Imaginary unit: i $(i^2 = -1)$

Complex number: $z = x + iy$

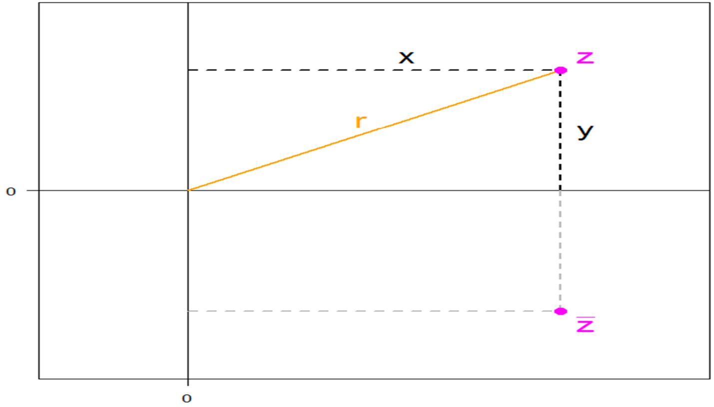
Cartesian coordinates: x (real part)

y (imaginary part)

Complex conjugate: $\bar{z} = x - iy$

Absolute value: $r = |z| = \sqrt{x^2 + y^2}$

$$z\bar{z} = (x+iy)(x-iy) = x^2 - i^2 y^2 = x^2 + y^2 = |z|^2$$

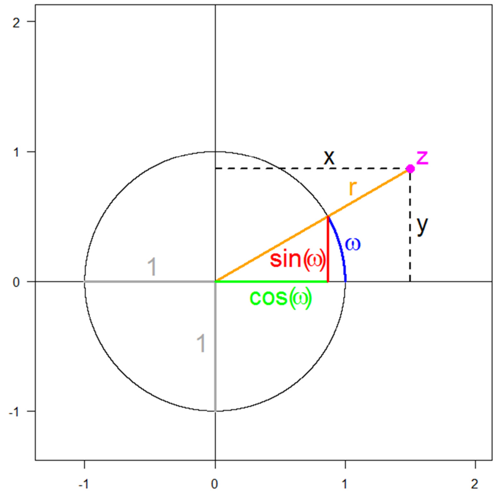


Polar coordinates: r (absolute value or modulus)

ω (argument or phase)

$$\cos(\omega) = \tfrac{x}{r} \Rightarrow x = r\cos(\omega)\,,\ \sin(\omega) = \tfrac{y}{r} \Rightarrow y = r\sin(\omega)$$

$$\tan(\omega) = \frac{\sin(\omega)}{\cos(\omega)} = \frac{y}{x} \ \Rightarrow \omega = \operatorname{atan}(\tfrac{y}{x})$$ [1]



[1] analogous if z not in 1st quadrant, e.g., $x<0$, $y>0 \Rightarrow \omega = \pi - \operatorname{atan}(|y/x|)$

Exercise: Show that the set $\mathbf{C}^n$ of n-dimensional complex vectors together with vector addition defined by

$$z+w=\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}+\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}=\begin{pmatrix} z_1+w_1 \\ \vdots \\ z_n+w_n \end{pmatrix}$$

is a commutative group. FG

Exercise: Show that the set $\mathbf{C}^n$ together with vector addition defined as above and scalar multiplication defined by

$$\lambda z=\lambda\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}=\begin{pmatrix} \lambda z_1 \\ \vdots \\ \lambda z_n \end{pmatrix}$$

is a complex vector space, i.e.,

$$\lambda(\mu z)=(\lambda\mu)z,$$
$$1z=z,$$
$$\lambda(z+w)=(\lambda z)+(\lambda w),$$
$$(\lambda+\mu)z=(\lambda z)+(\mu z).$$

FV

Exercise: Show that the inner product of two complex vectors z and w defined by

$$\langle z,w\rangle=w^*z=\overline{(w_1,\ldots,w_n)}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}=\sum_{t=1}^{n} z_t\overline{w_t},$$

satisfies

(i) $\langle v+w,z\rangle=\langle v,z\rangle+\langle w,z\rangle$,

(ii) $\langle z,v+w\rangle=\langle z,v\rangle+\langle z,w\rangle$,

(iii) $\langle \lambda z,w\rangle=\lambda\langle z,w\rangle$,

(iv) $\langle v,\lambda w\rangle=\overline{\lambda}\langle v,w\rangle$,

(v) $\langle z,w\rangle=\overline{\langle w,z\rangle}$,

(vi) $\langle z,z\rangle\geq 0$,

(vii) $\langle z,z\rangle=0 \Leftrightarrow z=0$. FI

Two complex vectors z and w are said to be orthogonal, if

$$\langle z,w\rangle=0.$$

Exercise: Show that the norm of a complex vector z defined by

$$\|z\| = \sqrt{\langle z, z \rangle} = \sqrt{\sum_{t=1}^{n} z_t \overline{z_t}} = \sqrt{\sum_{t=1}^{n} |z_t|^2}$$

satisfies

(i) $\|\lambda z\| = |\lambda| \, \|z\|$,

(ii) $\|z + w\| \le \|z\| + \|w\|$. FN

Hint: Use the Cauchy-Schwarz Inequality

$$|\langle z, w \rangle| \le \|z\| \, \|w\|.$$

Exercise: Prove the Pythagorean theorem

$$\langle z, w \rangle = 0 \Rightarrow \|z + w\|^2 = \|z\|^2 + \|w\|^2.$$ FP

Sinusoid: $g(t) = R \sin(\omega t + \phi)$

Parameters:
- R (amplitude)
- ω (frequency)
- ϕ (phase)

A sinusoid is periodic with period $p = \frac{2\pi}{\omega}$ because

$$\begin{aligned} g(t + \tfrac{2\pi}{\omega}) &= R \sin(\omega (t + \tfrac{2\pi}{\omega}) + \phi) \\ &= R \sin(\omega t + \phi + 2\pi) \\ &= R \sin(\omega t + \phi) \\ &= g(t). \end{aligned}$$

For fixed n, the frequencies $\omega_k = \frac{2\pi}{n} \cdot k$, $k=0,\ldots,[\frac{n}{2}]$ are called Fourier frequencies.

$\omega_1 = \frac{2\pi}{n} \cdot 1$ implies a period of $p_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\frac{2\pi}{n} \cdot 1} = n$.

$\omega_2 = \frac{2\pi}{n} \cdot 2$ implies a period of $p_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{\frac{2\pi}{n} \cdot 2} = \frac{n}{2}$.

$\vdots$

Exercise: Use the Euler relation FE

$$e^{i\omega} = \cos(\omega) + i\sin(\omega)$$

to show that $e^{i(\omega+2\pi)} = e^{i\omega}$, $\overline{e^{i\omega}} = e^{-i\omega}$, $e^{i(\omega_j-\omega_k)n} = 1$.

The n vectors FB

$$e_k = \frac{1}{\sqrt{n}}\begin{pmatrix} e^{i\omega_k 1} \\ \vdots \\ e^{i\omega_k n}\end{pmatrix},\ k=0,\ldots n-1,$$

constitute an orthonormal basis for $\mathbf{C}^n$ because

$$\langle e_k, e_k\rangle = \frac{1}{n}\sum_{t=1}^{n} e^{i\omega_k t}\overline{e^{i\omega_k t}} = \frac{1}{n}\sum_{t=1}^{n} e^{i\omega_k t}e^{-i\omega_k t} = \frac{1}{n}\sum_{t=1}^{n} e^{0} = 1,$$

$$\langle e_j, e_k\rangle = \frac{1}{n}\sum_{t=1}^{n} e^{i\omega_j t}\overline{e^{i\omega_k t}} = \frac{1}{n}\sum_{t=1}^{n} e^{i\omega_j t}e^{-i\omega_k t} = \frac{1}{n}\sum_{t=1}^{n} (e^{i(\omega_j-\omega_k)})^t$$

$$= \frac{1}{n}e^{i(\omega_j-\omega_k)}\sum_{t=0}^{n-1}(e^{i(\omega_j-\omega_k)})^t = \frac{1}{n}e^{i(\omega_j-\omega_k)}\frac{1-e^{i(\omega_j-\omega_k)n}}{1-e^{i(\omega_j-\omega_k)}}$$

$$= 0 \quad \text{if } j \neq k.$$

Thus, any $x \in \mathbf{C}^n$ has a representation of the form FS

$$x = \sum_{k=0}^{n-1}\lambda_k e_k\ .$$

Taking inner products of each side we obtain

$$\langle x, e_k\rangle = \left\langle \sum_{j=0}^{n-1}\lambda_j e_j, e_k\right\rangle = \sum_{j=0}^{n-1}\lambda_j\langle e_j, e_k\rangle = \lambda_k\langle e_k, e_k\rangle = \lambda_k,$$

$$\lambda_k = \langle x, e_k\rangle = e_k^* x = \frac{1}{\sqrt{n}}\sum_{t=1}^{n} x_t e^{-i\omega_k t},$$

$$\|x\|^2 = \langle x, x\rangle = \sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\lambda_k\bar{\lambda}_j\langle e_k, e_j\rangle = \sum_{k=0}^{n-1}\lambda_k\bar{\lambda}_k = \sum_{k=0}^{n-1}|\lambda_k|^2\ .$$

Exercise: Show that $\frac{1}{n}\sum_{t=1}^{n}(x_t-\bar{x})^2 = \frac{1}{n}\sum_{k\neq 0}|\lambda_k|^2$. FF

Note: For a given time series $x_1,\ldots,x_n$ observed at times $1,\ldots,n$ or time intervals $(0,1),\ldots,(n-1,n)$, the size of $|\lambda_k|$ therefore indicates how much of the sample variance can be attributed to frequency ω_k.

Suppose that $x \in \mathbf{R}^n$. If $1 \le k < \frac{n}{2}$, then AZ

$$e^{i\omega_{n\text{-}k}t} = e^{i\frac{2\pi(n-k)}{n}t} = e^{i(2\pi - \frac{2\pi k}{n})t} = e^{-i\frac{2\pi k}{n}t} = e^{-i\omega_k t} = \overline{e^{i\omega_k t}},$$

$$\lambda_{n-k} = \tfrac{1}{\sqrt{n}}\sum_{t=1}^{n} x_t e^{-i\omega_{n\text{-}k}t} = \tfrac{1}{\sqrt{n}}\sum_{t=1}^{n} x_t \overline{e^{-i\omega_k t}} = \overline{\tfrac{1}{\sqrt{n}}\sum_{t=1}^{n} x_t e^{-i\omega_k t}} = \overline{\lambda_k},$$

$$\begin{aligned}\lambda_k \frac{e^{i\omega_k t}}{\sqrt{n}} + \lambda_{n-k}\frac{e^{i\omega_{n-k}t}}{\sqrt{n}} &= \tfrac{\lambda_k}{\sqrt{n}} e^{i\omega_k t} + \overline{\tfrac{\lambda_k}{\sqrt{n}}}\,\overline{e^{i\omega_k t}} \\ &= (a_k + ib_k)(\cos(\omega_k t) + i\sin(\omega_k t)) \\ &\quad + (a_k - ib_k)(\cos(\omega_k t) - i\sin(\omega_k t)) \\ &= 2a_k\cos(\omega_k t) - 2b_k\sin(\omega_k t) \\ &= R_k\sin(\phi_k)\cos(\omega_k t) + R_k\cos(\phi_k)\sin(\omega_k t) \\ &= R_k\sin(\omega_k t + \phi_k),\end{aligned}$$ [2]

where R_k and ϕ_k are the polar coordinates of $-2b_k+2a_k i$, i.e.,

$$2a_k = R_k\sin(\phi_k),\ -2b_k = R_k\cos(\phi_k).$$

If $k=\frac{n}{2}$, then

$$\omega_k = \tfrac{2\pi k}{n} = \pi,\ e^{i\omega_k t} = e^{i\pi t} = \cos(\pi t) + \underbrace{\sin(\pi t)}_{=0} = \sin(\pi t + \tfrac{\pi}{2}),$$

$$\lambda_k = \tfrac{1}{\sqrt{n}}\sum_{t=1}^{n} x_t\cos(\pi t) = \tfrac{1}{\sqrt{n}}\sum_{t=1}^{n} x_t(-1)^t \in \mathbf{R},$$

$$\tfrac{\lambda_k}{\sqrt{n}} e^{i\omega_k t} = R_k\cos(\pi t) = R_k\sin(\pi t + \tfrac{\pi}{2}) = R_k\sin(\omega_k t + \phi_k).$$

If $k=0$, then

$$\omega_k = \tfrac{2\pi 0}{n} = 0,\ e^{i\omega_k t} = e^{i0t} = 1,$$

$$\lambda_k = \tfrac{1}{\sqrt{n}}\sum_{t=1}^{n} x_t\cdot 1 = \sqrt{n}\,\tfrac{1}{n}\sum_{t=1}^{n} x_t = \sqrt{n}\,\bar{x} \in \mathbf{R},$$

$$\tfrac{\lambda_k}{\sqrt{n}} e^{i\omega_k t} = \bar{x}.$$

Thus,

$$x_t = \tfrac{1}{\sqrt{n}}\left(\lambda_0 + \sum_{k=1}^{n-1}\lambda_k e^{i\omega_k t}\right) = \bar{x} + \sum_{k=1}^{[n/2]} R_k\sin(\omega_k t + \phi_k).$$

---

[2] $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

The **periodogram** of $x_1,\ldots,x_n$ is defined by

$$I(\omega)=\frac{1}{2\pi n}\left|\sum_{t=1}^{n} x_t e^{-i\omega t}\right|^2 .$$

For $1\le k<\frac{n}{2}$,

$$I(\omega_k)=\frac{1}{2\pi}\left|\lambda_k\right|^2=\frac{1}{2\pi}(a_k^2+b_k^2)=\frac{1}{8\pi}((2a_k)^2+(-2b_k)^2)=\frac{1}{8\pi}R_k^2 .$$

It will be shown later that for any nonzero Fourier frequency ω_k, $I(\omega_k)$ can be written as

$$I(\omega_k)=\frac{1}{2\pi}\sum_{j=-(n-1)}^{n-1}\hat{\gamma}(j)\, e^{-i\omega_k j} ,$$

where

$$\hat{\gamma}(j)=\frac{1}{n}\sum_{t=1}^{n-|j|}(x_t-\overline{x})(x_{t+|j|}-\overline{x})$$

is the sample autocovariance at lag j.

If the observations $x_1,\ldots,x_n$ come from a stationary process x, the periodogram $I(\omega)$ may be regarded as a sample analogue of the function

$$f(\omega)=\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\gamma(j)\, e^{-i\omega j} ,$$

which is called the **spectral density** of the process x.

The stationarity of the process x implies that all x_t have the same mean and the same variance and the autocovariances

$$\gamma(j)=\mathrm{Cov}(x_t,x_{t-j})$$

depend only on j but not on t.

Exercise: Show that $\int_{-\pi}^{\pi} e^{i\omega k}\, d\omega = 0$ if $k \neq 0$. A0

Assuming that the interchange of summation and integration is justified we can derive the **spectral representation of the autocovariance function** γ of a stationary process x with spectral density f as follows:

$$\int_{-\pi}^{\pi} e^{i\omega k} f(\omega) d\omega = \int_{-\pi}^{\pi} e^{i\omega k} \tfrac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma(j)\, e^{-i\omega j}\, d\omega$$

$$= \tfrac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma(j) \int_{-\pi}^{\pi} e^{i\omega(k-j)}\, d\omega$$

$$= \tfrac{1}{2\pi} \gamma(k) \int_{-\pi}^{\pi} e^{i\omega(k-k)}\, d\omega$$

$$= \gamma(k)$$

Remark: Let

$$f_J(\omega) = \sum_{j=-J}^{J} \gamma(j) \cos(\omega j).$$

If

$$g(\omega) = \sum_{j=-\infty}^{\infty} |\gamma(j)| < \infty,$$

then

$$\int_{-\pi}^{\pi} g(\omega) d\omega < \infty$$

and, by the dominant convergence theorem,

$$\lim_{J\to\infty} \int_{-\pi}^{\pi} f_J(\omega) d\omega = \int_{-\pi}^{\pi} \lim_{J\to\infty} f_J(\omega) d\omega,$$

because

$$|f_J(\omega)| \leq \sum_{j=-J}^{J} |\gamma(j) \cos(\omega j)| \leq \sum_{j=-J}^{J} |\gamma(j)| \leq g(\omega).$$

AR

Remark: It follows from

$$f(\omega)=\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\gamma(j)\,e^{-i\omega j}$$

and
$$\gamma(j)=\int_{-\pi}^{\pi}e^{i\omega j}f(\omega)d\omega$$

that the spectral density f and the autocovariance function γ contain the identical information.

Given n observations $x_1,\ldots,x_n$, the periodogram

$$I(\omega)=\frac{1}{2\pi}\sum_{j=-(n-1)}^{n-1}\hat{\gamma}(j)\,e^{-i\omega j}$$

is a very erratic estimator for the spectral density, because the sample autocovariances

$$\hat{\gamma}(j)=\frac{1}{n}\sum_{t=1}^{n-|j|}(x_t-\bar{x})(x_{t+|j|}-\bar{x})$$

contain very few products $(x_t-\bar{x})(x_{t+|j|}-\bar{x})$ if $|j|$ is large.

An obvious improvement is to give less weight to the more variable sample autocovariances.

An estimator of the type

$$\hat{f}(\omega)=\frac{1}{2\pi}\sum_{j=-(n-1)}^{n-1}w_j\hat{\gamma}(j)\,e^{-i\omega j}$$

is called a **weighted covariance estimator**. A widely used estimator is the **Bartlett estimator** which uses the triangle weights

$$w_j=\begin{cases}1-\frac{|j|}{M} & \text{if } |j|<M,\\ 0 & \text{else.}\end{cases}$$

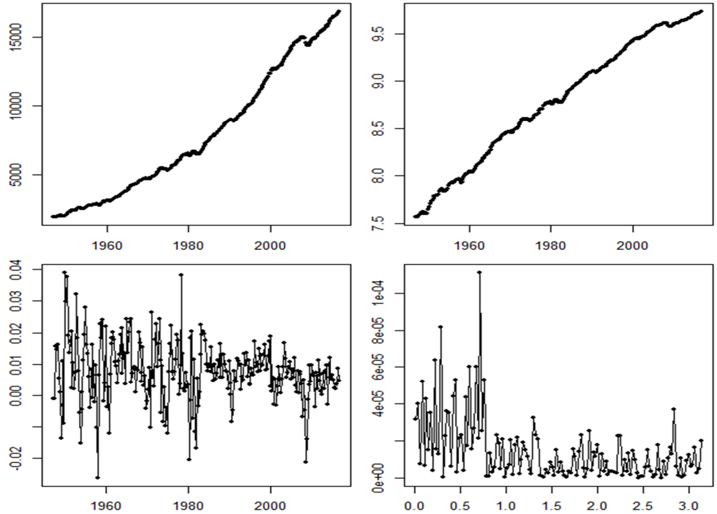
The truncation point M is an important parameter for controlling the smoothness of the estimator.

An alternative method of smoothing the periodogram is to take weighted averages over neighboring frequencies. A widely used **smoothed periodogram estimator** is the **modified Daniell smoother**, which differs from a simple moving average of the periodogram only in that the first and the last weight are only half as large as the others.

Exercise: Spectral analysis of the postwar US GDP

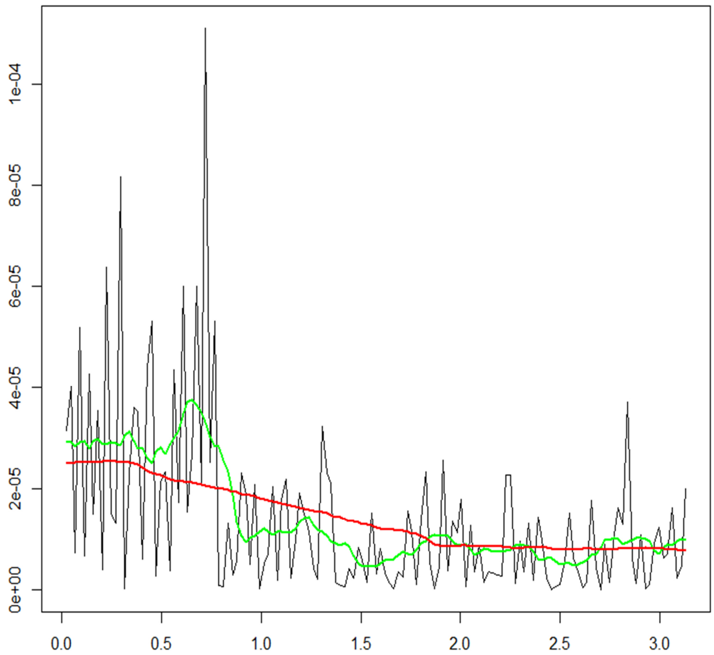
- Create a working directory, say **C:\GDPq**, for the analysis of the quarterly US GDP.
- Download the real Gross Domestic Product (quarterly, seasonally adjusted) as a csv file from the website of the Federal Reserve Bank of St. Louis into your working directory. The downloaded file **GDPC1.csv** consists of two columns (dates and GDP values).
- Import the data into R and plot the GDP, the log GDP, the differenced log GDP, and the periodogram of the differences.

```
setwd("C:/GDPq")  # comment: set working directory
D <- read.csv("GDPC1.csv")  # import data
d <- D[,1] # 1st column of D: dates
v <-D[,2] # 2nd column of D: GDP values
d <- as.Date(d) # convert character strings to dates
N <- length(v) # N = no. of quarters = length of vector v
par(mfrow=c(2,2))    # subsequent plots in 2x2 array
par(mar=c(2,2,1,1))  # set narrow margins for plots
plot(d,v,pch=20) # plot GDP values against dates
y <- log(v); plot(d,y,pch=20) # plot character: solid circle
r <- y[2:N]-y[1:(N-1)]; n <- N-1 # n = no. of differences
plot(d[2:N],r,pch=20,type="o") # overplot points&lines
h <- spec.pgram(r,taper=0,detrend=F,fast=F,plot=F)
c <- 2*pi; f <- c*h$freq # Fourier fr. between 0 and pi
pg <- h$spec/c; plot(f,pg,type="o",pch=20) # periodogr.
```



• Smooth the periodogram with the modified Daniell smoother.

```
par(mfrow=c(1,1)) # single plot
plot(f,pg,type="l") # only lines, no points
h  <- spec.pgram(r,taper=0,detrend=F,fast=F,plot=F,
                                   spans=13)
lines(c*h$freq,h$spec/c,col="green",lwd=2)
# add line to existing plot with line width twice as wide
h  <- spec.pgram(r,taper=0,detrend=F,fast=F,plot=F,
                                     spans=101)
lines(c*h$freq,h$spec/c,col="red",lwd=2)
```



The higher the span (the total number of terms in the moving average), the smoother the estimate.