Consider a discrete-time random walk

$$x_t = u_1 + \ldots + u_t$$

with i.i.d. $N(0,\sigma^2)$ increments u_t.

Exercise: Show that **U1**

(i) $x_t \sim N(0, t\sigma^2)$,

(ii) $s<t \Rightarrow x_t - x_s \sim N(0,(t-s)\sigma^2)$,

(iii) $q<r\leq s<t \Rightarrow x_r - x_q$ and $x_t - x_s$ are independent.

Exercise: Show that the discrete-time random walk satisfies the difference equation

$$x_t = \phi x_{t-1} + u_t$$

with $x_0=0$ and $\phi=1$. **U2**

Exercise: Show that **UL**

$$\frac{1}{n}\left(\sqrt{n}\,\bar{x}\right) = n^{-\frac{3}{2}} \sum_{t=1}^{n} x_t \xrightarrow{L} N(0, \tfrac{\sigma^2}{3}).$$

Solution:

$$\sum_{t=1}^{n} x_t = u_1 + (u_1+u_2) + \ldots + (u_1+\ldots+u_n)$$

$$= n u_1 + (n-1) u_2 + \ldots + u_n$$

$$\sim N(0,(n^2+(n-1)^2+\ldots+1^2)\sigma^2)$$

$$= N(0, \tfrac{n(n+1)(2n+1)}{6}\sigma^2)$$

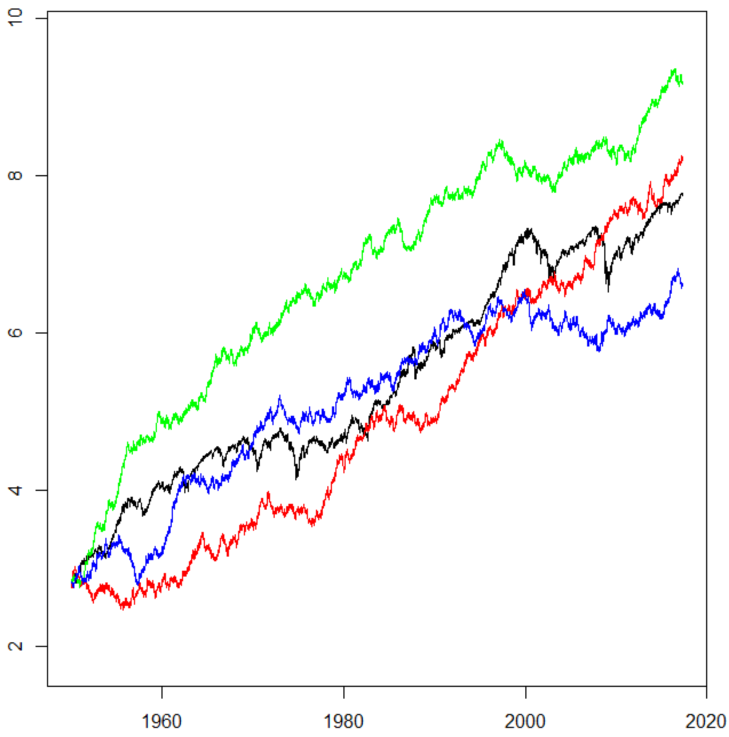
$$\Rightarrow n^{-\frac{3}{2}} \sum_{t=1}^{n} x_t \sim N(0, \underbrace{\frac{n(n-1)(2n+1)}{6n^3}}_{\to \frac{\sigma^2}{3}} \sigma^2).$$

Exercise: Compare the daily quotes of the S&P 500 index (symbol: ^GSPC) with realizations of a random walk.

Download the historical prices from Yahoo!Finance as a csv file **^GSPC.csv** into the working directory **C:\SP500.**

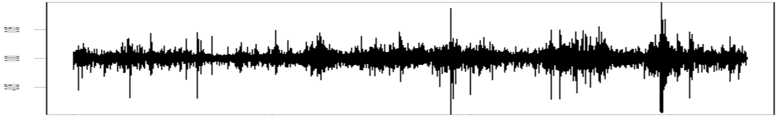
Import the data into R and plot the log closing prices and 3 realizations of a random walk (with drift) with matching parameters (starting value, mean and variance of log returns).

```
setwd("C:/SP500")   # R uses / as path separator
Y <- read.csv("^GSPC.csv",header=T,na.strings="null")
# in the downloaded file, missing values are represented
# by the string "null" rather than by the symbol NA
Y <- na.omit(Y)   # rows with missing values are omitted
N <- nrow(Y); D <- as.Date(Y[,1])   # dates in column 1
cl <- log(Y[,6])   # adjusted close prices in column 6
r <- cl[2:N]-cl[1:(N-1)]; n <- N-1# n (log) returns
my <- mean(r); sigma <- sd(r)          # sample moments
par(mar=c(2,2,1,1)); COL <- c("red","green","blue")
plot(D,cl,type="l",ylim=range(cl)+c(-1,2))
for (j in 1:3) { u <- rnorm(n,m=my,sd=sigma)
    x <- cumsum(c(cl[1],u)); lines(D,x,col=COL[j]) }
```



Exercise: Compare the (log) returns with matching Gaussian increments and resampled returns.

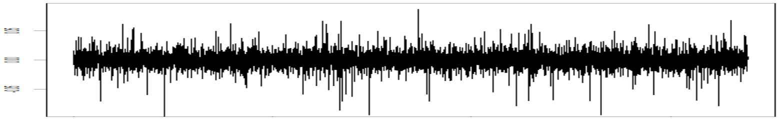
```
par(mar=c(0.1,2,0.1,0.1)); YL <- c(-0.09,0.09)
plot(r,type="l",ylim=YL)   # returns
```



```
u <- rnorm(n,m=my,sd=sigma); plot(u,t="l",ylim=YL)
```

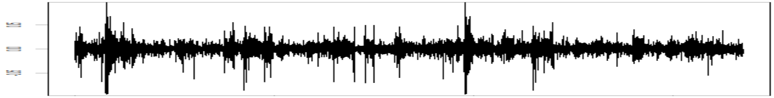
Obviously, the density of the returns has more probability mass near the center as well as in the tails than a normal density. A more realistic sample of synthetic returns can be obtained by resampling the given returns.

```
u <- sample(r,size=n,replace=T); plot(u,t="l",ylim=YL)
```

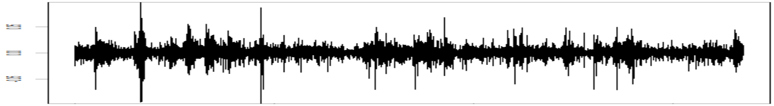


Resampling blocks of returns rather than individual returns produces clusters of different volatility.

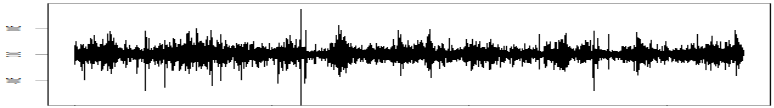
```
# k=n/b nonoverlapping blocks of length b:
b <- 250; k <- trunc(n/b); K  <- sample(1:k,k,replace=T)
u <- NULL; for (j in K) u <- c(u,r[(j-1)*b+1:b])
```



```
# k=n/b overlapping blocks of length b:
b <- 250; k <- trunc(n/b); K  <- sample(0:(n-b),k,T)
u <- NULL; for (j in K) u <- c(u,r[j+1:b])
```



```
# stationary bootstrap: blocks of random length
q <- 0.996; K <- sample(1:n,n,T) # mean length=1/(1-q)
for (i in 2:n) if (runif(1)<q) # increase block with prob. q
K[i] <- ifelse(K[i-1]!=n,K[i-1]+1,1); u <- r[K]
```



Given n observations $x_1,\ldots,x_n$ from a discrete-time random walk, a simple continuous-time process can be defined by

$$x_n(t)=x_{[tn]},\ t\in[0,1],$$

where $x_0=0$ and $[tn]$ is the greatest integer less than or equal to tn.

Exercise: Show that as $n\to\infty$ UC

(i) $\mathrm{Var}(x_n(1))\to\infty$

and (ii) $0<n^{\alpha}\mathrm{Var}(x_n(1))<\infty \Leftrightarrow \alpha=-\frac{1}{2}$.

Exercise: Show that the continuous-time process UN

$$x_n^*(t)=n^{-\frac{1}{2}}x_n(t)=n^{-\frac{1}{2}}x_{[tn]},\ t\in[0,1],$$

satisfies (i) $x_n^*(0)=0$,

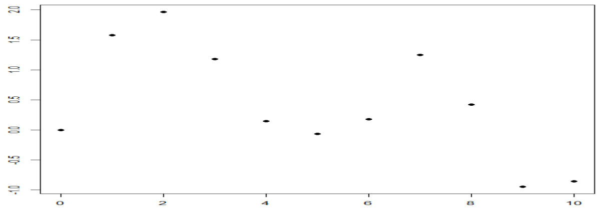
(ii) $x_n^*(t)\xrightarrow{L}\mathrm{N}(0,t\sigma^2)$,

(iii) $s<t\Rightarrow x_n^*(t)-x_n^*(s)\xrightarrow{L}N(0,(t-s)\,\sigma^2)$,

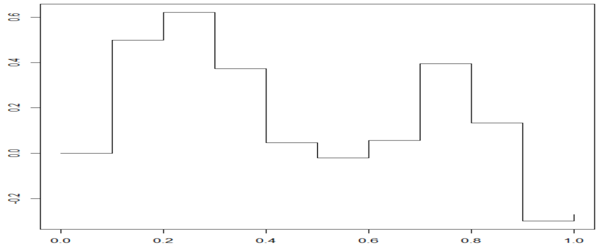
(iv) $q<r\le s<t\Rightarrow x_n^*(r)-x_n^*(q)$ and $x_n^*(t)-x_n^*(s)$ are independent if n is sufficiently large.

Exercise: Plot realizations of the discrete-time process x_t and the continuous-time process $x_n^*(t)$ for $\sigma^2=1$ and $n=10$.

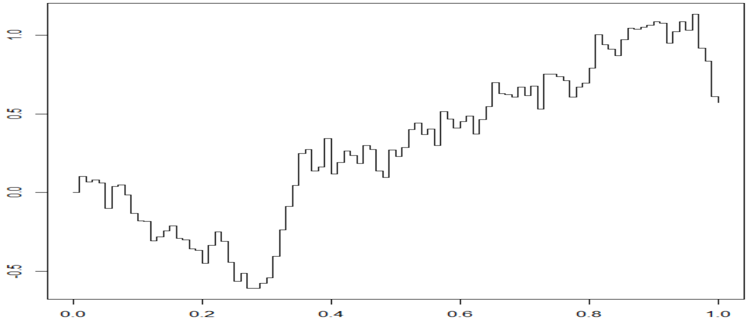
```
par(mar=c(2,2,0.1,0.1),pch=19); n <- 10
t <- c(0:n); x <- cumsum(c(0,rnorm(n))); plot(t,x)
```

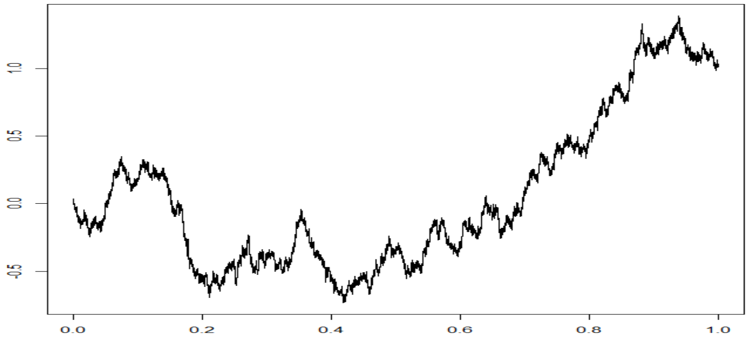


```
t <- t/n; x <- x/n^0.5; plot(t,x,type="s")
# "s": stair steps (move first horizontal, then vertical)
```



Exercise: Plot realizations of the process $x_n^*(t)$ for $\sigma^2=1$ and $n=100$, 100 000.





As n increases the height of the jumps in the graph of $x_n^*(t)$ decreases. We can therefore expect continuity in the limit. Of course, this does not imply smoothness.

A continuous-time process $B(t)$, $t\in[0,1]$, is called **Brownian motion** with variance σ^2 if

(i) $B(0)=0$,

(ii) $B(t)\sim \mathrm{N}(0,t\sigma^2)$,

(iii) $s<t \Rightarrow B(t)-B(s)\sim N(0,(t-s)\,\sigma^2)$,

(iv) $q<r\leq s<t \Rightarrow B(r)-B(q)$ and $B(t)-B(s)$ are independent.

Brownian motion with variance 1 is called **standard Brownian motion** or **Wiener process**.

Exercise: Show that UB

$$s<t \Rightarrow \mathrm{Cov}(B(s),B(t))=s\sigma^2.$$

Solution:

$$\begin{aligned}\mathrm{Cov}(B(s),B(t)) &= \mathrm{Cov}(B(s),(B(t)-B(s))+B(s)) \\ &= \mathrm{Cov}(B(s),B(t)-B(s))+\mathrm{Cov}(B(s),B(s)) \\ &= \mathrm{Cov}(B(s)-B(0),B(t)-B(s))+\mathrm{Var}(B(s)) \\ &= 0+s\sigma^2\end{aligned}$$

It can be shown that any realization of Brownian motion is everywhere continuous and nowhere differentiable with probability 1.

Indeed, for $0<h\to 0$ we have

$$\begin{aligned}E(B(t+h)-B(t))^2 &= \mathrm{Var}(B(t+h)-B(t)) \\ &= ((t+h)-t)\sigma^2 \\ &= h\sigma^2 \to 0\end{aligned}$$

and

$$\begin{aligned}E\left(\frac{B(t+h)-B(t)}{h}\right)^2 &= \tfrac{1}{h^2}h\sigma^2 \\ &= \tfrac{\sigma^2}{h} \to \infty.\end{aligned}$$

UD

For any fixed $0<\tau\leq 1$, the central limit theorem applied to the mean

$$\frac{1}{[\tau n]}\sum_{t=1}^{[\tau n]} u_t$$

of the fraction $u_1,\ldots,u_{[\tau n]}$ of the whole sample $u_1,\ldots,u_n$ gives

$$\sqrt{[\tau n]}\,\frac{1}{[\tau n]}\sum_{t=1}^{[\tau n]} u_t \xrightarrow{L} N(0,\sigma^2)$$

and

$$x_n^*(\tau)=\frac{1}{\sqrt{n}}\sum_{t=1}^{[\tau n]} u_t=\sqrt{\frac{[\tau n]}{n}}\,\frac{\sqrt{[\tau n]}}{[\tau n]}\sum_{t=1}^{[\tau n]} u_t \xrightarrow{L} \underbrace{\sqrt{\tau}N(0,\sigma^2)}_{=N(0,\tau\sigma^2)}.$$

In contrast, the **functional central limit theorem** concerns the asymptotic behavior of x_n^* regarded as a stochastic function of τ, i.e.,

$$x_n^* \xrightarrow{L} B,$$

where B is Brownian motion with variance σ^2.

For the extension of convergence in law to random functions, it is required, among other conditions, that

$$(x_n^*(\tau_1),\ldots,x_n^*(\tau_k))^T \xrightarrow{L} (B(\tau_1),\ldots,B(\tau_k))^T$$

for any $0\leq\tau_1<\ldots<\tau_k\leq 1$.

The continuous mapping theorem (CMT):

For a sequence of random variables x_t and a continuous function g, we have

$$x_n \xrightarrow{L} x \;\Rightarrow\; g(x_n) \xrightarrow{L} g(x).$$

Analogously, we have for a sequence of stochastic functions f_n and a continuous functional g,

$$f_n \xrightarrow{L} f \;\Rightarrow\; g(f_n) \xrightarrow{L} g(f).$$

Functionals map a function into a real number and a stochastic function into a random variable, respectively.

Examples: (i) $g(f)=f(0)$, (ii) $g(f)=\int_0^1 f(\tau)d\tau$

Example: $g_1(f) = \int_0^1 f(\tau)d\tau$

$$g_1(x_n^*) = \int_0^1 x_n^*(\tau)d\tau = \frac{1}{\sqrt{n}} \int_0^1 x_{[\tau n]}d\tau$$

$$= \frac{1}{\sqrt{n}} \left(\int_0^{\frac{1}{n}} x_{[\tau n]}d\tau + \ldots + \int_{\frac{n-1}{n}}^1 x_{[\tau n]}d\tau \right)$$

$$= \frac{1}{\sqrt{n}} \left(\int_0^{\frac{1}{n}} x_0 d\tau + \ldots + \int_{\frac{n-1}{n}}^1 x_{n-1}d\tau \right) = n^{-\frac{3}{2}} \sum_{t=1}^n x_{t-1}$$

$$x_n^* \xrightarrow{L} B \Rightarrow g_1(x_n^*) = \int_0^1 x_n^*(\tau)d\tau \xrightarrow{L} g_1(B) = \int_0^1 B(\tau)d\tau$$

$$n^{-\frac{3}{2}} \sum_{t=1}^n x_{t-1} \xrightarrow{L} N(0, \tfrac{\sigma^2}{3}) \Rightarrow \int_0^1 B(\tau)d\tau \sim N(0, \tfrac{\sigma^2}{3})$$

RM

Example: $g_2(f) = \int_0^1 (f(\tau))^2 d\tau$

$$g_2(x_n^*) = \int_0^1 (x_n^*(\tau))^2 d\tau = \frac{1}{n} \int_0^1 x_{[\tau n]}^2 d\tau$$

$$= \frac{1}{n} \left(\int_0^{\frac{1}{n}} x_{[\tau n]}^2 d\tau + \ldots + \int_{\frac{n-1}{n}}^1 x_{[\tau n]}^2 d\tau \right)$$

$$= \frac{1}{n} \left(\int_0^{\frac{1}{n}} x_0^2 d\tau + \ldots + \int_{\frac{n-1}{n}}^1 x_{n-1}^2 d\tau \right) = \frac{1}{n^2} \sum_{t=1}^n x_{t-1}^2 .$$

$$x_n^* \xrightarrow{L} B \Rightarrow g_2(x_n^*) = \int_0^1 (x_n^*(\tau))^2 d\tau \xrightarrow{L} g_2(B) = \int_0^1 (B(\tau))^2 d\tau$$

R2

Under the unit root hypothesis RU

$$H_0: \phi = 1,$$

the expected value of the denominator of the statistic

$$\hat{\phi} - \phi = \hat{\phi} - 1 = \tfrac{1}{n-1} \sum_{t=2}^{n} x_{t-1} u_t \,/\, \tfrac{1}{n-1} \sum_{t=2}^{n} x_{t-1}^2$$

is given by

$$\tfrac{1}{n-1} \sum_{t=2}^{n} E(u_1 + \ldots + u_{t-1})^2 = \tfrac{\sigma^2}{n-1} \sum_{t=2}^{n} (t-1) = \tfrac{n\sigma^2}{2},$$

which implies that we need to multiply $\hat{\phi} - 1$ by n in order to obtain a nondegenerate asymptotic distribution.

The estimator $\hat{\phi}$ is called a **superconsistent** estimator, because it converges to $\phi = 1$ at a faster rate than usual.

We have

$$n(\hat{\phi} - 1) = \tfrac{1}{n} \sum_{t=2}^{n} x_{t-1} u_t \,/\, \tfrac{1}{n^2} \sum_{t=2}^{n} x_{t-1}^2$$

$$= \tfrac{1}{n} (u_1 u_2 + (u_1 + u_2) u_3 + \ldots + (u_1 + \ldots + u_{n-1}) u_n) \,/\, g_2(x_n^*)$$

$$= \tfrac{1}{n} \left(\tfrac{1}{2} \left(\sum_{t=1}^{n} u_t \right)^2 - \tfrac{1}{2} \sum_{t=1}^{n} u_t^2 \right) \,/\, \int_0^1 (x_n^*(\tau))^2 \, d\tau$$

$$\approx \tfrac{1}{2} \left((x_n^*(1))^2 - \sigma^2 \right) \,/\, \int_0^1 (x_n^*(\tau))^2 \, d\tau$$

$$= g_3(x_n^*)$$

$$\xrightarrow{L} \quad g_3(B) = \tfrac{1}{2} \left((B(1))^2 - \sigma^2 \right) \,/\, \int_0^1 (B(\tau))^2 \, d\tau$$

$$= \tfrac{1}{2} \left((W(1))^2 - 1 \right) \,/\, \int_0^1 (W(\tau))^2 \, d\tau .$$

We cannot use the statistics $(n-1)(\hat{\phi}-1)$ or $n(\hat{\phi}-1)$ to test the unit root hypothesis

$$H_0\text{: } \phi=1$$

against the alternative hypothesis

$$H_A\text{: } \phi<1$$

unless we have critical values.

For the calculation of critical values, we do not need to use the asymptotic distribution of the respective test statistic. Instead, we can use Monte Carlo techniques. First, we can generate m pseudo-random samples

$$u_1(j),\ldots,u_n(j),\ j=1,\ldots,m,$$

of $N(0,1)$ variates and then compute $\hat{\phi}$ for each sample. Finally, order statistics are used to estimate the quantiles of interest.

<u>Exercise:</u> Find critical values for the test statistic $n(\hat{\phi}-1)$. Use n=25, 50, 100, 1000, m=1000,100000, and α=0.05.

```
m <- 1000; n <- 25; n1 <- n-1; phi1 <- rep(0,m)
for (i in 1:m)
  { u <- rnorm(n); x <- cumsum(u)[1:n1]
    phi1[i] <- sum(x*u[2:n])/sum(x*x)  }  # phi1=phi-1
q <- quantile(phi1,probs=0.05); cr.val <- n*q; cr.val
-7.654794
```

Analogously, we obtain the remaining values (α=0.05):

n	for m=1000	for m=100 000
25	-7.7	-7.3
50	-7.7	-7.8
100	-8.1	-7.8
1000	-8.5	-8.1

Clearly, the values obtained with m=100000 are more reliable than those obtained with m=1000.

Furthermore, we can use the critical values obtained for large values of n, e.g., n=1000, as estimates of the critical values of the asymptotic distribution.

Exercise: Test H_0: ϕ=1 for a synthetic AR(1) series.

```
x <- arima.sim(list(order=c(1,0,0),ar=0.7),n=25) # φ=0.7
n <- 25; x1 <- x[1:(n-1)]
phi <- sum(x1*x[2:n])/sum(x1*x1)
n*(phi-1)
-8.7104
```

The unit root hypothesis H_0 can be rejected, because the value of the test statistic is less than the critical value for a sample size of 25, i.e., -8.7104 < -7.3.

Exercise: Suppose that $x_1,\ldots,x_n$ are non-stochastic and $u_1,\ldots,u_n$ are uncorrelated with common mean 0 and variance σ^2. Show that in the linear regression model

$$y_t = \beta x_t + u_t$$

the variance of the OLS estimator $\hat{\beta}$ is given by

$$\mathrm{var}(\hat{\beta}) = \sigma^2 / \sum_{t=2}^{n} x_{t-1}^2 .$$

RV

Another way of testing the unit root hypothesis is to write the model

$$x_t=\phi x_{t-1}+u_t$$

as

$$\Delta x_t=x_t-x_{t-1}=\phi x_{t-1}+u_t-x_{t-1}=\phi^* x_{t-1}+u_t,$$

where $\phi^*=\phi-1$, and reject H_0 if the value of the OLS estimator

$$\hat{\phi}^* = \sum_{t=2}^{n} x_{t-1}\Delta x_t \Big/ \sum_{t=2}^{n} x_{t-1}^2$$

or, alternatively, the conventional OLS t-ratio

$$t = \hat{\phi}^* / \sqrt{\widehat{\mathrm{var}}(\hat{\phi}^*)} ,$$

where

$$\widehat{\mathrm{var}}(\hat{\phi}^*) = \frac{1}{n-2}\sum_{t=2}^{n}(\Delta x_t - \hat{\phi}^* x_{t-1})^2 \Big/ \sum_{t=2}^{n} x_{t-1}^2 ,$$

is much smaller than 0. The test based on t is called **Dickey-Fuller test**. Clearly, t has neither a t-distribution nor a limiting normal distribution if ϕ=1.

A more realistic model is obtained by introducing additional lags in order to allow for serial correlation:

$$x_t=\phi_1 x_{t-1}+\ldots+\phi_p x_{t-p}+u_t$$

Writing the model as

$$\begin{aligned}\Delta x_t&=(\phi_1-1)\,x_{t-1}+\ldots+\phi_p\,x_{t-p}+u_t\\ &=[(\phi_1+\ldots+\phi_p-1)-(\phi_2+\ldots+\phi_p)]\,x_{t-1}\\ &\qquad+[(\phi_2+\ldots+\phi_p)-(\phi_3+\ldots+\phi_p)]\,x_{t-2}\\ &\qquad\vdots\\ &\qquad+[(\phi_{p-1}+\phi_p)-\phi_p]\,x_{t-(p-1)}\\ &\qquad+\phi_p\,x_{t-p}+u_t\\ &=(\phi_1+\ldots+\phi_p-1)\,x_{t-1}-(\phi_2+\ldots+\phi_p)\Delta x_{t-1}-\ldots-\phi_p\Delta x_{t-(p-1)}+u_t\\ &=\phi^* x_{t-1}+\delta_1\Delta x_{t-1}+\ldots+\delta_p\Delta x_{t-p-1}+u_\mathrm{t}\end{aligned}$$

we see that the unit root hypothesis

$$\Phi(1)=1-\phi_1-\ldots-\phi_p=0$$

is equivalent to the hypothesis H_0: $\phi^*=0$. RL

Including also a constant term and a linear time trend, we obtain an even more general model:

$$\Delta x_t=\alpha+\beta t+\phi^* x_{t-1}+\delta_1\Delta x_{t-1}+\ldots+\delta_p\Delta x_{t-p}+u_t$$

The test of the hypothesis H_0: $\phi^*=0$, which is based on the conventional OLS t-ratio for ϕ^*, is called **augmented Dickey-Fuller (ADF) test**.

In practice, it is extremely hard to decide whether a constant term and a time trend should be included and how many lags should be included. Unfortunately, different model specifications typically produce different test results.

Exercise: Apply an augmented Dickey-Fuller test to the log S&P500 series created above.

Hint:

```
library(tseries)  # the package tseries is loaded
help(adf.test)
```