A family X_t, $t\in\mathbf{Z}$, of random vectors X_t: $\Omega\to\mathbf{R}^k$ defined on a probability space (Ω,A,P) is called a **stationary process** if the mean vectors

$$EX_t = E\begin{pmatrix} X_{t1} \\ \vdots \\ X_{tk} \end{pmatrix} = \begin{pmatrix} EX_{t1} \\ \vdots \\ EX_{tk} \end{pmatrix}$$

and the autocovariance matrices

$$\text{cov}(X_t,X_{t-h})=E(X_t-EX_t)(X_{t-h}-EX_{t-h})^T$$

are independent of t.

The **autocovariance function** of a stationary process is defined by

$$\Gamma(h)=\text{cov}(X_0,X_{0-h}).$$

Exercise: Show that $\Gamma(-h)=\Gamma^T(h)$. MS

A stationary process is called **white noise** if its auto-covariance function satisfies

$$\Gamma(h)=0 \quad \forall\ h\neq 0.$$

Since $\Gamma(0)$ does not have to be a diagonal matrix, any two components of white noise can be correlated with each other contemporaneously.

A stationary process X is called a **first order autoregressive process** (or **AR(1) process**) if it can be expressed as

$$X_t = \begin{pmatrix} X_{t1} \\ \vdots \\ X_{tk} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \cdots & \phi_{1k} \\ \vdots & \ddots & \vdots \\ \phi_{k1} & \cdots & \phi_{kk} \end{pmatrix} \begin{pmatrix} X_{(t-1)1} \\ \vdots \\ X_{(t-1)k} \end{pmatrix} + \begin{pmatrix} U_{t1} \\ \vdots \\ U_{tk} \end{pmatrix}$$

$$= \Phi X_{t-1} + U_t,$$

where U is white noise with mean vector 0.

Each component of an AR(1) process depends not only on lagged values of itself but also on lagged values of the other components.

Substituting in an AR(1) equation

$$X_t = \Phi X_{t-1} + U_t$$

first $\Phi X_{t-2}+U_{t-1}$ for X_{t-1} and then $\Phi X_{t-3}+U_3$ for X_{t-2}, … gives

$$\begin{aligned} X_t &= \Phi(\Phi X_{t-2}+U_{t-1})+U_t = \Phi^2 X_{t-2}+\Phi U_{t-1}+U_t \\ &= \Phi^2(\Phi X_{t-3}+U_{t-2})+\Phi U_{t-1}+U_t = \Phi^3 X_{t-3}+\Phi^2 U_{t-2}+\Phi U_{t-1}+U_t \\ &\vdots \\ &= \Phi^m X_{t-m} + \sum_{j=0}^{m-1} \Phi^j U_{t-j} \end{aligned}$$

Suppose that Φ is diagonalizable[1], i.e., there is an invertible matrix C such that $\Lambda = C^{-1}\Phi C$ is a diagonal matrix. It then follows from

$$\Lambda = C^{-1}\Phi C \Leftrightarrow \Phi = C\Lambda C^{-1}$$

that

$$\begin{aligned} \Phi^2 &= \Phi\Phi = C\Lambda C^{-1}C\Lambda C^{-1} = C\Lambda^2 C^{-1} \\ \Phi^3 &= \Phi^2\Phi = C\Lambda^2 C^{-1}C\Lambda C^{-1} = C\Lambda^3 C^{-1} \\ &\vdots \\ \Phi^m &= C\Lambda^m C^{-1}. \end{aligned}$$

Thus,

$$\Phi^m = C\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{pmatrix}^m C^{-1} = C\begin{pmatrix} \lambda_1^m & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k^m \end{pmatrix} C^{-1}$$

will vanish as $m \to \infty$ only if the not necessarily real numbers $\lambda_1,\ldots,\lambda_k$ have modulus less than 1. MR

[1] For example, any Hermitian matrix (a complex square matrix that is equal to its own conjugate transpose) is diagonalizable. Thus, real symmetric matrices are diagonalizable.

A $k \times k$ matrix Φ is diagonalizable if and only if it has k linearly independent eigenvectors (**diagonalization theorem**).

Proof:

Suppose that $c_1,\ldots,c_k$ are linearly independent eigenvectors with eigenvalues $\lambda_1,\ldots,\lambda_k$. Then

$$\Phi = \Phi(c_1,\ldots,c_k)(c_1,\ldots,c_k)^{-1}$$

$$= (\Phi c_1,\ldots,\Phi c_k)(c_1,\ldots,c_k)^{-1}$$

$$= (\lambda_1 c_1,\ldots,\lambda_k c_k)(c_1,\ldots,c_k)^{-1}$$

$$= (c_1,\ldots,c_k)\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{pmatrix}(c_1,\ldots,c_k)^{-1}.$$

If

$$\Phi = (c_1,\ldots,c_k)\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{pmatrix}(c_1,\ldots,c_k)^{-1},$$

then

$$(\Phi c_1,\ldots,\Phi c_k) = \Phi(c_1,\ldots,c_k)$$

$$= (c_1,\ldots,c_k)\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{pmatrix}$$

$$= (\lambda_1 c_1,\ldots,\lambda_k c_k).$$

MD

Exercise: Show that if a 2×2 matrix Φ has 2 different eigenvalues λ_1 and λ_2, the corresponding eigenvectors c_1 and c_2 will be linearly independent.

Solution: Since λ_1 and λ_2 are different, at least one of them, say λ_1, is not equal to zero. Assuming that the antithesis

$$c_1=\nu c_2 \text{ for some } \nu\neq0$$

is valid, we obtain

$$\Phi c_1=\nu\Phi c_2,$$

$$\lambda_1 c_1=\nu\lambda_2 c_2,$$

and

$$c_1=\frac{\lambda_2}{\lambda_1}\nu c_2,$$

Since $\frac{\lambda_2}{\lambda_1}\neq1$ and c_1, $c_2\neq0$, this is in contradiction with the antithesis.

ME

Exercise: Find the eigenvalues λ_1 and λ_2 of the matrix

$$\Phi=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}.$$

Hint: If c is an eigenvector with eigenvalue λ, i.e.,

$$\Phi c=\lambda c$$

or, equivalently,

$$(\lambda I-\Phi)c=0,$$

then the invertibility of the matrix $\lambda I-\Phi$ would imply that

$$c=(\lambda I-\Phi)^{-1}0=0,$$

which is inconsistent with the requirement that c must be a non-zero vector. The eigenvalues can therefore be found by solving the equation

$$\det(\lambda I-\Phi)=0.$$

MV

If Φ is a $k \times k$ matrix, the characteristic polynomial $\det(\lambda I - \Phi)$ has degree k. According to the fundamental theorem of algebra it has therefore k (complex) roots if each root is counted with its algebraic multiplicity. Since eigenvectors corresponding to different eigenvalues are independent, Φ can only be non-diagonalizable if there exists an eigenvalue with algebraic multiplicity $m_a > 1$ and geometric multiplicity $m_g < m_a$. The geometric multiplicity of an eigenvalue is the number of linearly independent eigenvectors with that eigenvalue.

<u>Exercise:</u> Show that the matrix MN

$$\Phi = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

is non-diagonalizable.

Hint: The eigenvectors corresponding to an eigenvalue λ can be found by solving the equation $(\lambda I - \Phi)c = 0$ for c.

The condition that all the eigenvalues of Φ are less than 1 in absolute value, i.e.,

$$|z| \geq 1 \Rightarrow \det(\Phi - zI) \neq 0,$$

is equivalent to

$$|z| \geq 1 \Rightarrow \det(-\tfrac{1}{z}(\Phi - zI)) \neq 0,$$

$$|z| \geq 1 \Rightarrow \det(I - \tfrac{1}{z}\Phi) \neq 0,$$

and $$|z| \leq 1 \Rightarrow \det(I - z\Phi) \neq 0.$$

<u>Remark:</u> If all roots of the polynomial $\det(I - z\Phi)$ lie outside of the unit circle, the sequence Φ, Φ^2, Φ^3, … is absolutely summable and

$$\sum_{j=0}^{\infty} \Phi^j U_{t-j}$$

converges (componentwise) in mean square to X_t.

Using lag-operator notation the equation

$$X_t - \Phi X_{t-1} = U_t$$

can also be written as

$$(I - \Phi L) X_t = U_t\,,$$

where $I - \Phi L$ is a matrix-valued polynomial. For example, in the bivariate case we have

$$(I - \Phi L)X_t = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} L \right) \begin{pmatrix} X_{t1} \\ X_{t2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \phi_{11}L & -\phi_{12}L \\ -\phi_{21}L & 1 - \phi_{22}L \end{pmatrix} \begin{pmatrix} X_{t1} \\ X_{t2} \end{pmatrix}$$

$$= \begin{pmatrix} (1 - \phi_{11}L)X_{t1} - \phi_{12}LX_{t2} \\ -\phi_{21}LX_{t1} + (1 - \phi_{22}L)X_{t2} \end{pmatrix}$$

$$= \begin{pmatrix} X_{t1} - \phi_{11}X_{(t-1)1} - \phi_{12}X_{(t-1)2} \\ X_{t2} - \phi_{21}X_{(t-1)1} - \phi_{22}X_{(t-1)2} \end{pmatrix}.$$

A stationary process X is called an **autoregressive process of order *p*** (or **AR(*p*) process**) if it can be expressed as

$$X_t = \Phi_1 X_{t-1} + \ldots + \Phi_p X_{t-p} + U_t$$

or, equivalently, as

$$X_t - \Phi_1 X_{t-1} - \ldots - \Phi_p X_{t-p} = U_t\,,$$

where U is white noise with mean vector 0.

Using lag-operator notation, the latter equation can also be written as

$$\Phi(L) X_t = U_t,$$

where

$$\Phi(L) = I - \Phi_1 L - \ldots - \Phi_p L^p$$

is a matrix-valued polynomial.

Let X be a general **linear process** represented by

$$X_t = \sum_{j=-\infty}^{\infty} \Psi_j U_{t-j} \,,$$

where U is white noise with $EU_t=0$ and $\mathrm{var}(U_t)=\Sigma$.
We have

$$EX_t = \sum_{j=-\infty}^{\infty} \Psi_j EU_{t-j} = 0$$

and

$$\Gamma_X(k) = \mathrm{cov}(X_t, X_{t-k}) = E \sum_{j=-\infty}^{\infty} \Psi_j U_{t-j} \left(\sum_{j=-\infty}^{\infty} \Psi_j U_{(t-k)-j} \right)^T$$

$$= \sum_{r=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \Psi_r EU_{t-r} U^T_{t-(j+k)} \Psi^T_j$$

$$= \sum_{j=-\infty}^{\infty} \Psi_{j+k} \Sigma \Psi^T_j \,.$$

Since neither EX_t nor $\mathrm{cov}(X_t, X_{t-k})$ depend on t, the process X is weakly stationary. PL

The spectral densities of U and X are given by

$$f_U(\omega) = \tfrac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-i\omega k} \, \Gamma_U(k) = \tfrac{1}{2\pi} \Sigma \,,$$

$$f_X(\omega) = \tfrac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-i\omega k} \, \Gamma_X(k)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-i\omega k} \sum_{j=-\infty}^{\infty} \Psi_{j+k} \Sigma \Psi^T_j$$

$$= \tfrac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-i\omega k} \sum_{j=-\infty}^{\infty} \Psi_j \Sigma \Psi^T_{j-k}$$

$$= \tfrac{1}{2\pi} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left(e^{-i\omega j} \Psi_j \Sigma \right) \Psi^T_{j-k} \, e^{i\omega(j-k)}$$

$$= \tfrac{1}{2\pi} \sum_{j=-\infty}^{\infty} \Psi_j \, e^{-i\omega j} \, \Sigma \sum_{k=-\infty}^{\infty} \Psi^T_{j+k} \, e^{i\omega(j+k)}$$

$$= \tfrac{1}{2\pi} \sum_{j=-\infty}^{\infty} \Psi_j \, e^{-i\omega j} \, \Sigma \left(\sum_{j=-\infty}^{\infty} \Psi_j \, e^{-i\omega j} \right)^{*}.$$

PD

A representation

$$(I-\Phi L)X_t = U_t$$

of an AR(1) process X is called **causal** if X_t can be expressed in terms of present and past shocks, i.e.,

$$X_t = \left(\sum_{j=0}^{\infty} \Phi^j L^j\right) U_t = \sum_{j=0}^{\infty} \Phi^j U_{t-j} \,.$$

Its spectral density is given by

$$f_X(\omega) = \frac{1}{2\pi} \sum_{j=0}^{\infty} \Phi^j e^{-i\omega j} \, \Sigma \left(\sum_{j=0}^{\infty} \Phi^j e^{-i\omega j}\right)^*$$

$$= \frac{1}{2\pi} (I - \Phi e^{-i\omega})^{-1} \, \Sigma \, ((I - \Phi e^{-i\omega})^{-1})^* \,.$$

Exercise: Derive the sum formula PG

$$\sum_{j=0}^{n} \Phi^j = (I-\Phi^{n+1})(I-\Phi)^{-1}$$

for a geometric series of matrices.

Hint: Multiply each side of the equation by $I-\Phi$.

Remark: Moreover, if all eigenvalues of Φ have modulus less than 1, we have

$$\sum_{j=0}^{\infty} \Phi^j = (I-\Phi)^{-1}.$$

Exercise: Show that PV

$$\sum_{j=0}^{\infty} \Phi^j e^{-i\omega j} = (I-\Phi e^{-i\omega})^{-1},$$

if all eigenvalues of Φ have modulus less than 1.

Analogously,

$$f_X(\omega) = \frac{1}{2\pi} (I-\Phi_1 e^{-i\omega} - \ldots - \Phi_p e^{-i\omega p})^{-1} \Sigma ((I-\Phi_1 e^{-i\omega} - \ldots - \Phi_p e^{-i\omega p})^{-1})^*$$

is the spectral density of an AR(p) process X with causal representation

$$(I-\Phi_1 L - \ldots - \Phi_p L^p) X_t = U_t \,.$$

Exercise: Reexamine the relationship between changes in the industrial production and changes in the duration of unemployment with parametric methods.

• Write an R function for the calculation of the spectral density of a vector autoregressive process.

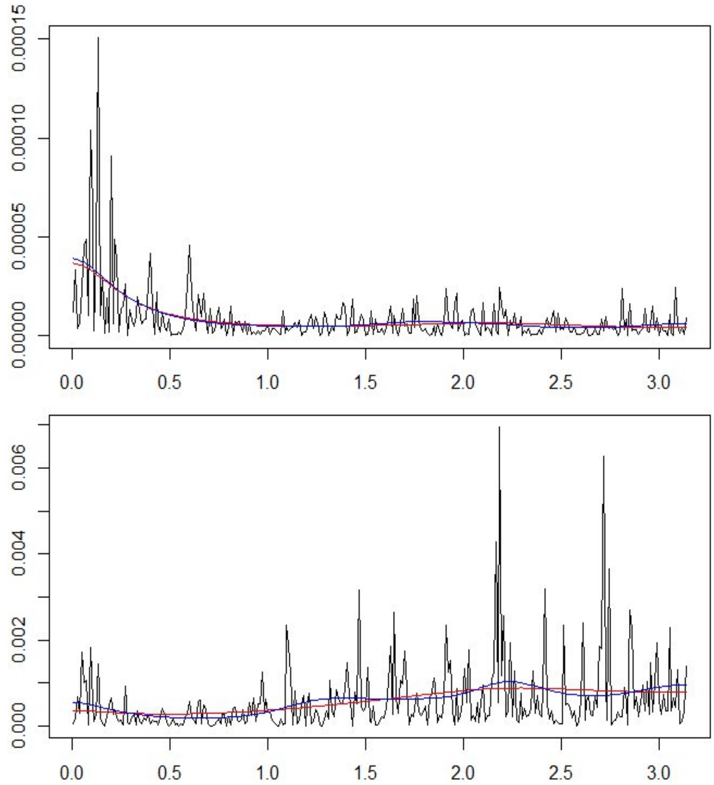
```
var.spec <- function(fr,AR.p) {
# fr    … vector of frequencies
# AR.p … AR(p) model estimated by R function ar
    nf <- length(fr); p <- AR.p$order
    sigma <- AR.p$var.pred; k <- length(sigma[1,])
    Id <- diag(1,nrow=k,ncol=k) # identity matrix
    sp <- array(dim=c(nf,k,k))
    for (w in 1:nf) {
        A <- Id
        for (l in 1:p) A <- A-AR.p$ar[l,,]*exp(-1i*fr[w]*l)
        A <- solve(A) # inverse of A
        sp[w,,] <- A%*%sigma%*%t(Conj(A)) }
    return(sp/(2*pi)) }
```

• Estimate AR models of order p=3 and p=6, respectively.

```
AR.3 <- ar(xy,order.max=3,aic=F,demean=T)
AR.6 <- ar(xy,order.max=6,aic=F,demean=T)
# aic=F … order is fixed, not selected automatically

# AR.3$ar: array of dim (3,2,2) with AR coefficients

AR.3$ar[1,,] # lag 1
              Series 1       Series 2
Series 1     0.2601357     0.007842615
Series 2    -0.5299257    -0.373324819
AR.3$ar[2,,] # lag 2
              Series 1       Series 2
Series 1     0.1446515     0.001099141
Series 2    -1.4206001    -0.173167764
AR.3$ar[3,,] # lag 3
              Series 1       Series 2
Series 1     0.1706967    -0.002957159
Series 2    -1.4166969    -0.011742439
AR.3$var.pred # variance not explained by AR model
              Series 1       Series 2
Series 1     4.423710e-05  -6.444694e-06
Series 2    -6.444694e-06   3.076618e-03
```

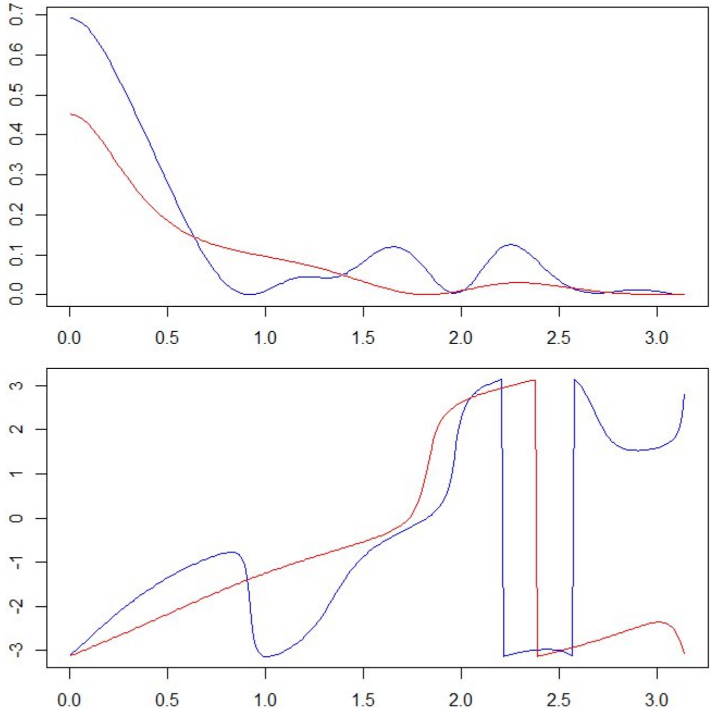
• Estimate the univariate spectral densities.

```
par(mfrow=c(2,1),mar=c(2,2,1,1))
p <- spec.pgram(xy[,1],taper=0,detr=F,fast=F,plot=F)
f <- p$freq*2*pi; plot(f,p$spec/(2*pi),type="l")
sp.3 <- var.spec(f,AR.3); lines(f,sp.3[,1,1],col="red")
sp.6 <- var.spec(f,AR.6); lines(f,sp.6[,1,1],col="blue")

p <- spec.pgram(xy[,2],taper=0,detr=F,fast=F,plot=F)
plot(f,p$spec/(2*pi),type="l")
lines(f,sp.3[,2,2],col="red")
lines(f,sp.6[,2,2],col="blue")
```



• Estimate the squared coherency and the phase spectrum.

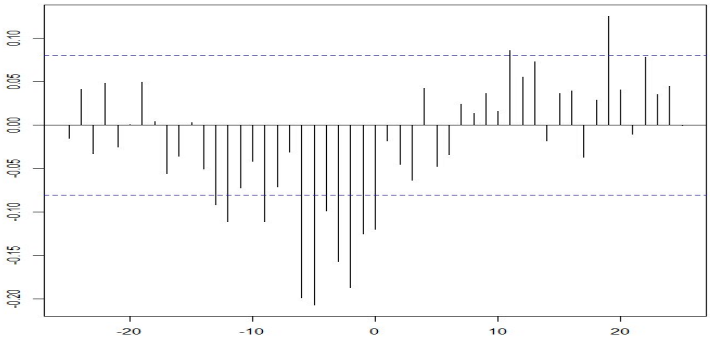
```
par(mfrow=c(2,1))
plot(f,Mod(sp.6[,1,2])^2/(sp.6[,1,1]*sp.6[,2,2]),type="l",
  col="blue")
lines(f,Mod(sp.3[,1,2])^2/(sp.3[,1,1]*sp.3[,2,2]),
  col="red")

plot(f,Arg(sp.6[,1,2]),type="l",col="blue")
lines(f,Arg(sp.3[,1,2]),type="l",col="red")
```



The squared coherency is large at the low frequencies. There the slope of the phase spectrum is approximately 2, which indicates that changes in the duration of unemployment lag two months behind changes in industrial production.

• Estimate the cross-correlation function.

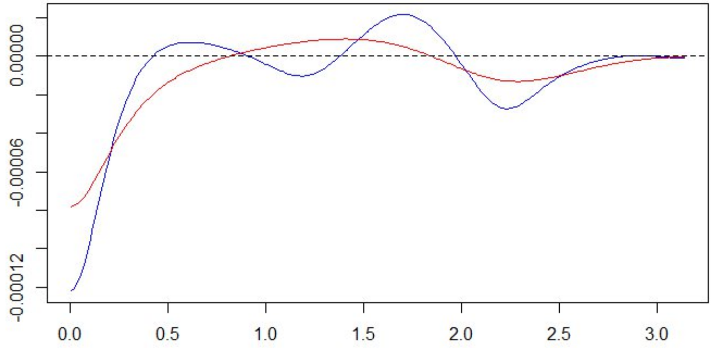
```
ccf(x,y,lag.max=25,type="correlation")
```

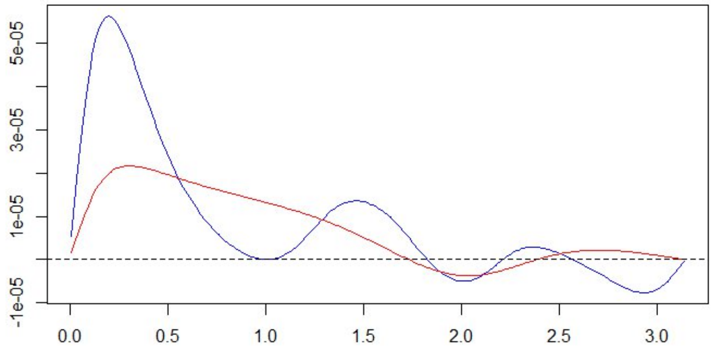


Significant negative correlations at small negative lags are in line with the results of the cross spectral analysis.

- Estimate the cospectrum and the quadrature spectrum.

```
par(mfrow=c(2,1))
plot(f,Re(sp.6[,1,2]),type="l",col="blue")
lines(f,Re(sp.3[,1,2]),type="l",col="red")
abline(h=0,lty="dashed") # dashed horizontal line


plot(f,-Im(sp.6[,1,2]),type="l",col="blue")
lines(f,-Im(sp.3[,1,2]),type="l",col="red")
abline(h=0,lty=2)
```





The cospectrum indicates that the overall negative relationship between the two variables is mainly due to the low frequencies.

The quadrature spectrum does not consistently deviate from zero enough to allow a meaningful interpretation.

A stationary process X is called an **autoregressive moving average process of order (p,q)** (or **ARMA(p,q) process**) if it can be expressed as

$$X_t = \Phi_1 X_{t-1} + \ldots + \Phi_p X_{t-p} + U_t + \Theta_1 U_{t-1} + \ldots + \Theta_q U_{t-q}$$

or, equivalently, as

$$X_t - \Phi_1 X_{t-1} - \ldots - \Phi_p X_{t-p} = U_t + \Theta_1 U_{t-1} + \ldots + \Theta_q U_{t-q},$$

where U is white noise with mean vector 0.

Using lag-operator notation, the latter equation can also be written as

$$\Phi(L) X_t = \Theta(L) U_t,$$

where

$$\Phi(L) = I - \Phi_1 L - \ldots - \Phi_p L^p$$

and

$$\Theta(L) = I + \Theta_1 L + \ldots + \Theta_q L^q$$

are matrix-valued polynomials.

An ARMA(p,0) process is an AR(p) process. An ARMA(0,q) process is also called a **moving average process of order q** (or **MA(q) process**).

The ARMA(p,q) equation

$$(I - \Phi_1 L - \ldots - \Phi_p L^p) X_t = (I + \Theta_1 L + \ldots + \Theta_q L^q) U_t$$

is said to be **causal** if

$$|z| \le 1 \Rightarrow \det(I - z\Phi_1 - \ldots - z^p \Phi_p) \ne 0.$$

It is said to be **invertible** if

$$|z| \le 1 \Rightarrow \det(I + z\Theta + \ldots + z^q \Theta_q) \ne 0.$$

Exercise: Show that the bivariate AR(1) process

$$\begin{pmatrix} X_{t1} \\ X_{t2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{(t-1)1} \\ X_{(t-1)2} \end{pmatrix} = \begin{pmatrix} U_{t1} \\ U_{t2} \end{pmatrix}$$

is causal and invertible. PC

Exercise: Show that the bivariate MA(2) process

$$\begin{pmatrix} X_{t1} \\ X_{t2} \end{pmatrix} = \begin{pmatrix} U_{t1} \\ U_{t2} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} U_{(t-1)1} \\ U_{(t-1)2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U_{(t-2)1} \\ U_{(t-2)2} \end{pmatrix}$$

is causal and invertible. PI

Exercise: Show that the bivariate ARMA(1,1) process

$$\begin{pmatrix} X_{t1} \\ X_{t2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} X_{(t-1)1} \\ X_{(t-1)2} \end{pmatrix} = \begin{pmatrix} U_{t1} \\ U_{t2} \end{pmatrix} + \begin{pmatrix} 0 & \frac{5}{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_{(t-1)1} \\ U_{(t-1)2} \end{pmatrix}$$

is causal and invertible. PA

It does not make sense to estimate the parameter matrices $\Phi_1,\ldots,\Phi_p,\Theta_1,\ldots,\Theta_q$, and Σ of an ARMA(p,q) process if they are not unique.

To ensure identifiability in the univariate case, where $\Phi(L)$ and $\Theta(L)$ are just scalar polynomials, we must require, in addition to causality and invertibility, that $\Phi(z)$ and $\Theta(z)$ have no common zeros. For example, the equation

$$(1-\tfrac{1}{4}L^2)X_t=(1+\tfrac{1}{2}L)U_t$$

can be written more parsimoniously as

$$(1-\tfrac{1}{2}L)X_t=U_t,$$

because the polynomials

$$1-\tfrac{1}{4}z^2=(1+\tfrac{1}{2}z)(1-\tfrac{1}{2}z)$$

and

$$1+\tfrac{1}{2}z$$

have a common zero.

In the multivariate case, the matrix-valued polynomials $\Phi(z)$ and $\Theta(z)$ can have a common left factor even if $\det(\Phi(z))$ and $\det(\Theta(z))$ have no common zero. To avoid the difficulties involved in the identification of multivariate ARMA processes, many time series analysts use only multivariate AR models for the modeling of multivariate time series.

<u>Exercise:</u> Show that the equation PU

$$\begin{pmatrix} X_{t1} \\ X_{t2} \end{pmatrix} - \begin{pmatrix} 0 & \phi+\theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{(t-1)1} \\ X_{(t-1)2} \end{pmatrix} = \begin{pmatrix} U_{t1} \\ U_{t2} \end{pmatrix} + \begin{pmatrix} 0 & -\theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_{(t-1)1} \\ U_{(t-1)2} \end{pmatrix}$$

can be written more parsimoniously as

$$\begin{pmatrix} X_{t1} \\ X_{t2} \end{pmatrix} - \begin{pmatrix} 0 & \phi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{(t-1)1} \\ X_{(t-1)2} \end{pmatrix} = \begin{pmatrix} U_{t1} \\ U_{t2} \end{pmatrix}$$

although the polynomials

$$\det(\Phi(z)) = \det\left(I - \begin{pmatrix} 0 & \phi+\theta \\ 0 & 0 \end{pmatrix} z\right)$$

and

$$\det(\Theta(z)) = \det\left(I + \begin{pmatrix} 0 & -\theta \\ 0 & 0 \end{pmatrix} z\right)$$

have no common zero.

<u>Hint:</u> Multiply both $\Phi(z)$ and $\Theta(z)$ by $\Theta^{-1}(z) = \begin{pmatrix} 1 & \theta z \\ 0 & 1 \end{pmatrix}$.

<u>Remark:</u> The inverse of the matrix-valued polynomial $\Theta(z)$ is also a matrix-valued polynomial. Its determinant is a constant unequal to zero. Such a matrix-valued polynomial is called **unimodular**.