ROTA–BAXTER OPERATORS AND POST-LIE ALGEBRA STRUCTURES ON SEMISIMPLE LIE ALGEBRAS

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Abstract. Rota–Baxter operators R of weight 1 on $\mathfrak{n}$ are in bijective correspondence to post-Lie algebra structures on pairs $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n}$ is complete. We use such Rota–Baxter operators to study the existence and classification of post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n}$ is semisimple. We show that for semisimple $\mathfrak{g}$ and $\mathfrak{n}$, with $\mathfrak{g}$ or $\mathfrak{n}$ simple, the existence of a post-Lie algebra structure on such a pair $(\mathfrak{g}, \mathfrak{n})$ implies that $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic, and hence both simple. If $\mathfrak{n}$ is semisimple, but $\mathfrak{g}$ is not, it becomes much harder to classify post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, or even to determine the Lie algebras $\mathfrak{g}$ which can arise. Here only the case $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$ was studied. In this paper we determine all Lie algebras $\mathfrak{g}$ such that there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

1. Introduction

Rota–Baxter operators were introduced by G. Baxter [3] in 1960 as a formal generalization of integration by parts for solving an analytic formula in probability theory. Such operators $R\colon A \to A$ are defined on an algebra A by the identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy)$$

for all $x, y \in A$, where λ is a scalar, called the *weight* of R. These operators were then further investigated, by G.-C. Rota [31], Atkinson [1], Cartier [17] and others. In the 1980s these operators were studied in integrable systems in the context of classical and modified Yang–Baxter equations [34, 4]. Since the late 1990s, the study of Rota–Baxter operators has made great progress in many areas, both in theory and in applications [26, 2, 23, 21, 22, 5, 20].

Post-Lie algebras and post-Lie algebra structures also arise in many areas, e.g., in differential geometry and the study of geometric structures on Lie groups. Here post-Lie algebras arise as a natural common generalization of pre-Lie algebras [24, 27, 33, 6, 7, 8] and LR-algebras [9, 10], in the context of nil-affine actions of Lie groups, see [11]. A detailed account of the differential geometric context of post-Lie algebras is also given in [19]. On the other hand, post-Lie algebras have been introduced by Vallette [35] in connection with the homology of partition posets and the study of Koszul operads. They have been studied by several authors in various contexts, e.g., for algebraic operad triples [29], in connection with modified Yang–Baxter equations, Rota–Baxter operators, universal enveloping algebras, double Lie algebras, R-matrices, isospectral flows, Lie-Butcher series and many other topics [2, 19, 20]. There are several results on the existence and classification of post-Lie algebra structures, in particular on commutative post-Lie algebra structures [13, 14, 15].

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It is well-known [2] that Rota–Baxter operators R of weight 1 on $\mathfrak{n}$ are in bijective correspondence to post-Lie algebra structures on pairs $(\mathfrak{g},\mathfrak{n})$, where $\mathfrak{n}$ is complete. In fact, RB-operators always yield PA-structures. So it is possible (and desirable) to use results on RB-operators for the existence and classification of post-Lie algebra structures.

The paper is organized as follows. In section 2 we give basic definitions of RB-operators and PA-structures on pairs of Lie algebras. We summarize several useful results. For a complete Lie algebra $\mathfrak{n}$ there is a bijection between PA-structures on $(\mathfrak{g},\mathfrak{n})$ and RB-operators of weight 1 on $\mathfrak{n}$. The PA-structure is given by $x\cdot y=\{R(x),y\}$. Here we study the kernels of R and $R+\mathrm{id}$. If $\mathfrak{g}$ and $\mathfrak{n}$ are not isomorphic, then both R and $R+\mathrm{id}$ have a non-trivial kernel. Moreover, if one of $\mathfrak{g}$ or $\mathfrak{n}$ is not solvable, then at least one of $\ker(R)$ and $\ker(R+\mathrm{id})$ is non-trivial.
In section 3 we complete the classification of PA-structures on pairs of semisimple Lie algebras $(\mathfrak{g},\mathfrak{n})$, where either $\mathfrak{g}$ or $\mathfrak{n}$ is simple. We already have shown the following in [11]. If $\mathfrak{g}$ is simple, and there exists a PA-structure on $(\mathfrak{g},\mathfrak{n})$, then also $\mathfrak{n}$ is simple, and we have $\mathfrak{g}\cong\mathfrak{n}$ with $x\cdot y=0$ or $x\cdot y=[x,y]$. Here we deal now with the case that $\mathfrak{n}$ is simple. Again it follows that $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic. The proof via RB-operators uses results of Koszul [28] and Onishchik [30]. We also show a result concerning semisimple decompositions of Lie algebras. Suppose that $\mathfrak{g}=\mathfrak{s}_1+\mathfrak{s}_2$ is the vector space sum of two semisimple subalgebras of $\mathfrak{g}$. Then $\mathfrak{g}$ is semisimple. As a corollary we show that the existence of a PA-structure on $(\mathfrak{g},\mathfrak{n})$ for $\mathfrak{g}$ semisimple and $\mathfrak{n}$ complete implies that $\mathfrak{n}$ is semisimple.
In section 4 we determine all Lie algebras $\mathfrak{g}$ which can arise by PA-structures on $(\mathfrak{g},\mathfrak{n})$ with $\mathfrak{n}=\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C})$. This turns out to be much more complicated than the case $\mathfrak{n}=\mathfrak{sl}_2(\mathbb{C})$, which we have done in [11]. By Theorem 3.3 of [12], $\mathfrak{g}$ cannot be solvable unimodular. On the other hand, the result we obtain shows that there are more restrictions than that.

2. Preliminaries

Let A be a nonassociative algebra over a field K in the sense of Schafer [32], with K-bilinear product $A\times A\to A$, $(a,b)\mapsto ab$. We will assume that K is an arbitrary field of characteristic zero, if not said otherwise.

Definition 2.1. Let $\lambda\in K$. A linear operator $R\colon A\to A$ satisfying the identity

$$R(x)R(y)=R\big(R(x)y+xR(y)+\lambda xy\big) \tag{1}$$

for all $x,y\in A$ is called a *Rota–Baxter operator on A of weight λ*, or just *RB-operator*.

Two obvious examples are given by $R=0$ and $R=\lambda\,\mathrm{id}$, for an arbitrary nonassociative algebra. These are called the *trivial* RB-operators. The following elementary lemma was shown in [23], Proposition 1.1.12.

Lemma 2.2. *Let R be an RB-operator on A of weight λ. Then $-R-\lambda\,\mathrm{id}$ is an RB-operator on A of weight λ, and $\lambda^{-1}R$ is an RB-operator on A of weight 1 for all $\lambda\neq 0$.*

It is also easy to verify the following results.

Proposition 2.3. [5] *Let R be an RB-operator on A of weight λ and $\psi\in\mathrm{Aut}(A)$. Then $R^{(\psi)}=\psi^{-1}R\psi$ is an RB-operator on A of weight λ.*

Proposition 2.4. [23] *Let B be a countable direct sum of an algebra A. Then the operator R defined on B by*

$$R((a_1,a_2,\ldots,a_n,\ldots))=(0,a_1,a_1+a_2,a_1+a_2+a_3,\ldots)$$

is an RB-operator on B of weight 1.

Proposition 2.5. *Let $B = A \oplus A$ and $\psi \in \operatorname{Aut}(A)$. Then the operator R defined on B by*

$$R((a_1, a_2)) = (0, \psi(a_1)) \tag{2}$$

is an RB-operator on B of weight 1. *Furthermore the operator R defined on B by*

$$R((a_1, a_2)) = (-a_1, -\psi(a_1)) \tag{3}$$

is an RB-operator on B of weight 1.

Proof. Let $x = (a_1, a_2)$ and $y = (b_1, b_2)$. Then we have

$$\begin{aligned} R(R(x)y + xR(y) + \lambda xy) &= R((0, \psi(a_1)b_2 + (0, a_2\psi(b_1)) + (a_1b_1, a_2b_2)) \\ &= (0, \psi(a_1b_1)) \\ &= (0, \psi(a_1)\psi(b_1)) \\ &= R(x)R(y). \end{aligned}$$

The second claim follows similarly. □

Proposition 2.6. [26] *Let $A = A_1 \oplus A_2$, R_1 be an RB-operator of weight λ on A_1, R_2 be an RB-operator of weight λ on A_2. Then the operator $R\colon A \to A$ defined by $R((a_1, a_2)) = (R_1(a_1), R_2(a_2))$ is an RB-operator of weight λ on A.*

Proposition 2.7. [23] *Let $A = A_1 \dot{+} A_2$ be the direct vector space sum of two subalgebras. Then the operator R defined on A by*

$$R(a_1 + a_2) = -\lambda a_2 \tag{4}$$

for $a_1 \in A_1$ and $a_2 \in A_2$ is an RB-operator on A of weight λ.

We call such an operator *split*, with subalgebras A_1 and A_2. Note that the set of all split RB-operators on A is in bijective correspondence with all decompositions $A = A_1 \dot{+} A_2$ as a direct sum of subalgebras.

Lemma 2.8. [5] *Let R be an RB-operator of nonzero weight λ on an algebra A. Then R is split if and only if $R(R + \lambda \operatorname{id}) = 0$.*

Lemma 2.9. *Let $A = A_- \dot{+} A_0 \dot{+} A_+$ be a direct vector space sum of subalgebras of A. Suppose that R is an RB-operator of weight λ on A_0, A_- is an $(R + \operatorname{id})(A_0)$-module and A_+ is an $R(A_0)$-module. Define an operator P on A by*

$$P_{|A_-} = 0,\ P_{|A_0} = R,\ P_{|A_+} = -\lambda \operatorname{id}. \tag{5}$$

Then P is an RB-operator on A of weight λ.

Definition 2.10. Let P be an RB-operator on A defined as above such that not both A_- and A_+ are zero. Then P is called *triangular-split*.

We also recall the definition of post-Lie algebra structures on a pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ over K, see [11].

Definition 2.11. Let $\mathfrak{g}=(V,[\,,])$ and $\mathfrak{n}=(V,\{\,,\})$ be two Lie brackets on a vector space V over K. A *post-Lie algebra structure*, or *PA-structure* on the pair $(\mathfrak{g},\mathfrak{n})$ is a K-bilinear product $x\cdot y$ satisfying the identities:

$$x\cdot y-y\cdot x=[x,y]-\{x,y\} \tag{6}$$

$$[x,y]\cdot z=x\cdot(y\cdot z)-y\cdot(x\cdot z) \tag{7}$$

$$x\cdot\{y,z\}=\{x\cdot y,z\}+\{y,x\cdot z\} \tag{8}$$

for all $x,y,z\in V$.

Define by $L(x)(y)=x\cdot y$ the left multiplication operator of the algebra $A=(V,\cdot)$. By (8), all $L(x)$ are derivations of the Lie algebra $(V,\{,\})$. Moreover, by (7), the left multiplication

$$L\colon \mathfrak{g}\to \operatorname{Der}(\mathfrak{n})\subseteq \operatorname{End}(V),\ x\mapsto L(x)$$

is a linear representation of $\mathfrak{g}$.
If $\mathfrak{n}$ is abelian, then a post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$ corresponds to a *pre-Lie algebra structure* on $\mathfrak{g}$. In other words, if $\{x,y\}=0$ for all $x,y\in V$, then the conditions reduce to

$$x\cdot y-y\cdot x=[x,y],$$
$$[x,y]\cdot z=x\cdot(y\cdot z)-y\cdot(x\cdot z),$$

i.e., $x\cdot y$ is a *pre-Lie algebra structure* on the Lie algebra $\mathfrak{g}$, see [11].

Definition 2.12. Let $x\cdot y$ be a PA-structure on $(\mathfrak{g},\mathfrak{n})$. If there exists a $\varphi\in\operatorname{End}(V)$ such that

$$x\cdot y=\{\varphi(x),y\}$$

for all $x,y\in V$, then $x\cdot y$ is called an *inner* PA-structure on $(\mathfrak{g},\mathfrak{n})$.

The following result is proved in [2], Corollary 5.6.

Proposition 2.13. *Let $(\mathfrak{n},\{,\},R)$ be a Lie algebra together with a Rota–Baxter operator R of weight 1, i.e., a linear operator satisfying*

$$\{R(x),R(y)\}=R(\{R(x),y\}+\{x,R(y)\}+\{x,y\})$$

for all $x,y\in V$. Then

$$x\cdot y=\{R(x),y\}$$

defines an inner PA-structure on $(\mathfrak{g},\mathfrak{n})$, where the Lie bracket of $\mathfrak{g}$ is given by

$$[x,y]=\{R(x),y\}-\{R(y),x\}+\{x,y\}. \tag{9}$$

Note that $\ker(R)$ is a subalgebra of $\mathfrak{n}$. For $x,y\in\ker(R)$ we have $R(\{x,y\})=0$. Recall that a Lie algebra is called *complete*, if it has trivial center and only inner derivations.

Proposition 2.14. *Let $\mathfrak{n}$ be a Lie algebra with trivial center. Then any inner PA-structure on $(\mathfrak{g},\mathfrak{n})$ arises by a Rota–Baxter operator of weight 1. Furthermore, if $\mathfrak{n}$ is complete, then every PA-structure on $(\mathfrak{g},\mathfrak{n})$ is inner.*

Proof. The first claim follows from Proposition 2.10 in [11]. By Lemma 2.9 in [11] every PA-structure on $(\mathfrak{g},\mathfrak{n})$ with complete Lie algebra $\mathfrak{n}$ is inner. The result can also be derived from the proof of Theorem 5.10 in [2]. □

Corollary 2.15. *Let $\mathfrak{n}$ be a complete Lie algebra. Then there is a bijection between PA-structures on $(\mathfrak{g},\mathfrak{n})$ and RB-operators of weight 1 on $\mathfrak{n}$.*

As we have seen, any inner PA-structure on $(\mathfrak{g},\mathfrak{n})$ with $Z(\mathfrak{n})=0$ arises by a Rota–Baxter operator of weight 1. For Lie algebra $\mathfrak{n}$ with non-trivial center this need not be true.

Example 2.16. *Let (e_1,e_2,e_3) be a basis of V and $\mathfrak{n}=\mathfrak{r}_2(K)\oplus K$ with $\{e_1,e_2\}=e_2$. Then*

$$\varphi=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ \alpha & \beta & \gamma\end{pmatrix}$$

defines an inner PA-structure on $(\mathfrak{g},\mathfrak{n})$ by $x\cdot y=\{\varphi(x),y\}$ with $\mathfrak{g}=\mathfrak{n}$, i.e., with $[e_1,e_2]=e_2$. But φ is not always a Rota–Baxter operator of weight 1 for $\mathfrak{n}$. It is easy to see that this is the case if and only if $\beta=0$.

Proposition 2.17. *Let $x\cdot y$ be an inner PA-structure arising from an RB-operator R on $\mathfrak{n}$ of weight 1. Then R is also an RB-operator of weight 1 on $\mathfrak{g}$, i.e., it satisfies*

$$[R(x),R[y)]=R([R(x),y]+[x,R(y)]+[x,y])$$

for all $x,y\in V$.

Proof. Because of $R([x,y])=\{R(x),R(y)\}$ and the definition of $[x,y]$ we have

$$\begin{aligned}R([R(x),y]+[x,R(y)]+[x,y])&=\{R(R(x)),R(y)\}+\{R(x),R(R(y))\}+\{R(x),R(y)\}\\&=[R(x),R(y)]\end{aligned}$$

for all $x,y\in V$. □

Corollary 2.18. *Let $x\cdot y=\{R(x),y\}$ be a PA-structure on $(\mathfrak{g},\mathfrak{n})$ defined by an RB-operator R of weight 1 on $\mathfrak{n}$. Denote by $\mathfrak{g}_i$ be the Lie algebra structure on V defined by*

$$\begin{aligned}[x,y]_0&=\{x,y\},\\ [x,y]_{i+1}&=[R(x),y]_i-[R(y),x]_i+[x,y]_i,\end{aligned}$$

for all $i\ge 0$. Then R defines a PA-structure on each pair $(\mathfrak{g}_{i+1},\mathfrak{g}_i)$.

We have $[x,y]_1=[x,y]$, and both R and $R+\mathrm{id}$ are Lie algebra homomorphisms from $\mathfrak{g}_{i+1}$ to $\mathfrak{g}_i$, see Proposition 7 in [34]. Hence we obtain a composition of homomorphisms

$$\mathfrak{g}_i\ \underset{R+\mathrm{id}}{\xrightarrow{R}}\ \mathfrak{g}_{i-1}\ \underset{R+\mathrm{id}}{\xrightarrow{R}}\ \cdots\ \underset{R+\mathrm{id}}{\xrightarrow{R}}\ \mathfrak{g}_0$$

So the kernels $\ker(R^i)$ and $\ker((R+\mathrm{id})^i)$ are ideals in $\mathfrak{g}_j$ for all $1\le i\le j$.

For a Lie algebra $\mathfrak{g}$, denote by $\mathfrak{g}^{(i)}$ the derived ideals defined by $\mathfrak{g}^{(1)}=\mathfrak{g}$ and $\mathfrak{g}^{(i+1)}=[\mathfrak{g}^{(i)},\mathfrak{g}^{(i)}]$ for $i\ge 1$. An immediate consequence of Proposition 2.13 is the following observation.

Proposition 2.19. *Let $x\cdot y=\{R(x),y\}$ be a PA-structure on $(\mathfrak{g},\mathfrak{n})$ defined by an RB-operator R of weight 1 on $\mathfrak{n}$. Then we have $\dim\mathfrak{g}^{(i)}\le\dim\mathfrak{n}^{(i)}$ for all $i\ge 1$.*

Corollary 2.20. *Let $x\cdot y$ be a PA-structure on $(\mathfrak{g},\mathfrak{n})$, where $\mathfrak{n}$ is complete. Then we have $\dim\mathfrak{g}^{(i)}\le\dim\mathfrak{n}^{(i)}$ for all $i\ge 1$. In particular, if $\mathfrak{n}$ is solvable, so is $\mathfrak{g}$, and if $\mathfrak{g}$ is perfect, so is $\mathfrak{n}$.*

Proof. By Corollary 2.15 this follows from the proposition. □

Proposition 2.21. *Let $x\cdot y=\{R(x),y\}$ be a PA-structure on $(\mathfrak{g},\mathfrak{n})$ defined by an RB-operator R of weight 1 on $\mathfrak{n}$. Then the following holds.*

(1) *If* $\mathfrak{g}$ *and* $\mathfrak{n}$ *are not isomorphic, then both* R *and* $R+\mathrm{id}$ *have a non-trivial kernel.*
(2) *If either* $\mathfrak{g}$ *or* $\mathfrak{n}$ *is not solvable, then at least one of the operators* R *and* $R+\mathrm{id}$ *has a non-trivial kernel.*

Proof. For (1), assume that $\ker(R)=0$. Then $R\colon \mathfrak{g}\to\mathfrak{n}$ is invertible, hence an isomorphism. This is a contradiction. The same is true for $R+\mathrm{id}$. For (2) assume that $\ker(R)=\ker(R+\mathrm{id})=0$. Then R and $R+\mathrm{id}$ are isomorphisms from $\mathfrak{g}$ to $\mathfrak{n}$, and $\mathfrak{g}\cong\mathfrak{n}$. Then we can apply a result of Jacobson [25] to the automorphism $\psi:=(R+\mathrm{id})\circ R^{-1}$ of $\mathfrak{n}$, because $\mathfrak{n}$ is not solvable. We obtain a nonzero fixed point $x\in\mathfrak{n}$, so that

$$0=\psi(x)-x=(R+\mathrm{id})R^{-1}(x)-x=R^{-1}(x).$$

Since R is bijective, $x=0$, a contradiction. □

Corollary 2.22. *Let* $\mathfrak{n}$ *be a simple Lie algebra and* R *be an invertible RB-operator of nonzero weight* λ *on* $\mathfrak{n}$*. Then we have* $R=-\lambda\,\mathrm{id}$*.*

Proof. By rescaling we may assume that R has weight 1. We obtain a PA-structure on $(\mathfrak{g},\mathfrak{n})$ by Proposition 2.13, with Lie bracket (9) on $\mathfrak{g}$. Since $\mathfrak{n}$ is not solvable, either R or $R+\mathrm{id}$ have a nontrivial kernel. But $\ker(R)=0$ by assumption, so that $\ker(R+\mathrm{id})$ is a nontrivial ideal of $\mathfrak{n}$. Hence we have $R+\mathrm{id}=0$. □

3. PA-structures on pairs of semisimple Lie algebras

We will assume that all algebras in this section are finite-dimensional. Let $x\cdot y$ be a PA-structure on $(\mathfrak{g},\mathfrak{n})$ over $\mathbb{C}$, where $\mathfrak{g}$ is simple and $\mathfrak{n}$ is semisimple. Then $\mathfrak{n}$ is also simple, and both $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic, see Proposition 4.9 in [11]. We have a similar result for $\mathfrak{n}$ simple and $\mathfrak{g}$ semisimple. However, its proof is more difficult than the first one.

Theorem 3.1. *Let* $x\cdot y$ *be a PA-structure on* $(\mathfrak{g},\mathfrak{n})$ *over* $\mathbb{C}$*, where* $\mathfrak{n}$ *is simple and* $\mathfrak{g}$ *is semisimple. Then* $\mathfrak{g}$ *is also simple, and both* $\mathfrak{g}$ *and* $\mathfrak{n}$ *are isomorphic.*

Proof. By Corollary 2.15 we have $x\cdot y=\{R(x),y\}$ for an RB-operator R of weight 1 on $\mathfrak{n}$. Assume that $\mathfrak{g}$ and $\mathfrak{n}$ are not isomorphic. By Proposition 2.21 (2) both $\ker(R)$ and $\ker(R+\mathrm{id})$ are proper nonzero ideals of $\mathfrak{g}$, with $\ker(R)\cap\ker(R+\mathrm{id})=0$. So we have

$$\mathfrak{g}=\ker(R)\oplus\ker(R+\mathrm{id})\oplus\mathfrak{s}$$

with a semisimple ideal $\mathfrak{s}$. We have $\mathfrak{n}=\mathrm{im}(R)+\mathrm{im}(R+\mathrm{id})$ because of $x=R(-x)+(R+\mathrm{id})(x)$ for all $x\in\mathfrak{n}$, and

$$\mathrm{im}(R)\cong\mathfrak{g}/\ker(R)\cong\ker(R+\mathrm{id})\oplus\mathfrak{s},$$
$$\mathrm{im}(R+\mathrm{id})\cong\mathfrak{g}/\ker(R+\mathrm{id})\cong\ker(R)\oplus\mathfrak{s}.$$

This yields a semisimple decomposition

$$\mathfrak{n}=(\ker(R+\mathrm{id})\oplus\mathfrak{s})+(\ker(R)\oplus\mathfrak{s}).$$

Suppose that $\mathfrak{s}$ is nonzero. Then both summands are not simple. This is a contradiction to Theorem 4.2 in Onishchik's paper [30], which says that at least one summand in a semisimple decomposition of a simple Lie algebra must be simple. Hence we obtain $\mathfrak{s}=0$, $\mathrm{im}(R)=\ker(R+\mathrm{id})$, $\mathrm{im}(R+\mathrm{id})=\mathrm{im}(R)$ and

$$\mathfrak{n}=\mathrm{im}(R)\dot{+}\,\mathrm{im}(R+\mathrm{id}).$$

Then the main result of Koszul's note [28] implies that $\mathfrak{n} = \operatorname{im}(R) \oplus \operatorname{im}(R + \mathrm{id})$, which is a contradiction to the simplicity of $\mathfrak{n}$. Hence $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic. □

If $\mathfrak{g}$ is semisimple with only two simple summands, we can prove the same result for any field K of characteristic zero.

Proposition 3.2. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n}$ is semisimple, and $\mathfrak{g} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ is the direct sum of two simple ideals of $\mathfrak{g}$. Then $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic.*

The proof is the same as before. The only argument where we needed the complex numbers, was the result of [30], which we do not need here.

Let $\mathfrak{n} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ be a direct sum of two simple isomorphic ideals $\mathfrak{s}_1$ and $\mathfrak{s}_2$. We would like to find all RB-operators of weight 1 on $\mathfrak{n}$ such that $\mathfrak{g}$ with bracket (9) is isomorphic to $\mathfrak{n}$.

Proposition 3.3. *All PA-structures on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{g} \cong \mathfrak{n} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ simple isomorphic ideals of $\mathfrak{n}$, arise by the trivial RB-operators or by one of the following RB-operators R on $\mathfrak{n}$, and $\psi \in \operatorname{Aut}(\mathfrak{n})$,*

$$
\begin{aligned}
R((s_1, s_2)) &= (-s_1, -\psi(s_1)),\\
R((s_1, s_2)) &= (0, \psi(s_1)),\\
R((s_1, s_2)) &= (-s_1, 0)),
\end{aligned}
$$

up to permuting the factors and application of $\varphi(R) = -R - \mathrm{id}$ to these operators.

Proof. By Proposition 2.5 and Proposition 2.7 the given operators are RB-operators of weight 1 on $\mathfrak{n}$, because R is. By Proposition 2.21 at least one of $\ker(R)$ and $\ker(R + \mathrm{id})$ is nonzero. Suppose first that both $\ker(R)$ and $\ker(R+\mathrm{id})$ are zero. Then we have $\mathfrak{g} = \ker(R) \oplus \ker(R+\mathrm{id})$ and $\mathfrak{n} = \ker(R) \dotplus \ker(R + \mathrm{id})$. It is easy to see that $\ker(R)$ coincides with $\mathfrak{s}_1$ or $\mathfrak{s}_2$ by using the Theorem of Koszul [28]. Applying φ if necessary, we can assume that $\ker(R) = \mathfrak{s}_2$. Then again by Koszul's result we have $R((s_1, s_2)) = (\psi_1(s_1), \psi_2(s_1))$ or $R((s_1, s_2)) = (\psi_1(s_1), 0))$ for some $\psi_1, \psi_2 \in \operatorname{Aut}(\mathfrak{n})$. Since $\operatorname{im}(R) = \ker(R + \mathrm{id})$ we either have $R((s_1, s_2)) = (-s_1, -\psi(s_1))$ or $R((s_1, s_2)) = (-s_1, 0)$.
In the second case, one of the kernels is zero. Applying φ if necessary, we may assume that $\ker(R + \mathrm{id}) = 0$ and $\ker(R) = \mathfrak{s}_1$. Then $\mathfrak{g}/\ker(R)$ is a simple Lie algebra, and $-R - \mathrm{id}$ is an invertible RB-operator of weight 1 on $\mathfrak{g}/\ker(R)$. By Corollary 2.22 we obtain $-R - \mathrm{id} = -\mathrm{id}$, hence $R = 0$ on $\mathfrak{g}/\ker(R)$. This implies $R^2 = 0$ on $\mathfrak{g}$. The projections of $\operatorname{im}(R)$ to $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are either zero or an isomorphism on one factor. So we have $R((s, 0)) = (0, \psi(s))$ or $R((s, 0)) = (\psi_1(s), \psi_2(s))$ for some automorphisms ψ, ψ_1, ψ_2. But the second operator does not satisfy $R^2 = 0$, and hence is impossible. Therefore we are done. □

Proposition 3.4. *Let $x \cdot y = \{R(x), y\}$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ defined by an RB-operator R of weight 1 on $\mathfrak{n}$. Let $\mathfrak{n}_1 = \ker(R^n)$, $\mathfrak{n}_2 = \ker(R + \mathrm{id})^n$, $\mathfrak{n}_3 = \operatorname{im}(R^n) \cap \operatorname{im}((R + \mathrm{id})^n)$ for $n = \dim(V)$. Then $\mathfrak{n} = \mathfrak{n}_1 \dotplus \mathfrak{n}_2 \dotplus \mathfrak{n}_3$ with $\{\mathfrak{n}_1, \mathfrak{n}_3\} \subseteq \mathfrak{n}_1$, $\{\mathfrak{n}_2, \mathfrak{n}_3\} \subseteq \mathfrak{n}_2$, and $\mathfrak{n}_3$ is solvable.*

Proof. We first show by induction that $\ker(R^i)$ is a subalgebra of $\mathfrak{n}$, and that

$$
\{\ker(R^i), \operatorname{im}((R + \mathrm{id})^i)\} \subseteq \ker(R^i)
$$

for all $i \geq 1$. The case $i = 1$ goes as follows. We already know that $\ker(R)$ is a subalgebra of $\mathfrak{n}$. So we have to show that $\{\ker(R), \operatorname{im}(R + \mathrm{id})\} \subseteq \ker(R)$. Let $x \in \ker(R)$ and $y \in \mathfrak{n}$. Then by (6) we have

$$\begin{aligned}\{x,(R+\mathrm{id})(y)\} &= \{x,R(y)\}+\{x,y\}\\ &= [x,y]+\{y,R(x)\}\\ &= [x,y],\end{aligned}$$

which is in $\ker(R)$, since this is an ideal in $\mathfrak{g}$. For the induction step $i\mapsto i+1$ consider the iteration of the Lie bracket (9) for all $i\geq 0$, given by

$$[x,y]_i=[x,y]_{i+1}-[R(x),y]_i-[x,R(y)]_i$$

for all $i\geq 0$. Then

$$\begin{aligned}\{x,y\} &= [x,y]_1-[R(x),y]_0-[x,R(y)]_0\\ &= [x,y]_2-[R^2(x),y]_0-2[R(x),y]_0-2[R(x),R(y)]_0-2[x,R(y)]_0-[x,R^2(y)]_0\end{aligned}$$

and so on. Define a degree of a term $[R^l(x),R^k(y)]_m$ by $l+k+m$, and let $x,y\in\ker(R^{i+1})$. We can iterate the brackets, until the degree of every summand on the right-hand side will be greater than $3i$, so that all summands either have a term $R^l(x)$ with $l>i$, or a term $R^k(y)$ with $k>i$, or all summands lie in $[\ker(R^{i+1}),\ker(R^{i+1})]_{i+1}$. By induction hypothesis, such terms will vanish for $l>i$ or $k>i$, and since $\ker(R^{i+1})$ is an ideal in $\mathfrak{g}_{i+1}$, we have $\{x,y\}\in\ker(R^{i+1})$, so that $\ker(R^{i+1})$ is a subalgebra of $\mathfrak{n}$. The induction step for the second claim follows similarly.

Since the image of a subalgebra under the action of an RB-operator is a subalgebra, $\mathfrak{n}_1$, $\mathfrak{n}_2$ and their intersection $\mathfrak{n}_3$ are subalgebras of $\mathfrak{n}$. We want to show that $\mathfrak{n}=\mathfrak{n}_1\dot{+}\mathfrak{n}_2\dot{+}\mathfrak{n}_3$. Because of $\ker(R^n)\cap\mathrm{im}(R^n)=0$ we have $\mathfrak{n}=\ker(R^n)\dot{+}\mathrm{im}(R^n)$. In the same way we have $\mathfrak{n}=\ker((R+\mathrm{id})^n)\dot{+}\mathrm{im}((R+\mathrm{id})^n)$. We obtain

$$\mathrm{im}(R^n)\cap\ker((R+\mathrm{id})^n)\dot{+}\mathrm{im}(R^n)\cap\mathrm{im}((R+\mathrm{id})^n)\subseteq\mathrm{im}(R^n).$$

We claim that $\ker((R+\mathrm{id})^n)\subseteq\mathrm{im}(R^n)$, so that we have equality above. Indeed, for $x\in\ker((R+\mathrm{id})^n)$ we have by the binomial formula

$$x+\binom{n}{n-1}R(x)+\cdots+\binom{n}{1}R^{n-1}(x)=-R^n(x)\in\mathrm{im}(R^n).$$

Applying R^{n-1} we obtain $R^{n-1}(x)\in\mathrm{im}(R^n)$ and

$$x+nR(x)+\cdots+\binom{n}{2}R^{n-2}(x)\in\mathrm{im}(R^n).$$

Iterating this we obtain $x\in\mathrm{im}(R^n)$. This yields

$$\begin{aligned}\mathfrak{n} &= \ker(R^n)\dot{+}\mathrm{im}(R^n)\\ &= \ker(R^n)\dot{+}\ker((R+\mathrm{id})^n)\dot{+}\mathrm{im}(R^n)\cap\mathrm{im}((R+\mathrm{id})^n)\\ &= \mathfrak{n}_1\dot{+}\mathfrak{n}_2\dot{+}\mathfrak{n}_3.\end{aligned}$$

On $\mathfrak{n}_3$ both operators R and $R+\mathrm{id}$ are invertible. By Proposition 2.21 part (2) it follows that $\mathfrak{n}_3$ is solvable. □

Corollary 3.5. *The decomposition* $\mathfrak{n}=\mathfrak{n}_1\dot{+}\mathfrak{n}_2\dot{+}\mathfrak{n}_3$ *induces a decomposition* $\mathfrak{g}_i=\mathfrak{n}_1\dot{+}\mathfrak{n}_2\dot{+}\mathfrak{n}_3$ *for each* $i\geq 1$ *with the same properties as in the Proposition. The Lie algebras* $(\mathfrak{n}_j,[,]_i)$ *and* $(\mathfrak{n}_j,[,]_0)$ *are isomorphic for* $j=1,2,3$.

Proof. Since R and $R+\mathrm{id}$ are RB-operators on all $\mathfrak{g}_i$, we obtain the same decomposition with the same subalgebras. Note that $R+\mathrm{id}$ is invertible on $\mathfrak{n}_1$, R is invertible on $\mathfrak{n}_2$ and both are invertible on $\mathfrak{n}_3$. In order to show that $(\mathfrak{n}_1,[,]_i$ is isomorphic to $(\mathfrak{n}_1,[,]_0$, we consider a chain of isomorphisms

$$(\mathfrak{n}_1,[,]_n) \xrightarrow{R+\mathrm{id}} (\mathfrak{n}_1,[,]_{n-1}) \xrightarrow{R+\mathrm{id}} \cdots \xrightarrow{R+\mathrm{id}} (\mathfrak{n}_1,[,]_0).$$

In a similar way we can deal with $\mathfrak{n}_2$ and $\mathfrak{n}_3$. □

Note that Proposition 3.6 *is not correct. Hence the proof of Proposition* 3.7 *and* 3.8 *is invalid. However, the statement of both results is true and we have given a new proof of it in our paper* [16] *on decompositions of algebras and post-associative algebra structures.*

Proposition 3.6. *Let $\mathfrak{g}=\mathfrak{s}_1+\mathfrak{s}_2$ be the vector space sum of two complex semisimple subalgebras of $\mathfrak{g}$. Then $\mathfrak{g}$ is semisimple.*

Proof. Suppose that the claim is not true and let $\mathfrak{g}$ be a counterexample of minimal dimension. Then $\mathfrak{g}$ contains a nonzero abelian ideal $\mathfrak{a}$. Then we obtain

$$\mathfrak{g}/\mathfrak{a}=\mathfrak{s}_1/(\mathfrak{s}_1\cap\mathfrak{a})+\mathfrak{s}_2/(\mathfrak{s}_2\cap\mathfrak{a}).$$

Since $\mathfrak{s}_1\cap\mathfrak{a}$ is an abelian ideal $\mathfrak{s}_1$, it must be zero, i.e., $\mathfrak{s}_1\cap\mathfrak{a}=0$. In the same way we have $\mathfrak{s}_2\cap\mathfrak{a}=0$. Hence we obtain a semisimple decomposition of $\mathfrak{g}/\mathfrak{a}$ with $\dim(\mathfrak{g}/\mathfrak{a})<\dim(\mathfrak{g})$. If $\mathfrak{g}/\mathfrak{a}$ is semisimple, this is a contradiction to the minimality of the counterexample $\mathfrak{g}$. Otherwise we may assume that $\mathfrak{g}$ has 1-dimensional solvable radical. Then $\mathfrak{g}$ is reductive, and by Theorem 3.2 of [30], there are no semisimple decompositions of a complex reductive non-semisimple Lie algebra. Hence we are done. □

Proposition 3.7. *Let $x\cdot y=\{R(x),y\}$ be a PA-structure on $(\mathfrak{g},\mathfrak{n})$ over $\mathbb{C}$, where $\mathfrak{n}$ is simple, defined by an RB-operator R of weight* 1 *on $\mathfrak{n}$, with associated Lie algebras $\mathfrak{g}_i$ for $i=1,\ldots,n=\dim(V)$. Assume that $\mathfrak{g}_0=\mathfrak{n}$ and $\mathfrak{g}_n$ are semisimple. Then all $\mathfrak{g}_i$ are isomorphic to $\mathfrak{n}$.*

Proof. Since $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are kernels of homomorphisms, they are ideals in $\mathfrak{g}_n$. The quotient $\mathfrak{g}_n/(\mathfrak{n}_1+\mathfrak{n}_2)\cong\mathfrak{n}_3$ is semisimple and solvable by Proposition 3.4. Hence $\mathfrak{n}_3=0$, and we obtain $\mathfrak{g}_n=\ker(R^n)\oplus\ker((R+\mathrm{id})^n)$. Because of Corollary 3.5 we have the decomposition $\mathfrak{g}_i=\ker(R^n)\dotplus\ker((R+\mathrm{id})^n)$ for all $i<n$, where all Lie algebras $(\ker(R^n),[,]_i)$ are isomorphic, and all Lie algebras $(\ker((R{+}\mathrm{id})^n),[,]_i)$ are isomorphic. By Proposition 3.6 all $\mathfrak{g}_i$ are semisimple. By Koszul's result [28], all $\mathfrak{g}_i$ are isomorphic. □

Proposition 3.8. *Suppose that there is a post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$ over $\mathbb{C}$, where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is complete. Then $\mathfrak{n}$ must be semisimple.*

Proof. By Corollary 2.15 the PA-structure is given by $x\cdot y=\{R(x),y\}$, where R is an RB-operator of weight 1 on $\mathfrak{n}$. If at least one of $\ker(R)$ and $\ker(R+\mathrm{id})$ is trivial, we obtain $\mathfrak{g}\cong\mathfrak{n}$ by Proposition 2.21, part (1). Otherwise $\mathfrak{n}=\mathrm{im}(R)+\mathrm{im}(R+\mathrm{id})$ is the sum of two nonzero semisimple subalgebras. By Proposition 3.6 $\mathfrak{n}$ is semisimple. □

4. PA-structures on $(\mathfrak{g},\mathfrak{n})$ with $\mathfrak{n}=\mathfrak{sl}_2(\mathbb{C})\times\mathfrak{sl}_2(\mathbb{C})$

In [11], Proposition 4.7 we have shown that PA-structures with $\mathfrak{n}=\mathfrak{sl}_2(\mathbb{C})$ exist on $(\mathfrak{g},\mathfrak{n})$ if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, or to one of the solvable non-unimodular Lie algebras $\mathfrak{r}_{3,\lambda}(\mathbb{C})$ for $\lambda\in\mathbb{C}\setminus\{-1\}$. In this section we want to show an analogous result for $\mathfrak{n}=\mathfrak{sl}_2(\mathbb{C})\times\mathfrak{sl}_2(\mathbb{C})$. Here we will use RB-operators on $\mathfrak{n}$ and an explicit classification by Douglas and Repka [18] of all subalgebras of $\mathfrak{n}$. This classification is up to inner automorphisms, but we will only need the subalgebras up to isomorphisms. Let us fix a basis $(X_1, Y_1, H_1, X_2, Y_2, H_2)$ of $\mathfrak{n}$ consisting of the following 4×4 matrices.

$$X_1 = E_{12},\ Y_1 = E_{21},\ H_1 = E_{11} - E_{22},\ X_2 = E_{34},\ Y_2 = E_{43},\ H_2 = E_{33} - E_{44}.$$

We use the following table.

Table 1: Complex 3-dimensional Lie algebras

$\mathfrak{g}$	Lie brackets
$\mathbb{C}^3$	$-$
$\mathfrak{n}_3(\mathbb{C})$	$[e_1, e_2] = e_3$
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_2$
$\mathfrak{r}_3(\mathbb{C})$	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + e_3$
$\mathfrak{r}_{3,\lambda}(\mathbb{C}),\ \lambda \neq 0$	$[e_1, e_2] = e_2,\ [e_1, e_3] = \lambda e_3$
$\mathfrak{sl}_2(\mathbb{C})$	$[e_1, e_2] = e_3,\ [e_1, e_3] = -2e_1,\ [e_2, e_3] = 2e_2$

Among the family $\mathfrak{r}_{3,\lambda}(\mathbb{C})$, $\lambda \neq 0$ there are still isomorphisms. In fact, $\mathfrak{r}_{3,\lambda}(\mathbb{C}) \cong \mathfrak{r}_{3,\mu}(\mathbb{C})$ if and only if $\mu = \lambda^{-1}$ or $\mu = \lambda$. The list of subalgebras $\mathfrak{h}$ of $\mathfrak{n}$ is given as follows. We first list the solvable subalgebras, then the semisimple ones and the subalgebras with a non-trivial Levi decomposition.

Table 2: Solvable subalgebras

$\dim(\mathfrak{h})$	Representative	Isomorphism type
1	$\langle X_1\rangle,\ \langle H_1\rangle,\ \langle X_1 + X_2\rangle,\ \langle X_1 + H_2\rangle,\ \langle H_1 + aH_2\rangle,\ a \in \mathbb{C}^*$	$\mathbb{C}$
2	$\langle X_1, X_2\rangle,\ \langle X_1, H_2\rangle,\ \langle H_1, H_2\rangle$	$\mathbb{C}^2$
2	$\langle X_1 + X_2, H_1 + H_2\rangle,\ \langle X_1, H_1 + X_2\rangle,\ \langle X_1, H_1 + aH_2\rangle,\ a \in \mathbb{C}$	$\mathfrak{r}_2(\mathbb{C})$
3	$\langle X_1, X_2, H_1 + \lambda H_2\rangle,\ \lambda \neq 0$	$\mathfrak{r}_{3,\lambda}(\mathbb{C}),\ \lambda \neq 0$
3	$\langle X_1, H_1, H_2\rangle,\ \langle X_1, H_1, X_2\rangle$	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$
4	$\langle X_1, H_1, X_2, H_2\rangle$	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$

Table 3: Semisimple subalgebras and Levi decomposable subalgebras

$\dim(\mathfrak{h})$	Representative	Isomorphism type
3	$\langle X_1, Y_1, H_1\rangle,\ \langle X_1 + X_2, Y_1 + Y_2, H_1 + H_2\rangle$	$\mathfrak{sl}_2(\mathbb{C})$
4	$\langle X_1, Y_1, H_1, H_2\rangle,\ \langle X_1, Y_1, H_1, X_2\rangle$	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$
5	$\langle X_1, Y_1, H_1, X_2, H_2\rangle$	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$

Theorem 4.1. *Suppose that there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras, and all these possibilities do occur:*

(1) $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.
(2) $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}_{3,\lambda}(\mathbb{C})$, $\lambda \neq -1$.
(3) $\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathfrak{r}_{3,\mu}(\mathbb{C})$, $(\lambda, \mu) \neq (-1,-1)$.
(4) $\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$.
(5) $\mathfrak{r}_2(\mathbb{C}) \oplus (\mathbb{C}^3 \ltimes \mathbb{C}) = \langle x_1, \ldots, x_6 \rangle$ *and Lie brackets, for* $\alpha \neq 0$, $\beta \neq 0, -1$

$$[x_1,x_2] = x_1,\ [x_3,x_6] = x_3,\ [x_4,x_6] = \alpha x_4,\ [x_5,x_6] = \beta x_5.$$

(6) $\mathbb{C} \oplus ((\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathbb{C}) \ltimes \mathbb{C}) = \langle x_1, \ldots, x_6 \rangle$ *and Lie brackets, for* $\lambda \neq 0$, $\alpha \neq 0, -1$,

$$[x_2,x_4] = x_2,\ [x_3,x_4] = \lambda x_3,\ [x_3,x_6] = x_3,\ [x_5,x_6] = \alpha x_5.$$

(7) $(\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathbb{C}^2) \ltimes \mathbb{C} = \langle x_1, \ldots, x_6 \rangle$ *and Lie brackets, for* $\lambda \neq 0$, $\alpha_1, \alpha_2 \neq 0$, *and* $(\lambda, \alpha_1, \alpha_2) \neq (-1, \alpha_1, -\alpha_1 - 1)$,

$$[x_1,x_3] = x_1,\ [x_2,x_3] = \lambda x_2,\ [x_2,x_6] = \alpha_1 x_2,\ [x_4,x_6] = x_4,\ [x_5,x_6] = \alpha_2 x_5.$$

(8) $(\mathbb{C}^2 \oplus \mathbb{C}^2) \ltimes \mathbb{C}^2 = \langle x_1, \ldots, x_6 \rangle$ *and Lie brackets*

$$\begin{aligned} &[x_1,x_5] = x_1, \quad [x_2,x_5] = \alpha_2 x_2,\ [x_3,x_5] = \alpha_4 x_3,\ [x_4,x_5] = \alpha_6 x_4, \\ &[x_1,x_6] = \alpha_1 x_1,\ [x_2,x_6] = \alpha_3 x_2,\ [x_3,x_6] = \alpha_5 x_3,\ [x_4,x_6] = \alpha_7 x_4, \end{aligned}$$

with one of the following conditions:

$$\begin{aligned} (a) \quad & \alpha_3 = 1,\ \alpha_5 = \alpha_1\alpha_7,\ \alpha_6 = \alpha_2\alpha_4,\ \alpha_1\alpha_2 \neq 1,\ \alpha_4, \alpha_7 \neq 0, -1, \\ (b) \quad & \alpha_4 = \alpha_1 - 1,\ \alpha_5 = -\alpha_1,\ \alpha_6 = \alpha_2(\alpha_1 - 1),\ \alpha_7 = \alpha_1\alpha_3 - \alpha_1^2\alpha_2 - \alpha_3, \\ & \alpha_3 - \alpha_1\alpha_2 \neq 0,\ \alpha_1 \neq 0, 1. \end{aligned}$$

Proof. By Corollary 2.15 it is enough to consider the RB-operators R of weight 1 on $\mathfrak{n}$. Then $\ker(R)$ and $\ker(R + \mathrm{id})$ are ideals in $\mathfrak{g}$. If R is trivial, or one of the kernels is trivial, then we have $\mathfrak{g} \cong \mathfrak{n}$, which is type (1). So we assume that R is non-trivial, both $\ker(R)$ and $\ker(R+\mathrm{id})$ are non-zero, and $\dim(\ker(R)) \geq \dim(\ker(R + \mathrm{id}))$. Then, for $\mathfrak{n} \not\cong \mathfrak{g}$, either $\mathfrak{g}$ has a non-trivial Levi decomposition, or $\mathfrak{g}$ is solvable.

Case 1: Assume that $\mathfrak{g}$ has a non-trivial Levi decomposition, i.e., that $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{r}$. We claim that $\mathfrak{sl}_2(\mathbb{C})$ is a direct summand of $\mathfrak{g}$, i.e., $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}$, and that $\mathfrak{r}$ is not isomorphic to $\mathfrak{r}_3(\mathbb{C})$. Then we can argue as follows. Because of Remark 2.12 of [12], $\mathfrak{g}$ cannot be unimodular, except for $\mathfrak{g} \cong \mathfrak{n}$. Thus $\mathfrak{r}$ cannot be unimodular, so that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}_{3,\lambda}(\mathbb{C})$ with $\lambda \neq -1$. On the other hand, all such algebras do arise by Proposition 2.6 and Proposition 4.7 of [11].

Case 1a: Suppose that $\mathfrak{sl}_2(\mathbb{C})$ is not contained in $\ker(R)$, $\ker(R + \mathrm{id})$ as a subalgebra. Then $\dim(\ker(R + \mathrm{id})) = 1$ and $\dim(\ker(R)) \in \{1, 2\}$. Let us assume, both have dimension 1. The other case goes similarly. Then we have $\mathfrak{r} = \langle x_1, x_2, x_3 \rangle$, $\ker(R) = \langle x_1 \rangle$ and $\ker(R + \mathrm{id}) = \langle x_2 \rangle$. Furthermore $\mathrm{im}(R) \cong \mathfrak{sl}_2(\mathbb{C}) \ltimes \langle x_2, x_3 \rangle$ and $\mathrm{im}(R + \mathrm{id}) \cong \mathfrak{sl}_2(\mathbb{C}) \ltimes \langle x_1, x_3 \rangle$ are 5-dimensional subalgebras of $\mathfrak{n}$. By table 3, $\mathfrak{sl}_2(\mathbb{C})$ is a direct summand of them. This implies that $\mathfrak{sl}_2(\mathbb{C})$ is also a direct summand in $\mathfrak{g}$. Since both $\ker(R)$ and $\ker(R + \mathrm{id})$ are ideals in $\mathfrak{r}$, we can exclude that $\mathfrak{r}$ is isomorphic to $\mathfrak{r}_3(\mathbb{C})$, and we are done.

Case 1b: $\mathfrak{sl}_2(\mathbb{C})$ is contained in one of $\ker(R)$, $\ker(R + \mathrm{id})$. Without loss of generality we may assume that $\mathfrak{sl}_2(\mathbb{C}) \subseteq \ker(R)$. If $\ker(R) = \mathfrak{sl}_2(\mathbb{C})$, then $\mathfrak{sl}_2(\mathbb{C})$ is an ideal of $\mathfrak{g}$, and we have $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}$, where $\mathfrak{r} \cong \mathrm{im}(R) \leq \mathfrak{n}$ is not isomorphic to $\mathfrak{r}_3(\mathbb{C})$ by table 2, and we are done. Thus we may assume that $\dim(\ker(R)) \geq 4$. If R splits with subalgebras $\ker(R)$ and $\ker(R+\mathrm{id})$, then $\mathfrak{g} \cong \ker(R) \oplus \ker(R + \mathrm{id})$, and $\dim(\ker(R)) + \dim(\ker(R + \mathrm{id})) = 6$. By table 3, $\mathfrak{sl}_2(\mathbb{C})$ is a direct summand of $\ker(R)$, and hence of $\mathfrak{g}$. So we have again $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}$, and $\mathfrak{r}$ is not isomorphic to $\mathfrak{r}_3(\mathbb{C})$. If R is not split, it remains to consider the case $\dim(\ker(R)) = 4$ and $\dim(\ker(R + \mathrm{id})) = 1$. We have $\mathfrak{r} = \langle x, y, z\rangle$ with $\ker(R) = \mathfrak{sl}_2(\mathbb{C}) \oplus \langle x\rangle$, $\ker(R + \mathrm{id}) = \langle y\rangle$ and $[y, \mathfrak{sl}_2(\mathbb{C})] = 0$. Assume that $[z, \mathfrak{sl}_2(\mathbb{C})] \neq 0$. Then $\mathfrak{sl}_2(\mathbb{C})$ is not a direct summand of the 5-dimensional subalgebra $\mathrm{im}(R + \mathrm{id})$ of $\mathfrak{n}$, which is a contradiction to table 3. Thus we have $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}$. Since $\mathfrak{r}$ has two disjoint 1-dimensional ideals $\langle x\rangle$ and $\langle y\rangle$, it is not isomorphic to $\mathfrak{r}_3(\mathbb{C})$.

Case 2: Assume that $\mathfrak{g}$ is solvable. Then $\mathrm{im}(R)$ and $\mathrm{im}(R + \mathrm{id})$ are solvable subalgebras of $\mathfrak{n}$ of dimension at most 4 by table 2. So we have $\dim(\ker(R)) \geq \dim(\ker(R + \mathrm{id})) \geq 2$. Thus we have the following four cases:

$$\begin{aligned}
(2a) &\quad \dim(\ker(R)) = 4,\ \dim(\ker(R + \mathrm{id})) = 2,\\
(2b) &\quad \dim(\ker(R)) = 3,\ \dim(\ker(R + \mathrm{id})) = 3,\\
(2c) &\quad \dim(\ker(R)) = 3,\ \dim(\ker(R + \mathrm{id})) = 2,\\
(2d) &\quad \dim(\ker(R)) = 2,\ \dim(\ker(R + \mathrm{id})) = 2.
\end{aligned}$$

For the cases $(2a)$ and $(2b)$, R is split since the dimensions add up to 6. Then $\mathfrak{g}$ is a direct sum of two solvable subalgebras, which are both isomorphic to subalgebras of $\mathfrak{n}$. So we have $\mathfrak{n} = \ker(R) \dotplus \ker(R + \mathrm{id})$ and $\mathfrak{g} = \ker(R) \oplus \ker(R + \mathrm{id})$.

Case 2a: Since we have only $\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$ as 4-dimensional solvable subalgebra of $\mathfrak{n}$, we have $\mathfrak{g} \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^2$, which is of type (3) for $(\lambda, \mu) = (0, 0)$, or $\mathfrak{g} \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$, which is of type (4). Both cases can arise. For the first one we will show this in case $(2b)$. For the second, it follows from Proposition 2.7 with $\mathfrak{n} = \langle X_1, H_1, X_2, H_2\rangle \dotplus \langle Y_1, Y_2 + H_1\rangle$.

Case 2b: We have $\mathfrak{g} \cong \mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathfrak{r}_{3,\mu}(\mathbb{C})$. The case $(\lambda, \mu) = (-1, -1)$ cannot arise by Theorem 3.3 of [11]. The cases $(\lambda, \mu) = (-1, \mu)$ for $\mu \neq -1$ arise by Proposition 2.7 with

$$\mathfrak{n} = \langle X_1, X_2, H_1 - H_2\rangle \dotplus \langle Y_1, Y_2, H_1 + \mu H_2\rangle.$$

The other cases with $\lambda, \mu \neq -1$ arise by Proposition 2.6 and Proposition 4.7 of [11].

Case 2c: Here $\mathfrak{g}$ is isomorphic to $(\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})) \rtimes \mathbb{C}$ or $(\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathbb{C}^2) \rtimes \mathbb{C}$. In the first case, $\mathfrak{r}_2(\mathbb{C}) \rtimes \mathbb{C} \cong \mathrm{im}(R)$ is a solvable subalgebra of $\mathfrak{n}$, hence isomorphic to $\mathfrak{r}_{3,\nu}(\mathbb{C})$ by table 2. So $\mathbb{C}$ acts trivially on $\mathfrak{r}_2(\mathbb{C})$, and $\mathrm{im}(R + \mathrm{id}) \cong \mathfrak{r}_{3,\lambda}(\mathbb{C}) \rtimes \mathbb{C} \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$. Then $\mathfrak{g} \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$, which we have already considered in Case $(2a)$. For $(\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathbb{C}^2) \rtimes \mathbb{C}$ we need to distinguish $\lambda = 0$ and $\lambda \neq 0$.

Case 2c, $\lambda = 0$: By Proposition 2.3 we may assume that $\mathrm{im}(R + \mathrm{id}) = \langle X_1, H_1, X_2, H_2\rangle$. Since $\ker(R)$ is an ideal of $\mathrm{im}(R + \mathrm{id})$ isomorphic to $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$, we have $\ker(R) = \langle X_1, H_1, X_2\rangle$. Let us consider the characteristic polynomial χ_R of the linear operator R acting on $\mathfrak{n}$. By assumption on the kernels, $\chi_R(t) = t^3(t + 1)^2(t - \rho)$.

Case 2c, $\lambda = 0$, $\rho \neq 0, -1$: Then $R(x_6) = \rho x_6$ for $x_6 = H_2 + \alpha H_1 + \beta X_1 + \gamma X_2$. Since $\ker(R + \mathrm{id})$ is an abelian 2-dimensional subalgebra of $\mathfrak{n}$, we have

$$\ker(R + \mathrm{id}) = \langle Y_1 + \nu_1 X_1 + \nu_2 H_1, Y_2 + \nu_3 X_2 + \nu_4 H_2\rangle.$$

We want to compute $[x,y]$ for $x=x_6$ and $y\in\ker(R+\mathrm{id})$. By Proposition 2.13 we have, using $R(x_6)=\rho x_6$

$$\begin{aligned}[x,y]&=\{R(x),y\}-\{R(y),x\}+\{x,y\}\\&=\{R(x),y\}\\&=\rho\{x,y\}.\end{aligned}$$

For $x_6=H_2+\alpha H_1+\beta X_1+\gamma X_2$ and $y\in\ker(R+\mathrm{id})$ this yields, using the Lie brackets of $\mathfrak{n}$ in the standard basis $\{X_1,Y_1,H_1,X_2,Y_2,H_2\}$,

$$[x_6,Y_1+\nu_1X_1+\nu_2H_1]=\rho((2\alpha\nu_1-2\beta\nu_2)X_1-2\alpha Y_1+\beta H_1), \tag{10}$$

$$[x_6,Y_2+\nu_3X_2+\nu_4H_2]=\rho((2\nu_3-2\gamma\nu_4)X_2-2Y_2+\gamma H_2). \tag{11}$$

Since $\ker(R+\mathrm{id})$ is an ideal in $\mathfrak{g}$ and $\rho\neq 0$, both vectors lie again in $\ker(R+\mathrm{id})$. Comparing coefficients for the basis vectors we obtain

$$\beta=-2\alpha\nu_2,\ \alpha(\nu_1+\nu_2^2)=0,\ \gamma=-2\nu_4,\ \nu_3=-\nu_4^2.$$

Suppose that $\alpha=0$. Then $x_6=H_2-2\nu_4X_2$ and $\langle X_1,H_1\rangle\cong\mathfrak{r}_2(\mathbb{C})$ is a direct summand of $\mathfrak{g}$. Therefore $\mathfrak{g}\cong\mathfrak{r}_2(\mathbb{C})\oplus\mathbb{C}\oplus\mathfrak{r}_{3,\mu}(\mathbb{C})$ with $\mathbb{C}=\langle Y_1+\nu_1X_1+\nu_2H_1\rangle$, $\mathfrak{r}_{3,\mu}(\mathbb{C})=\langle X_2,H_2-2\nu_4X_2,Y_2+\nu_4H_2-\nu_4^2X_2\rangle$, $\mu=-(\rho+1)/\rho$, which we have already considered above. Hence we may assume that $\alpha\neq 0$ and $\nu_1=-\nu_2^2$. Consider a new basis for $\mathfrak{g}$ (note that we redefine x_6) given by

$$\begin{aligned}(x_1,\ldots,x_6)=(&X_1,-\frac{1}{2}H_1+\nu_2X_1,X_2,Y_1+\nu_2H_1-\nu_2^2X_1,Y_2+\nu_4H_2-\nu_4^2X_2,\\&\frac{1}{2\rho}(H_2+\alpha H_1-2\alpha\nu_2X_1-2\nu_4X_2)),\end{aligned}$$

with Lie brackets

$$[x_1,x_2]=x_1,\ [x_1,x_6]=-\frac{\rho+1}{\rho}\alpha x_1,\ [x_3,x_6]=-\frac{\rho+1}{\rho}x_3,\ [x_4,x_6]=\alpha x_4,\ [x_5,x_6]=x_6.$$

This algebra is of type (5), if we replace x_6 by $x_6+\frac{\alpha(\rho+1)}{\rho}x_2$. It arises for the triangular-split RB-operator R with $A_-=\ker(R)=\langle x_1,x_2,x_3\rangle$, $\ker(R+\mathrm{id})=\langle x_4,x_5\rangle$ and $A_0=\langle x_6\rangle$, where $x_6=H_2-2\nu_4X_2$, with the action $R(x_6)=\rho x_6$.

Case 2c, $\lambda=0$, $\rho=-1$: We may assume that there exists $x_6=Y_2+v$ such that $(R+\mathrm{id})(x_6)=\mu(H_2+\alpha H_1+\beta X_1+\gamma X_2)$ for some non-zero μ and some $\alpha,\beta,\gamma\in\mathbb{C}$. Since $\ker(R+\mathrm{id})$ is an abelian subalgebra we obtain $\alpha=\beta=0$ and $\ker(R+\mathrm{id})=\langle H_2+\gamma X_2,Y_1+\nu_1X_1+\nu_2H_1\rangle$. Then we may choose $x_6=Y_2+\kappa X_2+\nu_3H_1+\nu_4X_1$. Then

$$\begin{aligned}[x_6,H_2+\gamma X_2]&=\{R(x_6),H_2+\gamma X_2\}\\&=\{(R+\mathrm{id})(x_6)-x_6,H_2+\gamma X_2\}\\&=-\{Y_2+\kappa X_2,H_2+\gamma X_2\}\\&=-2Y_2+2\kappa X_2+\gamma H_2.\end{aligned}$$

This is not contained in $\ker(R+\mathrm{id})$, which is a contradiction to the fact that $\ker(R+\mathrm{id})$ is an ideal.

Case 2c, $\lambda=0$, $\rho=0$: Then we have $R(H_2)=\alpha H_1+\beta X_1+\gamma X_2\neq 0$ and $\ker(R+\mathrm{id})=\langle Y_1+\nu_1X_1+\nu_2H_1,Y_2+\nu_3X_2+\nu_4H_2\rangle$. Since $[H_2,Y_2+\nu_1X_1+\nu_2H_1]=\{\gamma X_2,Y_2+\nu_1X_2+\nu_2H_2\}$ is in $\ker(R+\mathrm{id})$, we obtain $\gamma=0$. Since $[H_2,Y_1+\nu_1X_1+\nu_2H_2]=\{\alpha H_1+\beta X_1,Y_1+\nu_1X_1+\nu_2H_1\}$

is in $\ker(R+\mathrm{id})$, we obtain $\alpha(\nu_1+\nu_2^2)=0$ and $\beta=-2\alpha\nu_2$. Since $R(H_2)\neq 0$ we have $\alpha\neq 0$, $\nu_1=-\nu_2^2$ and $R(H_2)=\alpha H_1-2\alpha\nu_2X_1$. Consider a new basis for $\mathfrak{g}$ given by

$$(x_1,\ldots,x_6)=(X_1,-\frac{1}{2}H_1+\nu_2X_1,X_2,Y_1+\nu_2H_1-\nu_2^2X_1,Y_2+\nu_3X_2+\nu_4H_2,-\frac{1}{2}H_2),$$

with Lie brackets

$$[x_1,x_2]=x_1,\ [x_1,x_6]=\alpha x_1,\ [x_3,x_6]=x_3,\ [x_4,x_6]=-\alpha x_4.$$

This algebra is of type (3), if we replace x_6 by $x_6-\alpha x_2$.

Case 2c, $\lambda\neq 0$: Then we have $\ker(R)=\langle X_1,X_2,-\frac{1}{2}(H_1+\lambda H_2)\rangle$. We again have $\chi_R(t)=t^3(t+1)^2(t-\rho)$, where we distinguish the cases $\rho\neq 0,-1$, $\rho=-1$ and $\rho=0$.

Case 2c, $\lambda\neq 0$, $\rho\neq 0,-1$: Then we may assume that $R(x_6)=\rho x_6$ for $x_6=H_2+\alpha H_1+\beta X_1+\gamma X_2$. As $\ker(R+\mathrm{id})$ is abelian, we have $\ker(R+\mathrm{id})=\langle Y_1+\nu_1X_1+\nu_2H_1,Y_2+\nu_3X_2+\nu_4H_2\rangle$. Since $V=\ker(R)\oplus\ker(R+\mathrm{id})\oplus\langle x_6\rangle$, the two elements $H_1+\lambda H_2$ and $H_2+\alpha H_1$ need to be linearly independent, i.e., $1-\alpha\lambda\neq 0$. By (10) and (11) we obtain $\gamma=-2\nu_4$, $\beta=-2\alpha\nu_2$, $\nu_3=-\nu_4^2$ and $\alpha(\nu_1+\nu_2^2)$. Suppose that $\alpha=0$. Then $x_6=H_2-2\nu_4X_2$. Consider a new basis for $\mathfrak{g}$ given by

$$(x_1,\ldots,x_6)=(Y_1+\nu_1X_1+\nu_2H_1,X_1,X_2,-\frac{1}{2}(H_1+\lambda H_2),Y_2-\nu_4^2X_2+\nu_4H_2,-\frac{1}{2(\rho+1)}H_2),$$

with Lie brackets

$$[x_2,x_4]=x_2,\ [x_3,x_4]=\lambda x_3,\ [x_3,x_6]=x_3,\ [x_4,x_6]=-\lambda\nu_4x_3,\ [x_5,x_6]=-\frac{\rho}{1+\rho}x_5.$$

This is an algebra of type (6), if we replace x_4 by $x_4+\lambda\nu_4x_3$.
Now we assume that $\alpha\neq 0$. Consider a new basis for $\mathfrak{g}$ given by

$$\begin{aligned}(x_1,\ldots,x_6)=(X_1,X_2,-\frac{1}{2}(H_1+\lambda H_2),Y_2-\nu_4^2X_2+\nu_4H_2,Y_1-\nu_2^2X_1+\nu_2H_1,\\ -\frac{1}{2(\rho+1)}(H_2-2\nu_4X_2+\alpha(H_1-2\nu_2X_1))),\end{aligned}$$

with Lie brackets

$$\begin{aligned}&[x_1,x_3]=x_1,\ [x_2,x_3]=\lambda x_2,\ [x_1,x_6]=\alpha x_1,\ [x_2,x_6]=x_2,\\ &[x_3,x_6]=-\alpha\nu_2x_1-\lambda\nu_4x_2,\ [x_4,x_6]=\delta x_4,\ [x_5,x_6]=\alpha\delta x_5,\end{aligned}$$

where $\delta=-\frac{\rho}{\rho+1}$. Replacing x_6 by $\frac{1}{\delta}(x_6-\alpha\nu_2x_1-\nu_4x_2-\alpha x_3)$ we obtain the Lie brackets

$$[x_1,x_3]=x_1,\ [x_2,x_3]=\lambda x_2,\ [x_2,x_6]=\alpha'x_2,\ [x_4,x_6]=x_4,\ [x_5,x_6]=\alpha x_5,$$

where

$$\alpha'=\frac{1-\alpha\lambda}{\delta}=\frac{(\rho+1)(\alpha\lambda-1)}{\rho}.$$

Note that $\alpha'\neq 0$ and $\alpha'\neq\alpha\lambda-1$ by assumption. In other words, $\alpha\neq\frac{\alpha'+1}{\lambda}$. Consider a new basis for $\mathfrak{g}$ given by

$$(x_1',\ldots,x_6')=(x_2,x_1,\frac{1}{\lambda}x_3,x_4,x_5,x_6-\frac{\alpha'}{\lambda}x_3),$$

with Lie brackets

$$[x_1',x_3']=x_1',\ [x_2',x_3']=\lambda'x_2',\ [x_2',x_6']=-\alpha'\lambda'x_2',\ [x_4',x_6']=x_4',\ [x_5',x_6']=\alpha x_5',$$

where $\lambda' = \frac{1}{\lambda}$. This is of type (7). Since $\mathfrak{r}_{3,\lambda}(\mathbb{C}) \cong \mathfrak{r}_{3,\lambda'}(\mathbb{C})$, one may check that we do not only have $\alpha \neq \frac{\alpha'+1}{\lambda}$, but also $\alpha \neq \lambda - \alpha'$. For $\frac{\alpha'+1}{\lambda} \neq \lambda - \alpha'$ we obtain no restriction for α. However, for $\frac{\alpha'+1}{\lambda} = \lambda - \alpha'$ we obtain $\lambda = -1$ or $\lambda = \alpha' + 1$, which excludes both $(\lambda, \alpha', \alpha) = (-1, \alpha', -\alpha' - 1)$ and $(\lambda, \alpha', \alpha) = (\lambda, \lambda - 1, 1)$. Rewriting this in the parameters of the Lie brackets from type (7), we obtain all cases except for $(\lambda, \alpha', \alpha) = (\lambda, \lambda - 1, 1)$ with $\lambda \neq -1$. These PA-structures arise by a triangular-split RB-operator with $A_- = \ker(R)$, $A_+ = \ker(R + \mathrm{id})$ and $A_0 = \langle x_6 \rangle$ with the action $R(x_6) = \rho x_6$, $\rho \neq 0, -1$.

Case 2c, $\lambda \neq 0, \rho = -1$: This leads to a contradiction in the same way as case 2*c* with $\lambda = 0, \rho = -1$.

Case 2c, $\lambda \neq 0, \rho = 0$: We have $R(H_2) = \alpha X_1 + \beta X_2 + \gamma(H_1 + \lambda H_2)$ and $\ker(R + \mathrm{id}) = \langle x_4, x_5 \rangle$ with $x_4 = Y_1 + \nu_1 X_1 + \nu_2 H_1$, $x_5 = Y_2 + \nu_3 X_2 + \nu_4 H_2$. Similarly to (10), (11) we obtain $R(H_2) = \gamma(H_1 - 2\nu_2 X_1) + \gamma\lambda(H_2 - 2\nu_4 X_2)$. This implies that $\gamma \neq 0$ and $x_4 = Y_1 - \nu_2^2 X_1 + \nu_2 H_1$, $x_5 = Y_2 - \nu_4^2 X_2 + \nu_4 H_2$. By setting $x_1 = X_1$, $x_2 = X_2$, $x_3 = -\frac{1}{2}(H_1 + \lambda H_2)$ and $x_6 = \frac{1}{2\gamma} H_2$ we obtain a new basis for $\mathfrak{g}$ with Lie brackets

$$[x_1, x_3] = x_1,\ [x_2, x_3] = \lambda x_2,\ [x_1, x_6] = -x_1,\ [x_2, x_6] = \delta x_2,$$
$$[x_3, x_6] = \nu_2 x_1 + \lambda^2 \nu_4 x_2,\ [x_4, x_6] = x_4,\ [x_5, x_6] = \lambda x_5,$$

where $\delta = -\frac{1+\lambda\gamma}{\gamma}$ with $\delta \neq -\lambda$. Replacing x_6 by $x_6 + \nu_2 x_1 + \lambda \nu_4 x_2 + x_3$ we obtain the brackets

$$[x_1, x_3] = x_1,\ [x_2, x_3] = \lambda x_2,\ [x_2, x_6] = \alpha_1 x_2,\ [x_4, x_6] = x_4,\ [x_5, x_6] = \lambda x_5$$

with $\alpha_1 = \delta + \lambda = -\frac{1}{\gamma}$. This is of type (7) with $\alpha_2 = \lambda$. It arises by the triangular-split RB-operator with $A_- = \langle x_1, x_2 \rangle$, $A_+ = \langle x_4, x_5 \rangle$ and $A_0 = \langle u, v \rangle$, with $u = \frac{1}{\gamma}(H_2 - 2\nu_4 X_2)$ and $v = H_1 - 2\nu_2 X_1 + \lambda(H_2 - 2\nu_4 X_2)$, and the action $R(u) = v$, $R(v) = 0$.

Case 2d: Suppose that one of the kernels $\ker(R)$ and $\ker(R + \mathrm{id})$ is non-abelian. Without loss of generality, let us assume that $\ker(R) \cong \mathfrak{r}_2(\mathbb{C})$. Write $\mathfrak{g} \cong (\ker(R) \oplus \ker(R + \mathrm{id})) \ltimes \langle a, b \rangle$. Then $\ker(R) \ltimes \langle a \rangle$ is a 3-dimensional solvable subalgebra of $\mathrm{im}(R + \mathrm{id})$. By table 2 we see that it is isomorphic to $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$. In this case there exist nonzero $a' \in \ker(R) \oplus \langle a \rangle$ and $b' \in \ker(R) \oplus \langle b \rangle$ such that $[a', \ker(R)] = [b', \ker(R)] = 0$. Then $\mathfrak{g} \cong \ker(R) \oplus (\ker(R + \mathrm{id}) \oplus \langle a', b' \rangle)$ with $\ker(R) \cong \mathfrak{r}_2(\mathbb{C})$, and $\ker(R + \mathrm{id}) \oplus \langle a', b' \rangle \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$ by Table 2. Hence we obtain $\mathfrak{g} \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$, which is of type (4).
So we may assume that $\ker(R) \cong \ker(R + \mathrm{id}) \cong \mathbb{C}^2$. Then the characteristic polynomial of R has the form $\chi_R(t) = t^2(t+1)^2(t - \rho_1)(t - \rho_2)$.

Case 2d, $\rho_1, \rho_2 \neq 0, -1$: Suppose first that either $\rho_1 \neq \rho_2$, or that $\rho_1 = \rho_2$ and the eigenspace is 2-dimensional. Then by Proposition 3.4, $\mathfrak{n} = \ker(R) \dotplus \ker(R + \mathrm{id}) \dotplus \langle x_5', x_6' \rangle$ with linearly independent eigenvectors x_5', x_6' corresponding to the eigenvalues ρ_1 and ρ_2. Since $\ker(R)$ is an abelian ideal in $\mathrm{im}(R + \mathrm{id}) = \langle X_1, H_1, X_2, H_2 \rangle$, we may assume that $\ker(R) = \langle X_1, X_2 \rangle$ and $[x_5', x_6'] = 0$. The decomposition $\mathfrak{n} = \ker(R + \mathrm{id}) \dotplus \mathrm{im}(R + \mathrm{id})$ shows that $\ker(R + \mathrm{id})$ has a basis $x_3 = Y_1 + \alpha H_1 + \nu_3 X_1$, $x_4 = Y_2 + \beta H_2 + \nu_4 X_2$. Since $[x_5', x_6'] = 0$, we have $x_5' = H_1 + \nu_1 X_1 + \xi_1(H_2 + \nu_2 X_2)$, $x_6' = H_2 + \nu_2 X_2 + \xi_2(H_1 + \nu_1 X_1)$ with $\xi_1 \xi_2 \neq 1$. So we have by (10) and (11) $x_3 = Y_1 - \frac{\nu_1}{2} H_1 - \frac{\nu_1^2}{4} X_1$, $x_4 = Y_2 - \frac{\nu_2}{2} H_2 - \frac{\nu_2^2}{4} X_2$. Consider a basis for $\mathfrak{g}$ given by

$$(x_1, \ldots, x_6) = (X_1, X_2, x_3, x_4, -\frac{1}{2(1+\rho_1)} x_5', -\frac{1}{2(1+\rho_2)} x_6'),$$

with Lie brackets

$$[x_1,x_5]=x_1,\ [x_1,x_6]=\xi_2 x_1,\ [x_2,x_5]=\xi_1 x_2,\ [x_2,x_6]=x_2,$$
$$[x_3,x_5]=\gamma x_3,\ [x_3,x_6]=\delta\xi_2 x_3,\ [x_4,x_5]=\gamma\xi_1 x_4,\ [x_4,x_6]=\delta x_4,$$

where $\gamma=-\frac{\rho_1}{\rho_1+1}$, $\delta=-\frac{\rho_2}{\rho_2+1}$ with $\gamma,\delta\neq 0,-1$ and $\xi_1\xi_2\neq 1$. This is type $(8a)$. It arises by the triangular-split RB-operator R with $A_-=\langle X_1,X_2\rangle$, $A_+=\langle x_3,x_4\rangle$ and $A_0=\langle x_5,x_6\rangle$, where R acts on A_0 by $R(x_5)=\rho_1 x_5$ and $R(x_6)=\rho_2 x_6$. Note that for $\nu_2=\xi_2=0$ and $\xi_1\neq 0$ we get type (7) without the restriction $(\lambda,\alpha_1,\alpha_2)\neq(\lambda,\lambda-1,1)$ for $\lambda\neq -1$, which we had in Case 2c, $\lambda\neq 0$, $\rho\neq 0,-1$.

Suppose now that $\rho_2=\rho_1\neq 0,-1$, and the eigenspace for ρ_1 is 1-dimensional. Let $R(x_5')=\rho_1 x_5'$ and $R(x_6')=x_5'+\rho_1 x_6'$. In the same way as before we have $x_5'=H_1+\nu_1 X_1+\xi(H_2+\nu_2 X_2)$, $x_6'=\kappa(H_2+\nu_2 X_2)$ with $\kappa\neq 0$ and $x_3=Y_1-\frac{\nu_1}{2}H_1-\frac{\nu_1^2}{4}X_1$, $x_4=Y_2-\frac{\nu_2}{2}H_2-\frac{\nu_2^2}{4}X_2$. Consider a basis for $\mathfrak{g}$ given by

$$(x_1,\ldots,x_6)=(X_1,X_2,x_3,x_4,-\frac{1}{2(1+\rho_1)}x_5',-\frac{1}{2(1+\rho_1)}x_6'),$$

with Lie brackets

$$[x_1,x_5]=x_1,\ [x_1,x_6]=(\gamma+1)x_1,\ [x_2,x_5]=\xi x_2,\ [x_2,x_6]=(\kappa+\xi+\gamma\xi)x_2,$$
$$[x_3,x_5]=\gamma x_3,\ [x_3,x_6]=-(\gamma+1)x_3,\ [x_4,x_5]=\gamma\xi x_4,\ [x_4,x_6]=(\kappa\gamma-\xi-\gamma\xi)x_4,$$

where $\gamma=-\frac{\rho_1}{\rho_1+1}\neq 0,-1$ and $\kappa\neq 0$. This is type $(8b)$. It arises by the triangular-split RB-operator R with $A_-=\langle X_1,X_2\rangle$, $A_+=\langle x_3,x_4\rangle$ and $A_0=\langle x_5,x_6\rangle$, where R acts on A_0 by $R(x_5)=\rho_1 x_5$ and $R(x_6)=x_5+\rho_1 x_6$.

Case 2d, $\rho_1=\rho_2=0$: We have $\mathfrak{g}=\ker(R+\mathrm{id})\dot{+}\operatorname{im}(R+\mathrm{id})$ and we can assume that $\ker(R)=\langle X_1,X_2\rangle$ and $\ker(R+\mathrm{id})=\langle Y_1+\nu_1X_1+\nu_2H_1,Y_2+\nu_3X_2+\nu_4H_2\rangle$. Suppose first that $R(v)=X_1$ and $R(w)=X_2$ for some v,w. Then

$$[Y_1+\nu_1X_1+\nu_2H_1,v]=\{Y_1+\nu_1X_1+\nu_2H_1,X_1\}=-H_1+2\nu_2X_1\in\ker(R+\mathrm{id}),$$

which is a contradiction. Otherwise we see from the possible Jordan forms of R that there exist v,w with $R(v)=\alpha X_1+\beta X_2\neq 0$ and $R(w)=v$. This leads to a contradiction in the same way.

Case 2d, $\rho_1=0,\rho_2\neq 0,-1$: This case is analagous to the second part of the case before.

Case 2d, $\rho_1=0,\rho_2=-1$: As above we may assume that $\operatorname{im}(R+\mathrm{id})=\langle X_1,X_2,H_1,H_2\rangle$ and $\ker(R)=\langle X_1,X_2\rangle$, and $\alpha H_1+\beta H_2+\gamma X_1+\delta X_2\in\ker(R+\mathrm{id})\cap\operatorname{im}(R+\mathrm{id})$ for some $\alpha,\beta,\gamma,\delta\in\mathbb{C}$. Since $\ker(R+\mathrm{id})$ is abelian, we may assume that $\ker(R+\mathrm{id})=\langle H_1+\nu_1X_1,Y_2+\nu_2X_2+\nu_3H_2\rangle$ for some $\nu_1,\nu_2,\nu_3\in\mathbb{C}$. Let $v\in\ker(R^2)$ such that $R(v)=\nu_4X_1+\nu_5X_2\neq 0$. Then

$$[v,Y_2+\nu_2X_2+\nu_3H_2]=\{\nu_4X_1+\nu_5X_2,Y_2+\nu_2X_2+\nu_3H_2\}=\nu_5(H_2-2\nu_3X_2)\in\ker(R+\mathrm{id})$$

implies that $\nu_5=0$. By $[v,H_1+\nu_1X_1]=\{\nu_4X_1,H_1+\nu_1X_1\}=-2\nu_4X_1\in\ker(R+\mathrm{id})$ we obtain $\nu_4=0$, which is a contradiction to $R(v)\neq 0$. □

Remark 4.2. The algebras from different types are non-isomorphic, except for algebras of type (8), which have intersections with type (3) and (7) for certain parameter choices.

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