A REMARK ON AN INEQUALITY FOR THE PRIME COUNTING FUNCTION

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Abstract. We note that the inequalities $0.92\frac{x}{\log(x)} < \pi(x) < 1.11\frac{x}{\log(x)}$ do not hold for all $x \geq 30$, contrary to some references. These estimates on $\pi(x)$ came up recently in papers on algebraic number theory.

1. Chebyshev's estimates for $\pi(x)$

Let $\pi(x)$ denote the number of primes not greater than x, i.e.,

$$\pi(x) = \sum_{p \leq x} 1.$$

One of the first works on the function $\pi(x)$ is due to Chebyshev. He proved (see [2]) in 1852 the following explicit inequalities for $\pi(x)$, holding for all $x \geq x_0$ with some x_0 sufficiently large:

$$c_1 \frac{x}{\log(x)} < \pi(x) < c_2 \frac{x}{\log(x)},$$

$$c_1 = \log(2^{1/2}3^{1/3}5^{1/5}30^{-1/30}) \approx 0.921292022934,$$

$$c_2 = \frac{6}{5}c_1 \approx 1.10555042752.$$

This can be found in many books on analytic number theory (see for example [1], [3], [11] and [14]). But it seems that this result is sometimes cited incorrectly: it is claimed that the estimates are valid for all $x \geq 30$. For example, in [6], page 21 we read that

$$c_1 \frac{x}{\log(x)} < \pi(x) < c_2 \frac{x}{\log(x)}, \quad \forall\, x \geq 30.$$

But a quick numerical computation shows that this is wrong. To give an example, take $x = 100$. Then we have $\pi(x) = 25$ and

$$c_2 \frac{x}{\log(x)} \approx 24.00672250690558538515780234 < 25.$$

Actually, the inequality is far from true for small x. We have the following result:

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Theorem 1.1. *Let* $c_2 \approx 1.10555042752$ *be Chebyshev's constant. Then the inequality*

$$\pi(x) < c_2 \frac{x}{\log(x)}$$

is true for all $x \geq 96098$. *For* $x = 96097$ *it is false.*

Proof. In [10] it is shown that

$$\pi(x) < \frac{x}{\log(x) - 1.11}, \quad x \geq 4.$$

The RHS is less or equal to $c_2 x / \log(x)$ if and only if

$$x \geq \exp\left(\frac{1.11 \cdot c_2}{c_2 - 1}\right) \approx 112005.18.$$

This shows the claim for $x \geq 112006$. Since $x/\log(x)$ is a monotonously increasing function it is enough to check the claimed estimate for intergers x in the intervall $[96098, 112006]$ by computer. For $x = 96097$ we have $\pi(96097) = 9260$ and $c_2 x / \log(x) \approx 9259.92$. □

The incorrect inequality was also used in a former version of Khare's proof of Serre's modularity conjecture for the level one case, see [8], [9]. Let $\mathbb{F}$ be a finite field of characteristic p. The conjecture stated that an odd, irreducible Galois representation $\rho\colon Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F})$ which is unramified outside p is associated to a modular form on $SL_2(\mathbb{Z})$. Khare's proof is an elaborate induction on p. Starting with a p for which the conjecture is known one wants to prove the conjecture for a larger prime P. Kahre's arguments do only work if P and p are not Fermat primes, and if

$$\frac{P}{p} \leq a$$

for certain values $a > 1$, close to 1. At this point Khare used the incorrect estimate on $\pi(x)$, as explained above. Fortunately the proof easily could be repaired by using better estimates on $\pi(x)$ provided by Rosser and Schoenfeld [12], and Dusart [4].
Indeed, P. Dusart proved inequalities for $\pi(x)$ which are much better than Chebyshev's estimates. He verifies this for smaller x numerically. Nevertheless he claims in his thesis [5], that Chebyshev gave the following inequality

$$0.92 \frac{x}{\log(x)} < \pi(x) < 1.11 \frac{x}{\log(x)}, \quad x \geq 30,$$

which is equally wrong.
The question is: where lies the origin for this error ? Chebyshev himself proved inequalities in [2] with his constants c_1 and $c_2 = \frac{6}{5} c_1$ indeed for all $x \geq 30$, but for inequalities involving $\psi(x) = \sum_{n \leq x} \Lambda(n)$ instead of $\pi(x)$. His estimates concerning $\psi(x)$ seem to be correct for all

$x \geq 30$. For example, he shows by elementary means that, for all $x \geq 30$,

$$\psi(x) < \frac{6}{5}c_1 x + \frac{5}{4\log(6)}\log^2(x) + \frac{5}{4}\log(x) + 1,$$
$$\psi(x) > c_1 x - \frac{5}{2}\log(x) - 1.$$

To derive from this inequalities on $\pi(x)$ for $x \geq 30$, we have to estimate

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log(x)}{\log(p)}\right] \log(p).$$

Using the estimates $[y] \leq y < [y] + 1 \leq 2[y]$ for $y \geq 1$ we obtain

$$\psi(x) \leq \pi(x)\log(x) \leq 2\psi(x), \quad x \geq 2.$$

On the RHS we cannot do easily much better than $2\psi(x)$. Hence we obtain

$$c_1 \frac{x}{\log(x)} < \pi(x) < 2c_2 \frac{x}{\log(x)}, \quad x \geq 30.$$

On the other hand we know that

$$\pi(x) = \frac{\psi(x)}{\log(x)} + O\left(\frac{x}{\log^2(x)}\right), \quad x \geq 2,$$

so that we obtain, as x tends to infinity,

$$(c_1 + o(1))\frac{x}{\log(x)} \leq \pi(x) \leq (c_2 + o(1))\frac{x}{\log(x)}.$$

Chebyshev used these estimates to prove Bertrand's postulate: each interval $(n, 2n]$ for $n \geq 1$ contains at least one prime. Moreover his results were a first step towards the proof of the prime number theorem.

2. Other estimates for $\pi(x)$

There are many interesting inequalities on the function $\pi(x)$. Let us first consider inequalities of the form

$$A\frac{x}{\log(x)} < \pi(x) < B\frac{x}{\log(x)}$$

for all $x \geq x_0$, where x_0 depends on the constant $A \leq 1$ and respectively on $B > 1$. On the LHS we can choose A equal to 1, if $x \geq 17$. In fact, we have [5]

$$\frac{x}{\log(x)} < \pi(x), \quad \forall\, x \geq 17.$$

Note that for $x = 16.999$ we have $x/\log(x) \approx 6.0000257$, but $\pi(x) = 6$. Consider the RHS of the above inequalities: if we want to hold such inequalities on $\pi(x)$ for all $x \geq x_0$ with a smaller x_0, we need to enlarge the constant B. Conversely, if we need this inequality for smaller B, we have to enlarge x_0. The prime number theorem ensures that we can choose B as close to 1 as we want, provided x_0 is sufficiently large. The following result of Dusart [4] enables us to derive adjusted versions for the above inequalities:

Theorem 2.1 (Dusart)**.** *For real x we have the following sharp bounds:*

$$\pi(x) \geq \frac{x}{\log(x)}\left(1 + \frac{1}{\log(x)} + \frac{1.8}{\log^2(x)}\right), \quad x \geq 32299,$$

$$\pi(x) \leq \frac{x}{\log(x)}\left(1 + \frac{1}{\log(x)} + \frac{2.51}{\log^2(x)}\right), \quad x \geq 355991.$$

One can derive, for example, the following inequalities.

$$\pi(x) < 1.095 \cdot \frac{x}{\log(x)}, \quad x \geq 284860,$$

$$\pi(x) < 1.25506 \cdot \frac{x}{\log(x)}, \quad x \geq 17.$$

Among other inequalities on $\pi(x)$ we mention the following ones:

$$\frac{x}{\log(x) - m} < \pi(x) < \frac{x}{\log(x) - M}$$

for all $x \geq x_0$ with real constants m and M. They have been studied by various authors. A good reference is the article [10]. There it is shown, for example, that

$$\pi(x) > \frac{x}{\log(x) - \frac{28}{29}}, \quad x \geq 3299,$$

$$\pi(x) < \frac{x}{\log(x) - 1.11}, \quad x \geq 4.$$

The second inequality can also be used to obtain results on our estimate $\pi(x) < B\frac{x}{\log(x)}$, in particular for smaller x, where the second inequality of Theorem 2.1 is not valid. However we have

$$\frac{x}{\log(x)}\left(1 + \frac{1}{\log(x)} + \frac{2.51}{\log^2(x)}\right) < \frac{x}{\log(x) - 1.11}, \quad x \geq 28516.$$

For $x > 10^6$ and $a = 1.08366$ we can use [10]

$$\pi(x) < \frac{x}{\log(x) - a}.$$

Here the upper bound of Dusart is better only as long as $x \geq 2846396$.
Finally we mention the book [13], providing many references on inequalities on $\pi(x)$, and the recent article [7], where lower and upper bounds for $\pi(x)$ of the form $\frac{n}{H_n - c}$ are discussed, where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

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