# COMMUTATIVE POST-LIE ALGEBRA STRUCTURES ON KAC–MOODY ALGEBRAS

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Abstract. We determine commutative post-Lie algebra structures on some infinite-dimensional Lie algebras. We show that all commutative post-Lie algebra structures on loop algebras are trivial. This extends the results for finite-dimensional perfect Lie algebras. Furthermore we show that all commutative post-Lie algebra structures on affine Kac–Moody Lie algebras are "almost trivial".

## Introduction

Recently there is a surge of interest in so-called post-Lie algebras and post-Lie algebra structures. One origin comes from the study of geometric structures on Lie groups, where post-Lie algebras arise as a common generalization of pre-Lie algebras [13, 14, 19, 2, 3, 4] and LR-algebras [5, 6]. Here pre-Lie algebras, also called left-symmetric algebras, Vinberg algebras, or Koszul–Vinberg algebras, have been studied intensively before. For a survey, see [4]. On the other hand, post-Lie algebras have been introduced by Vallette [22] in 2007 in connection with the homology of partition posets and the study of Koszul operads. Then they were studied by several authors in various contexts, e.g., for algebraic operad triples [17], in connection with modified Yang–Baxter equations, Rota–Baxter operators, universal enveloping algebras, double Lie algebras, $R$-matrices, isospectral flows, Lie–Butcher series and many other topics [1, 10, 12].

Concerning post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, the existence question and the classification is of particular interest. There have been many results obtained so far. In [7] we introduced a special class of post-Lie algebra structures, namely commutative ones. In this case, the two Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$ coincide, and we obtain a bilinear commutative product satisfying a certain compatibility condition with the Lie bracket, which can be considered as a generalization of the left-symmetric identity (for a precise definition, see the introductory section below).

Commutative post-Lie algebra structures, CPA-structures in short, are much more tractable, and we have obtained several existence and classification results [7, 8, 9]. Among other things we proved in [8] that any commutative post-Lie algebra structure on a finite-dimensional perfect Lie algebra over field of characteristic zero is trivial. Moreover we classified CPA-structures on certain classes of nilpotent Lie algebras. It is natural to study CPA-structures also for infinite-dimensional Lie algebras. In [20] and [21] this has been done already for the two-sided infinite-dimensional Witt algebra and some of its generalizations.

We want to continue these investigations in this paper. We will prove that CPA-structures on loop algebras are trivial, and "almost trivial" on Kac–Moody algebras; see Theorem 1 and Theorem 3 for exact formulations.

## 0. Definitions, notations and conventions

Let $A$ be a nonassociative algebra over a field $K$ in the sense of Schafer [18], with $K$-bilinear product $A \times A \to A$, $(a, b) \mapsto ab$. We will assume that $K$ is an arbitrary field of characteristic different from 2, if not said otherwise. Consider bilinear maps $\varphi : A \times A \to A$ such that for any

*Date*: March 25, 2019.

$a \in A$, the linear map $\varphi(a, \cdot) : A \to A$ is a derivation of $A$. In other words,

$$\varphi(a, bc) = \varphi(a, b)c + b\varphi(a, c)$$

for any $a, b, c \in A$. The set of such bilinear maps forms a vector space which will be denoted by $\mathcal{D}(A)$, and the subspace of such *symmetric* maps will be denoted by $\mathcal{D}_{comm}(A)$.

Recall that a *commutative post-Lie algebra structure* (or, *CPA-structure*) on a Lie algebra $L$ is a binary multiplication $\cdot$ on $L$ which lies in $\mathcal{D}_{comm}(L)$, i.e., with

$$x \cdot y = y \cdot x$$

and

$$x \cdot [y, z] = [x \cdot y, z] - [x \cdot z, y]$$

and which, additionally, satisfies the condition

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$$

for any $x, y, z \in L$. When evaluating CPA-structures on various classes of Lie algebras, it will be convenient to write this new multiplication as a bilinear map $\varphi : L \times L \to L$, i.e., the previous three conditions are written in the form

$$\varphi(x, y) = \varphi(y, x) \tag{1}$$

$$\varphi(x, [y, z]) = [\varphi(x, y), z] - [\varphi(x, z), y] \tag{2}$$

$$\varphi([x, y], z) = \varphi(x, \varphi(y, z)) - \varphi(y, \varphi(x, z)) \tag{3}$$

for any $x, y, z \in L$.

For a Lie algebra $L$, $\mathrm{Z}(L)$ denotes the center of $L$. If $\mathrm{Z}(L) = 0$, then $L$ is called *centerless*. If $[L, L] = L$, then $L$ is called *perfect*.

All unadorned tensor products are over the base field $K$. The symbol $\oplus$ denotes the direct sum in the category of vector spaces.

## 1. Twisted loop algebras

Given a Lie algebra $L$ and a commutative associative algebra $A$, the *current Lie algebra* $L \otimes A$ carries a multiplication uniquely defined by the formula

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab,$$

where $x, y \in L$, and $a, b \in A$. In the particular case $A = K[t, t^{-1}]$, the Laurent polynomial algebra, we speak of the *(untwisted) loop Lie algebra associated with* $L$.

Now let

$$L = \bigoplus_{\bar{i} \in \mathbb{Z}/n\mathbb{Z}} L_{\bar{i}} \tag{4}$$

be a $\mathbb{Z}/n\mathbb{Z}$-graded Lie algebra, and consider the Lie algebra

$$\widehat{L} = \bigoplus_{i \in \mathbb{Z}} (L_{i (mod\, n)} \otimes t^i). \tag{5}$$

This subalgebra of the loop Lie algebra $L \otimes K[t, t^{-1}]$ will be called a *twisted loop Lie algebra associated to the graded Lie algebra* $L$. The untwisted case is formally included in the twisted one, when $n = 1$ and the grading (4) consists of the single zero component.

The direct sum (5) is a $\mathbb{Z}$-grading, what will be crucial in what follows. More generally, let $L = \bigoplus_{g \in G} L_g$ be a Lie algebra graded by an abelian group $G$. Then both vector spaces $\mathcal{D}(L)$ and $\mathcal{D}_{comm}(L)$ inherit a $G$-grading from $L$. Indeed, let us say that a bilinear map $\varphi : L \times L \to L$ *has degree* $g \in G$, or, symbolically, $\deg \varphi = g$, if $\varphi(L_h, L_f) \subseteq L_{h+f+g}$ for any $h, f \in G$. Then:

**Proposition 1.** *For an arbitrary $G$-graded Lie algebra $L$, we have*

$$\mathcal{D}(L) = \bigoplus_{g \in G} \{\varphi \in \mathcal{D}(L) \mid \deg \varphi = g\},$$
$$\mathcal{D}_{comm}(L) = \bigoplus_{g \in G} \{\varphi \in \mathcal{D}_{comm}(L) \mid \deg \varphi = g\}.$$

The proof is standard, and follows almost verbatim the proof of a similar statement for derivations. See, for example, Proposition 1.1 in [11].

In the sequel, it will be convenient to make use of the following auxiliary definition. Let us say that the Lie algebra $L$ *satisfies the condition (*)* if any $\varphi \in \mathcal{D}_{comm}(L)$ such that $\varphi(x, \varphi(y,z)) = \varphi(y, \varphi(x,z))$ for any $x,y,z \in L$, vanishes.

**Proposition 2.** *Let $L$ be a $\mathbb{Z}/n\mathbb{Z}$-graded Lie algebra satisfying the condition (*), and $\widehat{L}$ the twisted loop algebra associated to $L$. Then there is a bijection between the sets of CPA-structures on $\widehat{L}$ and CPA-structures on $L$. Namely, any CPA-structure on $\widehat{L}$ is of the form*

$$(x \otimes t^i, y \otimes t^j) \mapsto \varphi(x,y) \otimes t^{i+j}, \tag{6}$$

*for any $x \in L_{i (mod\, n)}$, $y \in L_{j (mod\, n)}$ and $i,j \in \mathbb{Z}$, where $\varphi$ is a CPA-structure on $L$.*

*Proof.* Let $\Phi$ be a CPA-structure on $\widehat{L}$. By Proposition 1,

$$\Phi = \sum_{\ell} \Phi_\ell, \tag{7}$$

where $\Phi_\ell$ is an element of $\mathcal{D}_{comm}(\widehat{L})$ of degree $\ell$, i.e.

$$\Phi_\ell(x \otimes t^i, y \otimes t^j) = \varphi_\ell(x,y) \otimes t^{i+j+\ell}$$

for any $x \in L_{i (mod\, n)}$, $y \in L_{j (mod\, n)}$, and some bilinear map $\varphi_\ell : L \times L \to L$ of degree $\ell (mod\, n)$. The commutativity of $\Phi_\ell$ is equivalent to the commutativity of $\varphi_\ell$, and the condition (2) for $\Phi_\ell$ is equivalent to the same condition for $\varphi_\ell$, whence $\varphi_\ell \in \mathcal{D}_{comm}(L)$ for any $\ell$.

The condition (3) for $\Phi$ is equivalent to

$$\sum_{\ell} \varphi_\ell(x,[y,z]) \otimes t^{i+j+k+\ell} = \sum_m \sum_s \big(\varphi_m(x, \varphi_s(y,z)) - \varphi_s(y, \varphi_m(x,z))\big) \otimes t^{i+j+k+m+s} \tag{8}$$

for any $x \in L_{i (mod\, n)}$, $y \in L_{j (mod\, n)}$, $z \in L_{k (mod\, n)}$, and $i,j,k \in \mathbb{Z}$. Assume the sum (7) contains elements of positive degree, and let $N$ be the largest such degree. The maximal possible degree of a summand at the left-hand side of the equality (8) is $i+j+k+N$, while at the right-hand side the summand

$$\varphi_N(x, \varphi_N(y,z)) - \varphi_N(y, \varphi_N(x,z)) \tag{9}$$

has degree $i+j+k+2N$, and all other summands have a smaller degree. Consequently, the expression (9) vanishes for any $x$, $y$, $z$ belonging to arbitrary homogeneous components of $L$, and hence vanishes for any $x,y,z \in L$. But then $\varphi_N$ vanishes, a contradiction. The same reasoning shows that the sum (7) does not contain summands of negative degree, and hence $\Phi$ is of degree zero, i.e. of the form (6). In this situation, the condition (3) for $\Phi$ is equivalent to the condition (3) for $\varphi$. □

**Remark.** An analogous result can be obtained by replacing the Laurent polynomial algebra by the (ordinary) polynomial algebra $K[t]$, or any similar graded polynomial-like algebra.

**Lemma 1.** *Let $L$ be a Lie algebra such that:*

(i) *$L$ is centerless;*
(ii) *all derivations of $L$ are inner;*

(iii) *any linear map $\omega$ from $L$ to an abelian subalgebra of $L$, satisfying the condition*

$$[\omega(x), y] + [x, \omega(y)] = 0$$

*for any $x, y \in L$, vanishes.*

*Then $L$ satisfies the condition (*).*

*Proof.* Since all derivations of $L$ are inner, any map $\varphi \in \mathcal{D}_{comm}(L)$ has the form $\varphi(x, y) = [y, \omega(x)]$ for $x, y \in L$ and some linear map $\omega : L \to L$. Since $\varphi$ is symmetric, we have $[\omega(x), y]+[x, \omega(y)] = 0$. The condition $\varphi(x, \varphi(y, z)) = \varphi(y, \varphi(x, z))$ is equivalent then to $[[z, \omega(y)], \omega(x)] = [[z, \omega(x)], \omega(y)]$ what, together with the Jacobi identity and the fact that $L$ is centerless, implies $[\omega(x), \omega(y)] = 0$, i.e. $\omega(L)$ is an abelian subalgebra in $L$. Now (iii) implies that $\omega$, and hence $\varphi$, vanishes, i.e. $L$ satisfies the condition (*). □

Now we can prove one of our main results.

**Theorem 1.** *Let*

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} \mathfrak{g}_i$$

*be a $\mathbb{Z}/n\mathbb{Z}$-graded simple finite-dimensional complex Lie algebra. Then any CPA-structure on the associated twisted loop algebra $\widehat{\mathfrak{g}}$ vanishes.*

*Proof.* The Lie algebra $\mathfrak{g}$ satisfies the conditions of Lemma 1. Indeed, (i) is evident and (ii) is well-known. According to Lemma 6.1 in [16], *any* linear map $\omega : \mathfrak{g} \to \mathfrak{g}$ satisfying $[\omega(x), y]+[x, \omega(y)] = 0$ for any $x, y \in \mathfrak{g}$, vanishes, thus (iii) is satisfied. Therefore, $\mathfrak{g}$ satisfies the condition (*), and by Proposition 2 the set of CPA-structures on $\widehat{\mathfrak{g}}$ is in bijection with the set of CPA-structures on $\mathfrak{g}$. But any CPA-structure on $\mathfrak{g}$ vanishes according to Proposition 5.4 of [7], or Proposition 3.1 of [8]. □

**Remarks.**

(i) On practice, only the cases $n = 1$ (the untwisted case) and $n = 2, 3$ (the twisted case) may occur; see Chapter 8 in [15].

(ii) According to Theorem 3.3 of [8], CPA-structures vanish not only on $\mathfrak{g}$, but on any perfect finite-dimensional Lie algebra over a field of characteristic zero. The twisted loop algebra $\widehat{\mathfrak{g}}$ is perfect, but not finite-dimensional, so Theorem 1 is not covered, at least in a straightforward way, by that result.

## 2. Graded Lie algebras, Witt algebras, and current algebras

This section contains some comments on and alternative approaches to the results of the previous section. The proofs are omitted, and the results stated here will be not used in the next section.

In the previous section we took advantage of the graded structure of twisted loop algebras, which, being coupled with the nonlinear condition (3), implies a strong restriction on CPA-structures. One of the possible generalizations along these line is:

**Proposition 3.** *Let $L$ be a $\mathbb{Z}$-graded Lie algebra $L = \bigoplus_{i\in\mathbb{Z}} L_i$ satisfying the condition (*). Then any CPA-structure $\varphi$ on $L$ is of degree 0: $\varphi(L_i, L_j) \subseteq L_{i+j}$ for any $i, j \in \mathbb{Z}$.*

The proof is straightforward and is similar to the proof of Proposition 2. In the situation of twisted loop algebras it was more convenient to use a somewhat more specific Proposition 2, and not this general result.

However, Proposition 3 can be used to establish the vanishing of CPA-structures on Witt algebras in a somewhat different way than this was done in [20]. Recall that the Witt algebras are defined as Lie algebras over a field $K$ of characteristic zero, having a set of basis vectors $\{e_i\}$ with multiplication

$$[e_i, e_j] = (j - i)e_{i+j}.$$

Depending on whether the indices run over all integers, or over integers $\ge -1$, we get the two-sided or one-sided Witt algebra, respectively.

**Theorem 2.** *Any CPA-structure on a Witt algebra (one- or two-sided) vanishes.*

The proof is obtained by an easy combination of reasonings as in the proof of Corollary 1, the facts that all derivations of a Witt algebra are inner, and all abelian subalgebras are one-dimensional, and Proposition 3.

Picking up another thread in §1, let us outline an alternative approach to the proof of Theorem 1; it takes advantage of the fact that one of the defining conditions of the post-Lie algebra is linear, and employs a linear-algebraic technique from [23], used earlier in [23] and [24] to describe other kinds of linear structures on current Lie algebras, such as derivations, low-degree cohomology, Poisson structures, etc., in terms of some invariants of the tensor factors.

Before we formulate the corresponding statement, a few definitions are in order. Recall that the *centroid* of a Lie algebra $L$ is the space of linear maps $\varphi : L \to L$ commuting with inner derivations of $L$, i.e.

$$\varphi([x, y]) = [\varphi(x), y]$$

for any $x, y \in L$. A Lie algebra is called *central*, if its centroid coincides with multiplications by the elements of the base field.

The set of all bilinear maps $\varphi : L \times L \to L$ such that for any $x \in L$, the linear map $\varphi(x, \cdot) : L \to L$ belongs to the centroid of $L$, i.e.,

$$\varphi(x, [y, z]) = [\varphi(x, y), z]$$

for any $x, y, z \in L$, forms a vector space denoted by $\mathcal{C}(L)$.

For a vector space $V$, $\mathrm{End}(V)$ denotes the set of all linear maps $V \to V$.

**Proposition 4.** *Let $L$ be a centerless Lie algebra, $A$ an associative commutative algebra with unit, and one of $L$, $A$ is finite-dimensional. Then*

$$\mathcal{D}(L \otimes A) \simeq \Big(\mathcal{D}(L) \otimes \mathrm{End}(A)\Big) \oplus \Big(\mathcal{C}(L) \otimes \mathcal{D}(A)\Big). \tag{10}$$

*Each element of $\mathcal{D}(L \otimes A)$ can be written as a sum of maps of the form $\varphi \otimes \alpha$, where $\varphi : L \times L \to L$ and $\alpha : A \times A \to A$ are bilinear maps, of the two kinds:*

(i) *$\varphi \in \mathcal{D}(L)$, and $\alpha(a, b) = \beta(a)b$ for any $a, b \in A$ and some linear map $\beta : A \to A$;*
(ii) *$\varphi \in \mathcal{C}(L)$, and $\alpha \in \mathcal{D}(A)$.*

The proof is very similar to the proof of the formula from Corollary 2.2 in [23] expressing derivations of a current Lie algebra in terms of its tensor factors. The condition that $L$ is centerless is not crucial and can be removed at the expense of more laborious computations and cumbersome formulas.

Similarly, we have:

**Proposition 5.** *In the setup of Proposition 4, assume additionally that $L$ is central. Then*

$$\mathcal{D}_{comm}(L \otimes A) \simeq \mathcal{D}_{comm}(L) \otimes A.$$

*Each element of $\mathcal{D}_{comm}(L \otimes A)$ can be written as the sum of maps of the form $\varphi \otimes \alpha$, where $\varphi \in \mathcal{D}_{comm}(L)$, and $\alpha : A \times A \to A$ is a bilinear map of the form $\alpha(a, b) = abu$ for some $u \in A$.*

Then, using Proposition 5, we may impose on $\mathcal{D}_{comm}(L \otimes A)$ the additional condition (3) to try to get an analogous formula for the set of CPA-structures on $L \otimes A$. However, the nonlinearity of (3) makes the task much more difficult, and we seemingly have to abandon the generality of Propositions 4 and 5, and assume $A$ to be a more or less concrete algebra. This provides a somewhat alternative way to the results of §1, at least in the nontwisted case.

## 3. Kac–Moody algebras

We use the well-known realization of untwisted and twisted affine Kac–Moody algebras as extensions of current Lie algebras by a derivation and a central element:

$$\widehat{\mathfrak{g}} \oplus \mathbb{C}t\frac{\mathrm{d}}{\mathrm{d}\,t} \oplus \mathbb{C}z\,, \tag{11}$$

where $\widehat{\mathfrak{g}}$ denotes, as previously, the twisted loop algebra associated to a $\mathbb{Z}/n\mathbb{Z}$-graded simple finite-dimensional complex Lie algebra $\mathfrak{g}$. The Euler derivation $t\frac{\mathrm{d}}{\mathrm{d}\,t}$ acts on the current Lie algebra $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$, and, by restriction, on $\widehat{g} \subseteq \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$, as the derivation of the second tensor factor, i.e.

$$[x\otimes t^i, t\frac{\mathrm{d}}{\mathrm{d}\,t}] = ix\otimes t^i,$$

where $x \in L$ and $i \in \mathbb{Z}$. The element $z$ is central, and multiplication on $\widehat{\mathfrak{g}}$ is twisted by the $\mathbb{C}z$-valued "Kac–Moody cocycle" (whose concrete form is not important for us here).

**Lemma 2.** *Let $L$ be a perfect centerless Lie algebra such that any CPA-structure on $L$ vanishes, $D$ be an outer derivation of $L$, and $\mathscr{L} = L \oplus KD$ be the semidirect sum with $D$ acting on $L$. Then any CPA-structure on $\mathscr{L}$ vanishes.*

*Proof.* Let $\Phi$ be a CPA-structure on $\mathscr{L}$. We may write

$$\begin{aligned}\Phi(x,y) &= \varphi(x,y) + \lambda(x,y)D\\ \Phi(x,D) &= \psi(x) + \mu(x)D\\ \Phi(D,D) &= a + \eta D\end{aligned}$$

where $x,y \in L$, for some (bi)linear maps $\varphi : L\times L \to L$, $\lambda : L\times L\to K$, $\psi : L \to L$, $\mu : L \to K$, and $a \in L$, $\eta \in K$. Then the symmetricity of $\Phi$ implies the symmetricity of $\varphi$, and the condition (2) for $\Phi$, written for an arbitrary triple $x, y, z \in L$, is equivalent to

$$\varphi(x,[y,z]) = [\varphi(x,y),z] - [\varphi(x,z),y] + \lambda(x,y)D(z) - \lambda(x,z)D(y),$$

and to the condition $\lambda(L,[L,L]) = 0$. But since $L$ is perfect, $\lambda$ vanishes, and hence $\varphi \in \mathcal{D}_{comm}(L)$. Imposing on $\Phi$ the condition (3) leads to the conclusion that the restriction $\varphi = \Phi|_L$ is a CPA-structure on $L$, and hence vanishes.

The condition (2) for $\Phi$, written for the triple $D,x,y$, where $x,y \in L$, implies $\mu([L,L]) = 0$, whence $\mu$ vanishes. Then writing the same condition for the triple $x,D,y$, and taking into account all the vanishing conditions obtained so far, we get $\psi(L) \subseteq Z(L) = 0$, whence $\psi$ vanishes.

Finally, the condition (2) for $\Phi$, written for the triple $D,D,x \in L$, yields $\eta D = \mathsf{ad}\,a$. But since $D$ is an outer derivation, $\eta = 0$, $a \in Z(L) = 0$, and $\Phi$ vanishes identically. □

**Lemma 3.** *Let $L$ be a centerless Lie algebra such that any CPA-structure on $L$ vanishes, and $\mathscr{L} = L\oplus Kz$ a nontrivial one-dimensional central extension of $L$. Then the set of CPA-structures on $\mathscr{L}$ consists of $Kz$-valued symmetric bilinear maps vanishing whenever one of the arguments belongs to $[L,L]\oplus Kz$.*

*Proof.* Write the Lie bracket on $\mathscr{L}$ as $\{x,y\} = [x,y] + \xi(x,y)z$, where $x,y \in L$, $[\,\cdot\,,\cdot\,]$ is a Lie bracket on $L$, and $\xi$ is a (nontrivial) 2-cocycle on $L$.

Let $\Phi$ be a CPA-structure on $\mathscr{L}$. Similarly with the proof of Lemma 2, we may write

$$\begin{aligned}\Phi(x,y) &= \varphi(x,y) + \lambda(x,y)z\\ \Phi(x,z) &= \psi(x) + \mu(x)z\\ \Phi(z,z) &= a + \eta z\end{aligned}$$

where $x,y \in L$, and all the maps and elements occurring at the right-hand side have the same meaning as in the proof of Lemma 2.

The symmetricity of $\Phi$ implies the symmetricity of $\varphi$ and $\lambda$. The condition (2) for $\Phi$, written for the triple $x, y, z$, where $x, y \in L$, yields $\psi(L) \subseteq Z(L) = 0$, whence $\psi$ vanishes. Then the same condition written for the triple $z, x, y$, yields $a = 0$ and $\eta\xi(x, y) = -\mu([x, y])$. But since $\xi$ is a nontrivial (i.e., not equal to a coboundary) cocycle, the latter equality implies $\eta = 0$.

Further, the condition (2) for $\Phi$, written for the triple $x, y, t \in L$, yields

$$\varphi(x, [y, t]) = [\varphi(x, y), t] - [\varphi(x, t), y],$$

i.e., $\varphi \in \mathcal{D}_{comm}(L)$, and

$$\lambda(x, [y, t]) + \xi(y, t)\mu(x) = \xi(\varphi(x, y), t) - \xi(\varphi(x, t), y). \tag{12}$$

The condition (3) for $\Phi$, written for the triple $x, y, t \in L$, yields

$$\varphi([x, y], t) = \varphi(x, \varphi(y, t)) - \varphi(y, \varphi(x, t)),$$

i.e., $\varphi$ is a CPA-structure on $L$, whence $\varphi$ vanishes.

The condition (3) for $\Phi$, written for the triple $z, x, x$, where $x \in L$, yields $\mu(x)^2 = 0$, whence $\mu$ vanishes.

To summarize: the only nonzero values of $\Phi$ are given by $\Phi(x, y) = \lambda(x, y)z$, where $x, y \in L$. Moreover, (12) now implies $\lambda(L, [L, L]) = 0$. These are exactly the maps as specified in the statement of the lemma. It is straightforward to check that conversely, any such map is a CPA-structure on $\mathscr{L}$. ☐

**Theorem 3.** *CPA-structures on an affine Kac–Moody algebra written in the form (11), form a one-dimensional vector space, spanned by the map:*

$$\left(t\frac{\mathrm{d}}{\mathrm{d}\,t}, t\frac{\mathrm{d}}{\mathrm{d}\,t}\right) \mapsto z$$

$$\textit{all other pairs of basis vectors} \mapsto 0.$$

*Proof.* As by Theorem 1 any CPA-structure on $\widehat{\mathfrak{g}}$ vanishes, and the Euler derivation $t\frac{\mathrm{d}}{\mathrm{d}\,t}$ (in fact, any nonzero derivation of the Laurent polynomials extended to the loop algebra $\widehat{\mathfrak{g}}$) is an outer derivation of $\widehat{\mathfrak{g}}$, Lemma 2 implies that any CPA-structure on the semidirect sum $\widehat{\mathfrak{g}} \oplus \mathbb{C}t\frac{\mathrm{d}}{\mathrm{d}\,t}$ vanishes. This allows, in its turn, to apply Lemma 3 to $L = \widehat{\mathfrak{g}} \oplus \mathbb{C}t\frac{\mathrm{d}}{\mathrm{d}\,t}$. Since $[L, L] = \widehat{\mathfrak{g}}$, Lemma 3 yields the desired result. ☐

## Acknowledgement

We thank the referee for helpful remarks which have improved the presentation of this article. Dietrich Burde is supported by the Austrian Science Foundation FWF, grant P28079 and grant I3248.

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