HILBERT SPACE GEOMETRY

Definition: A **vector space** over $\mathbb{R}$ is a set V (whose elements are called **vectors**) together with a binary operation

$$+: V \times V \to V,$$

which is called **vector addition**, and an external binary operation

$$\cdot: \mathbb{R} \times V \to V,$$

which is called **scalar multiplication**, such that

(i) (V,+) is a commutative group
(whose neutral element is called **zero vector**)

and (ii) for all $\lambda,\mu \in \mathbb{R}$, $x,y \in V$:

$$\lambda(\mu x) = (\lambda\mu)x,$$
$$1x = x,$$
$$\lambda(x+y) = (\lambda x)+(\lambda y),$$
$$(\lambda+\mu)x = (\lambda x)+(\mu x),$$

where the image of $(x,y) \in V \times V$ under + is written as x+y and the image of $(\lambda,x) \in \mathbb{R} \times V$ under $\cdot$ is written as λx or as $\lambda \cdot x$.

Exercise: Show that the set $\mathbb{R}^2$ together with vector addition and scalar multiplication defined by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

and

$$\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix},$$

respectively, is a vector space.

Remark: Usually we do not distinguish strictly between a vector space $(V,+,\cdot)$ and the set of its vectors V. For example, in the next definition V will first denote the vector space and then the set of its vectors.

Definition: If V is a vector space and $M \subseteq V$, then the set of all linear combinations of elements of M is called **linear hull** or **linear span** of M. It is denoted by span(M). By convention, $\text{span}(\emptyset)=\{0\}$.

Proposition: If V is a vector space, then the linear hull of any subset M of V (together with the restriction of the vector addition to $M \times M$ and the restriction of the scalar multiplication to $\mathbb{R} \times M$) is also a vector space.

Proof: We only need to prove that span(M) contains the zero vector and that it is closed under vector addition and scalar multiplication:

$$M=\emptyset \Rightarrow \text{span}(M)=\{0\} \Rightarrow 0 \in \text{span}(M)$$
$$M \neq \emptyset \Rightarrow \exists x \in M: 0 \cdot x = 0 \in \text{span}(M)$$

$$x,y \in \text{span}(M) \Rightarrow x+y=1\cdot x+1\cdot y \in \text{span}(M)$$
$$x \in \text{span}(M),\ \lambda \in \mathbb{R} \Rightarrow \lambda \cdot x \in \text{span}(M)$$

The other properties of a vector space are satisfied for all elements of V and therefore also for all elements of $M \subseteq V$.

Definition: If a subset M of a vector space V is also a vector space, it is called a **linear subspace** of V.

<u>Definition:</u> An **inner product space** is a vector space V together with a function

$$\langle\ \rangle : V \times V \to \mathbb{R}$$

(called **inner product**) satisfying the following axioms:

For all x,y,z$\in$ V, $\lambda \in \mathbb{R}$

(i) $\langle x, y\rangle = \langle y, x\rangle$,

(ii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,

(iii) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$,

(iv) $\langle x, x\rangle \geq 0$,

(v) $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.

A **semi-inner product** satisfies (i) – (iv), but $\langle x, x\rangle$ can be zero if $x \neq 0$.

<u>Exercise:</u> Show that the inner product axioms (i)-(iii) imply that for all x,y,z,u$\in$ V, $\lambda,\mu,\nu,\xi \in \mathbb{R}$

$$\langle \lambda x + \mu y, \nu z + \xi u\rangle = \lambda\nu\langle x, z\rangle + \lambda\xi\langle x, u\rangle + \mu\nu\langle y, z\rangle + \mu\xi\langle y, u\rangle.$$

<u>Exercise:</u> Show that the vector space $\mathbb{R}^2$ together with the function $\langle\ \rangle$ defined by

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1 y_1 + x_2 y_2$$

is an inner product space.

Definition: The **norm (seminorm)** of an element x of an inner product space (semi-inner product space) is defined by

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Cauchy-Schwarz Inequality: If x and y are elements of an inner product space, then

$$|\langle x, y\rangle| \le \|x\|\|y\|.$$

Proof: $0 \le \langle \|y\|x \pm \|x\|y, \|y\|x \pm \|x\|y\rangle$

$$= \|y\|^2\langle x, x\rangle \pm 2\|x\|\|y\|\langle x, y\rangle + \|x\|^2\langle y, y\rangle$$

$$= 2\|x\|^2\|y\|^2 \pm 2\|x\|\|y\|\langle x, y\rangle$$

$$= 2\|x\|\|y\|(\|x\|\|y\| \pm \langle x, y\rangle)$$

$$\Rightarrow 0 \le \|x\|\|y\| \pm \langle x, y\rangle \Rightarrow \pm\langle x, y\rangle \le \|x\|\|y\|$$

Exercise: Let V be a semi-inner product space.
Show that for all x,y,z$\in$V, $\lambda \in \mathbb{R}$

(i) $\|x + y\| \le \|x\| + \|y\|$,
(ii) $\|\lambda x\| = |\lambda|\,\|x\|$,
(iii) $\|x\| \ge 0$,

and, if V is an inner product space, also

(iv) $\|x\| = 0 \Leftrightarrow x = 0$.

Lemma: The triangle inequality $\|x+y\| \le \|x\| + \|y\|$ implies that for all x and y

$$\|x-y\| \ge \big|\, \|x\| - \|y\| \,\big|.$$

Proof: $\|x\| = \|(x-y)+y\| \le \|x-y\| + \|y\| \Rightarrow \|x-y\| \ge \|x\| - \|y\|$

$\|y\| = \|(y-x)+x\| \le \|y-x\| + \|x\| \Rightarrow \|y-x\| \ge \|y\| - \|x\|$

Continuity of the Norm: If the sequence (x_n) of elements of an inner product space V converges in norm to $x \in V$, then the sequence $\|x_n\|$ converges to $\|x\|$, i.e.,.

$$\|x_n - x\| \to 0 \Rightarrow \|x_n\| \to \|x\|.$$

Proof: $0 \le \big|\, \|x_n\| - \|x\| \,\big| \le \|x_n - x\| \to 0$

Continuity of the Inner Product: If the sequences (x_n) and (y_n) of elements of an inner product space V converge in norm to $x \in V$ and $y \in V$, respectively, then the sequence $\langle x_n, y_n \rangle$ converges to $\langle x, y \rangle$, i.e.,

$$\|x_n - x\| \to 0,\ \|y_n - y\| \to 0 \Rightarrow \langle x_n, y_n \rangle \to \langle x, y \rangle.$$

Proof:

$$\begin{aligned} 0 \le |\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n - y \rangle + \langle x_n - x, y \rangle| \\ &\le |\langle x_n, y_n - y \rangle| + |\langle x_n - x, y \rangle| \\ &\le \underset{\|x\|}{\underset{\downarrow}{\|x_n\|}}\, \underset{0}{\underset{\downarrow}{\|y_n - y\|}} + \underset{0}{\underset{\downarrow}{\|x_n - x\|}}\, \|y\| \end{aligned}$$

<u>Definition:</u> An inner product space H is called a **Hilbert space**, if it is complete in the sense that every Cauchy sequence (x_n) of elements of H converges to some element $x \in H$, i.e.,

$$x_n, x_m \in H,\ \|x_m - x_n\| \to 0 \text{ as } m,n \to \infty \Rightarrow \exists x \in H:\ \|x_n - x\| \to 0.$$

<u>Example:</u> That the inner product space $\mathbb{R}^2$ is a Hilbert space can be seen as follows.

$$\left\| \begin{pmatrix} x_{m1} \\ x_{m2} \end{pmatrix} - \begin{pmatrix} x_{n1} \\ x_{n2} \end{pmatrix} \right\|^2 = (x_{m1}-x_{n1})^2 + (x_{m2}-x_{n2})^2 \to 0$$

$$\Rightarrow\ (x_{m1}-x_{n1})^2 \to 0,\ (x_{m2}-x_{n2})^2 \to 0$$

$$\Rightarrow\ \exists x_1, x_2 \in \mathbb{R}:\ x_{n1} \to x_1,\ x_{n2} \to x_2 \ \text{ (by the completeness of } \mathbb{R}\text{)}$$

$$\Rightarrow\ \left\| \begin{pmatrix} x_{n1} \\ x_{n2} \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|^2 = (x_{n1}-x_1)^2 + (x_{n2}-x_2)^2 \to 0$$

<u>Definition:</u> A linear subspace S of a Hilbert space is said to be a **closed subspace**, if

$$x_n \in S,\ \|x_n - x\| \to 0 \Rightarrow x \in S.$$

Exercise: Show that the intersection $\bigcap_{i\in I} S_i$ of a family of closed subspaces of a Hilbert space is also a closed subspace.

Definition: The **closed span** of a subset M of a Hilbert space is defined as the intersection of all closed subspaces which contain all elements of M. It is denoted by $\overline{\text{span}}(M)$.

Definition: Two elements x and y of an inner product space are said to be **orthogonal** ($x\perp y$), if $\langle x, y\rangle=0$.

Proposition: The **orthogonal complement**

$$M^{\perp}=\{x\in H:\ x\perp y\ \forall y\in M\}$$

of any subset M of a Hilbert space H is a closed subspace.

Proof: $M^{\perp}$ is a linear subspace, because

$$z\in M \Rightarrow \langle 0, z\rangle=0 \Rightarrow 0\perp z,$$

$$x,y\in M^{\perp},\ z\in M \Rightarrow \langle x+\mu, z\rangle=\langle x, z\rangle+\langle y, z\rangle=0 \Rightarrow x+y\perp z,$$

$$x\in M^{\perp},\ \lambda\in \mathbb{R},\ z\in M \Rightarrow \langle \lambda x, z\rangle=\lambda\langle x, z\rangle=0 \Rightarrow \lambda x\perp z.$$

Moreover, $M^{\perp}$ is even a closed subspace, because

$$x_n\in M^{\perp},\ \|x_n - x\|\to 0,\ z\in M \Rightarrow \langle x_n, z\rangle=0 \text{ for all } n$$

$$\Rightarrow \langle x, z\rangle=\lim\langle x_n, z\rangle=0.$$

Projection Theorem: If S is a closed subspace of a Hilbert space H, then each $x \in H$ can be uniquely represented as

$$x = \hat{x} + u,$$

where $\hat{x} \in S$ and $u \in S^{\perp}$. Furthermore, $\hat{x}$ (which is called the **projection** of x onto S) satisfies

$$\|x - \hat{x}\| < \|x - y\|$$

for any other element $y \in S$.

Definition: Let S be a closed subspace of a Hilbert space H. The mapping

$$P_S(x) = \hat{x}, \; x \in H,$$

where $\hat{x}$ is the projection of x onto S, is called the **projection mapping** of H onto S.

Properties of Projection Mappings: If S, S_1, S_2 are closed subspaces of a Hilbert space H, $x, y, x_n \in H$, and $\lambda, \mu \in \mathbb{R}$, then:

(i) $P_S(\lambda x + \mu y) = \lambda P_S(x) + \mu P_S(y)$

(ii) $x \in S \Leftrightarrow P_S(x) = x$

(iii) $x \in S^{\perp} \Leftrightarrow P_S(x) = 0$

(iv) $x = P_S(x) + P_{S^{\perp}}(x)$

(v) $S_1 \subseteq S_2 \Leftrightarrow P_{S_1}(P_{S_2}(x)) = P_{S_1}(x)$

(vi) $\|x_n - x\| \to 0 \Rightarrow \|P_S(x_n) - P_S(x)\| \to 0$

Proposition: If $M=\{u_1,\ldots,u_n\}$ is a set of mutually orthogonal elements of a Hilbert space H and $0\notin M$, then

$$P_{\overline{\text{span}}(M)}(x)=\sum_{k=1}^{n}\frac{\langle x,u_k\rangle}{\|u_k\|^2}u_k \quad \forall x\in H.$$

Proof: For each $u_j\in M$ we have

$$\left\langle x-\sum_{k=1}^{n}\frac{\langle x,u_k\rangle}{\|u_k\|^2}u_k,u_j\right\rangle=\langle x,u_j\rangle-\sum_{k=1}^{n}\frac{\langle x,u_k\rangle}{\|u_k\|^2}\langle u_k,u_j\rangle$$

$$=\langle x,u_j\rangle-\frac{\langle x,u_j\rangle}{\|u_j\|^2}\langle u_j,u_j\rangle$$

$$=0.$$

Thus

$$\left\langle x-\sum_{k=1}^{n}\frac{\langle x,u_k\rangle}{\|u_k\|^2}u_k,\ \sum_{k=1}^{n}\lambda_k u_k\right\rangle=0$$

for each linear combination $\sum_{k=1}^{n}\lambda_k u_k\in \text{span}(M)=\overline{\text{span}}(M)$.