbb{Z}.$$

Remark: For zero-mean elements X and Y of L^2 we have

$$X \perp Y \Leftrightarrow \mathrm{Cov}(X,Y) = 0.$$