On a refinement of Ado's Theorem

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In this paper we study the minimal dimension $\mu(\mathrm{g})$ of a faithful g –module for n –dimensional Lie algebras g . This is an interesting invariant of g which is difficult to compute. It is desirable to obtain good bounds for $\mu(\mathrm{g})$, especially for nilpotent Lie algebras. Such a refinement of Ado's theorem is required for solving a question of J. Milnor in the theory of affine manifolds. We will determine here $\mu(\mathrm{g})$ for certain Lie algebras and prove upper bounds in general. For nilpotent Lie algebras of dimension n , the bound $\mathrm{n}^{\mathrm{n}}+1$ is known. We now obtain $\mu(\mathrm{g}) < \frac{\alpha}{\sqrt{\mathrm{n}}}2^{\mathrm{n}}$ with some constant $\alpha \sim 2.76287$.

1. Introduction

Let $\mathfrak{g}$ be an n–dimensional Lie algebra over a field K of characteristic zero. Ado's Theorem states that there exists a faithful representation of $\mathfrak{g}$ of *finite* dimension. We consider the following integer valued invariant of $\mathfrak{g}$:

$$\mu(\mathfrak{g},K) := \min\{\dim_K M \mid M \text{ is a faithful } \mathfrak{g}\text{–module}\}$$

It follows from the proof of Ado's Theorem that $\mu(\mathfrak{g},K)$ can be bounded by a function depending *only* on n. We will write $\mu(\mathfrak{g})$ if the field is fixed.
Virtually nothing is known about $\mu(\mathfrak{g})$. Interest for a refinement of Ado's Theorem in this respect comes from the question whether a given solvmanifold or nilmanifold admits a left-invariant affine structure or not. In the 70's Milnor conjectured that every solvmanifold admits such an structure. In particular, if the conjecture was true, $\mu(\mathfrak{g}) \le n+1$ for all solvable Lie algebras. However, there are counterexamples in dimension 10 and 11 even in the nilpotent case [BU2]. There are filiform nilpotent Lie algebras without any affine structure.
In [REE] it is proved that $\mu(\mathfrak{g}) < n^k+1$ for nilpotent Lie algebras of dimension n and nilpotency class k . Then $\mu(\mathfrak{g}) < n^n+1$ independently of k . We will improve this bound by showing $\mu(\mathfrak{g}) < \frac{\alpha}{\sqrt{n}}2^n$ with $\alpha \sim 2.76287$.
In the following we will assume $\mathrm{char(K)}=0$ if not mentioned otherwise. Note however, that for prime characteristic p the invariant $\mu(\mathfrak{g})$ is also an integer by Iwasawa's Theorem. Moreover $\mathfrak{g}$ can be embedded in an associative algebra with identity over K whose dimension is at most p^m with $m=n^3$. This gives an upper bound for $\mu(\mathfrak{g})$ over K , see [BAH], § 6.2 .

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First estimates of μ were made in connection with *linearizable* Lie groups over $\mathbb{R}$ and $\mathbb{C}$. Any Lie group is *locally* linearizable by Ado's Theorem, but there exist nonlinearizable Lie groups, e.g., the simply connected universal covering group of $\mathbf{SL}_2(\mathbb{R})$. However, if G is simply connected and solvable of dimension n, then G is linearizable by a Theorem of Malcev and isomorphic to a Lie subgroup of T_m, the group of non–singular upper triangular matrices. It arises the question about the size of m.
For the problem it is interesting to consider filiform nilpotent Lie algebras. All known counterexamples to the Milnor conjecture belong to this class. The bound n^k+1 for $\mu(\mathfrak{g})$ in that case is very rough. We provide a better bound in Proposition 7. If $\mathfrak{g}$ is filiform with abelian commutator algebra, or is of dimension less than 10, then $\mathfrak{g}$ admits an affine structure and we obtain a sharp result for $\mu(\mathfrak{g})$ (see Proposition 5).
It is not known whether $\mu(\mathfrak{g})$ grows polynomially or exponentially in n for nilpotent Lie algebras. The proof of Ado's theorem using the universal enveloping algebra does not give a polynomial bound. If $\mathfrak{g}$ is a solvable of dimension n with $\ell-$ dimensional nilradical $\mathfrak{n}$, we conjecture that $\mu(\mathfrak{g}) \le \mu(\mathfrak{n}) + n - \ell$.
We remark that the question of minimal faithful linear representations is also interesting for $p-$ groups, see [WEH].

2. First examples

Let $\mathfrak{g}$ be a Lie algebra of dimension n. How does $\mu(\mathfrak{g})$ depend on n ?
If $\mathfrak{g}$ has trivial center then the adjoint representation is faithful, hence $\mu(\mathfrak{g}) \le n$.
Assume $\mathfrak{g}$ to be abelian. Then $\mathfrak{g}$ is just a vector space. Any faithful representation ϕ of $\mathfrak{g}$ into $\mathfrak{gl}(V)$, where V is a d–dimensional vector space, turns $\phi(\mathfrak{g})$ into an $n-$ dimensional commutative subalgebra of $M_d(K)$. Since ϕ is a monomorphism, $n \le d^2$. But, in fact, $n \le [(d^2+4)/4]$ is true:

Proposition 1. (Jacobson) *Let M be a commutative subalgebra of $M_d(K)$ over an arbitrary field K. Then $\dim M \le [\frac{d^2+4}{4}]$ and this bound is sharp.*

The proof for $K=\mathbb{C}$ is due to Schur. The result implies that a faithful $\mathfrak{g}$–module has dimension d with $n \le [(d^2+4)/4]$, i.e., $d \ge \lceil 2\sqrt{n-1}\,\rceil$ where $\lceil x \rceil$ denotes the ceiling of x. On the other hand, it is easy to construct commutative subalgebras M of $M_d(K)$ of dimension exactly equal to $[(d^2+4)/4]$. We denote $\mu(\mathfrak{g})$ here by $\mu(n)$ since the number is independent of the field for abelian Lie algebras. As a corollary we obtain the following proposition:

Proposition 2. *Let $\mathfrak{g}$ be an abelian Lie algebra of dimension n over an arbitrary field K. Then $\mu(n) = \lceil 2\sqrt{n-1}\,\rceil$.*

Note that $\mu(n) = n$ is not true for $n > 4$: Let $\mathfrak{g}$ be an abelian Lie algebra with basis $\{x_1, \dots, x_5\}$. A faithful representation $\lambda : \mathfrak{g} \to \mathfrak{gl}(V)$ of dimension 4 is given by $\lambda(x_1) = e_{13},\ \lambda(x_2) = e_{23},\ \lambda(x_3) = e_{14},\ \lambda(x_4) = e_{24},\ \lambda(x_5) = \mathrm{Id}$. Here $\{e_{ij} \mid i,j = 1,2,3,4\}$ denotes the canonical basis for the matrix algebra. In fact, $\mu(5) = 4$.

Let $\mathfrak{t}_d$ be the nilpotent Lie algebra of strictly upper triangular matrices of order d and dimension $n = d(d-1)/2$. Then $\mu(\mathfrak{t}_d) = d$, and this is even smaller than $\mu(n)$ in the abelian case.

Proposition 3. *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra of dimension n with 1-dimensional center. Then $n \equiv 1(2)$ and $\mu(\mathfrak{g}) = (n+3)/2$.*

Proof: Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$. By assumption, $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}$ is 1-dimensional. Hence the Lie algebra structure on $\mathfrak{g}$ is defined by a skew-symmetric bilinear form $U \wedge U \to K$ where U is the subspace of $\mathfrak{g}$ complementary to K . It follows from the classification of such forms that $\mathfrak{g}$ is isomorphic to a Heisenberg Lie algebra $\mathfrak{h}_m(K)$. This algebras are defined on a $(2m+1)$–dimensional vector space with basis $x_1, \ldots, x_m, y_1, \ldots, y_m, z$ and brackets $[x_i, y_i] = z$. It is well known that they have a faithful $(m+2)$–dimensional representation, see example 1.1.2 in [COG]. This means $\mu(\mathfrak{g}) \leq m+2 = (n+3)/2$. On the other hand, there are no faithful representations of smaller dimension for $\mathfrak{h}_m(K)$. Since we have not found a proof in the literature, we will give one:

Lemma 1. *For the Heisenberg Lie algebras, $\mu(\mathfrak{h}_m) = m+2$.*

Proof: We first observe two facts:

(1) If the center $\mathfrak{z}$ of a nilpotent Lie algebra $\mathfrak{g}$ is 1-dimensional, then a representation $\lambda : \mathfrak{g} \to \mathfrak{gl}(V)$ is *faithful* if and only if $\mathfrak{z}$ acts nontrivially.
(2) In case of (1) there exists a $v \in V \setminus 0$ such that $\lambda(z)v \neq 0$, where z is a generator of $\mathfrak{z}$. If V has minimal dimension, then V is spanned by v and all $\lambda(x)v$ for $x \in \mathfrak{g}$.

If $\ker(\lambda) \neq 0$ then it intersects the center $\mathfrak{z}$ nontrivially, since $\mathfrak{g}$ is nilpotent and $\ker(\lambda)$ is a nonzero ideal of $\mathfrak{g}$. Hence $\ker(\lambda)$ contains z , i.e., $\lambda(z) = 0$. If $\lambda(z) \neq 0$, then $\ker(\lambda) = 0$. This shows (1) . For the second assertion observe that v and $\lambda(x)v$ generate a faithful submodule W of V . By minimality it follows $W = V$.

Assume that λ is a faithful representation of $\mathfrak{h}_m(K)$ of minimal degree. Fix $v \in V$ with $\lambda(z)v \neq 0$. We have to show $\dim V \geq m+2$.

Consider the *evaluation map* $e_v : \mathfrak{h}_m \to V$, $x \mapsto \lambda(x)v$. Let $\mathfrak{a} = \ker(e_v), \mathfrak{b} = \operatorname{im}(e_v)$. It is clear that $\mathfrak{a}$ is a subalgebra of $\mathfrak{h}_m$, not containing z .

Claim: $\mathfrak{a}$ is abelian:

Let $x, y \in \mathfrak{a}$, then $[x, y] \in \mathfrak{a}$, i.e., $\lambda([x, y])v = 0$. On the other hand, $[x, y] \in \mathfrak{z}$ and $\lambda(z)v \neq 0$, hence $[x, y] = 0$. We have $\dim V \geq \dim \mathfrak{b} = \dim \mathfrak{h}_m - \dim \mathfrak{a}$. The number on the right hand side is minimal if $\mathfrak{a}$ is a maximal abelian subalgebra. However, any maximal abelian subalgebra of $\mathfrak{h}_m$ not containing z has dimension m . Hence $\dim \mathfrak{b} \geq m+1$.

Claim: $v \notin \mathfrak{b}$, i.e., $\dim V \geq \dim \mathfrak{b} + 1 \geq m+2$:

Assume $v \in \mathfrak{b}$: Then there exists an x not in $\mathfrak{a}$ and not in $\mathfrak{z}$ such that $\lambda(x)v = v$. (Since $\lambda(z)$ is a commutator of two upper triangular endomorphisms, by Lie's theorem it is nilpotent. Therefore $\lambda(z)v = v$ is impossible.) There must be some $y \in \mathfrak{a}$ such that

$[x, y] = z$. If not, x would commute with $\mathfrak{a}$ and $< \mathfrak{a}, x >= \mathfrak{a}$ because $\mathfrak{a}$ is maximal abelian. This implies $x \in \mathfrak{a}$ and $v = \lambda(x)v = 0$, contradicting the choice of v . We obtain

$$\lambda(z)v = [\lambda(x), \lambda(y)]v = \lambda(x)\lambda(y)v - \lambda(y)\lambda(x)v = 0,$$

by using $\lambda(y)v = 0$ and $\lambda(x)v = v$. This is a contradiction. □

Remark 1. If $\mathfrak{g}$ is a 2-step nilpotent Lie algebra of dimension n then $\mu(\mathfrak{g}) \leq n+1$, see proposition 4 . For two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ we have $\mu(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \leq \mu(\mathfrak{g}_1) + \mu(\mathfrak{g}_2)$. Here we may have a strict inequality: Let $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{h}_1$. Then $\sum_{i=1}^k \mu(\mathfrak{h}_1) = 3k$ whereas $\mu(\mathfrak{g}) \leq 2k+1$: $\mathfrak{g}$ has basis $\{x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k\}$ with brackets $[x_i, y_i] = z_i$. A faithful representation $\lambda : \mathfrak{g} \to \mathfrak{gl}(V)$ of dimension $2k+1$ is given by

$$\lambda(x_i) = e_{1,i+1},\ \lambda(z_i) = e_{1,i+k+1},\ \lambda(y_i) = e_{i+1,i+k+1}.$$

Here $\{e_{i,j} \mid i, j = 1, \dots, 2k+1\}$ denotes the canonical basis for the matrix algebra. We have $[e_{i,j}, e_{k,l}] = \delta_{jk} e_{i,l} - \delta_{il} e_{k,j}$.

3. Lie algebras with an affine structure

If $\mathfrak{g}$ is the Lie algebra of an n – dimensional connected Lie group G which admits a left-invariant affine structure, then $\mathfrak{g}$ is said to admit an *affine structure*. The left-invariant affine structures on G are in 1–1 correspondence to so called LSA–structures on $\mathfrak{g}$:

Definition 1. A *left-symmetric algebra structure* or *LSA–structure* in short *on* $\mathfrak{g}$ *over a field* K is a K –bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto x \cdot y$ satisfying the conditions $[x, y] = x \cdot y - y \cdot x$ and $(x, y, z) = (y, x, z)$, where $(x, y, z) = x \cdot (y \cdot z) - (x \cdot y) \cdot z$ denotes the associator of $x, y, z \in \mathfrak{g}$.

For Lie algebras admitting an affine structure a stronger version of Ado's theorem holds (see [BU2]):

Lemma 2. *If* $\mathfrak{g}$ *admits an LSA–structure then* $\mu(\mathfrak{g}) \leq n+1$.

Which Lie algebras do admit an LSA–structure ? This is a difficult question, in particular for solvable Lie algebras. Semisimple Lie algebras over characteristic zero do *not* admit LSA–structures. This is no longer true for prime characteristic. LSA-structures for certain reductive Lie algebras can be classified ([BU1]). In the nilpotent case we have ([BU2]):

Proposition 4. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra of dimension* n *satisfying one of the following conditions:*

(1) $n < 8$.
(2) $\mathfrak{g}$ *is* p *–step nilpotent with* $p < 4$.
(3) $\mathfrak{g}$ *is* $\mathbb{Z}$ *–graded, i.e., has a nonsingular derivation.*

Then $\mathfrak{g}$ *admits an LSA–structure and* $\mu(\mathfrak{g}) \leq n+1$.

However, there exist nilpotent Lie algebras $\mathfrak{g}$ with $\mu(\mathfrak{g}) > n+1$, see [BU2]. These are the counterexamples to the Milnor conjecture. They are all filiform nilpotent, i.e., of step $n-1$.
On the other hand, it is often possible to find an LSA–structure on filiform Lie algebras. Consider the following construction:

Let $\mathfrak{g}$ be an $n-$ dimensional filiform Lie algebra with structure constants $\gamma_{i,j}^k$. Define an index set

$$\mathcal{D}_0 := \{(k,s)\in \mathbb{N}^2 \mid 2\le k\le [n/2],\ 2k+1\le s\le n\}$$

and set $\mathcal{D}=\mathcal{D}_0$ if n is odd, $\mathcal{D}=\mathcal{D}_0\cup\{(n/2,n)\}$, if n is even. Since $\mathfrak{g}$ is isomorphic to an infinitesimal deformation of the standard graded filiform L by a $2-$ cocycle $\psi\in H^2(L,L)$, we can obtain a special form for the structure constants of $\mathfrak{g}$ (see [BU3]):

Lemma 3. *Let $\mathfrak{g}$ be a complex filiform nilpotent Lie algebra of dimension n . Then there exists a basis $\{e_1,\dots,e_n\}$ such that*

(a) $[e_1,e_i]=e_{i+1}$ *for* $i\ge 2$
(b) *The structure constants in* $[e_i,e_j]=\sum_k \gamma_{i,j}^k e_k$ *(for* $i\ge 2$ *) can be written as*

$$\gamma_{i,j}^k=\sum_{l=0}^{[(j-i-1)/2]}(-1)^l\binom{j-i-l-1}{l}\alpha_{i+l,k-j+i+2l+1}$$

where the constants $\alpha_{i,j}$ *are zero for all pairs* (i,j) *not in* $\mathcal{D}$ *.*

We set $e_k=0$ for $k>n$, whereas $\gamma_{i,j}^k$ need not be zero in this case. There are $(n-3)^2/4$ structure constants $\alpha_{i,j}$ if n is odd, and $\frac{1}{4}(n-2)(n-4)+1$ otherwise. The formula above can be used to define filiform Lie algebras, but the Jacobi identity is not satisfied automatically (unless $n<8$).

Definition 2. Let $\mathfrak{g}$ be as above and set $A:=\operatorname{ad}(e_1)$, $B:=\operatorname{ad}(e_2)$. Let C be the linear map defined by $Ce_i=\zeta_i e_n$ with $\zeta_i\in\mathbb{C}$. We define linear maps $\lambda(e_i)$ as follows:

$$\begin{aligned}\lambda(e_1)&=A\\ \lambda(e_2)&=A^tBA+C\\ \lambda(e_i)&=[\lambda(e_1),\lambda(e_{i-1})],\ i\ge 3\end{aligned}$$

They define an LSA–structure on $\mathfrak{g}$ if and only if

$$(I)\qquad \operatorname{ad}(e_i)e_j=\lambda(e_i)e_j-\lambda(e_j)e_i$$

$$(II)\qquad \lambda([e_i,e_j])=[\lambda(e_i),\lambda(e_j)]$$

If (I) and (II) are satisfied we call this the *standard LSA–structure.* Note that $Ae_i=e_{i+1}$, $A^te_i=e_{i-1}$, $Be_i=[e_2,e_i]$ and $AC=0$.

Under which conditions on ζ_i and $\mathfrak{g}$ do equations $(I), (II)$ hold ? We would like to determine the filiform Lie algebras admitting a standard LSA–structure.

Lemma 4. *With the notations of definition* 2 *we have:*

(a) *For* $i \geq 2$

$$\mathrm{ad}(e_i) = \sum_{k=0}^{i-2} (-1)^k \binom{i-2}{k} A^{i-k-2} B A^k$$

$$\lambda(e_i) = (-1)^i (A^t B A^{i-1} + C A^{i-2}) + \sum_{k=0}^{i-3} (-1)^k \binom{i-3}{k} A^{i-k-3} B A^{k+1}$$

(b) *Property* (1) *is satisfied if and only if for* $k = 0, 1, \ldots, [(n-1)/2]$

$$\zeta_{2k+1} = \gamma_{2,2k+2}^{n+1} = \sum_{l=0}^{k} (-1)^{l-1} \binom{2k-l}{l-1} \alpha_{l+1,n-2k+2l}$$

(c) *Property* (2) *is satisfied if and only if a system of certain linear equations in the* ζ_{2k} *holds.*

For a proof see [BU3].
This construction provides an LSA–structure for many filiform Lie algebras. However, not all admit a standard LSA–structure. The linear equations in the ζ_{2k} do not have a solution in all cases.

Example 1. Let $\mathfrak{g}$ be a complex filiform Lie algebra of dimension 7 . Then there is a basis $\{e_1, \ldots, e_7\}$ such that

$$\begin{aligned}
[e_1, e_i] &= e_{i+1},\ i \geq 2 \\
[e_2, e_3] &= \alpha_{2,5} e_5 + \alpha_{2,6} e_6 + \alpha_{2,7} e_7 \\
[e_2, e_4] &= \alpha_{2,5} e_6 + \alpha_{2,6} e_7 \\
[e_2, e_5] &= (\alpha_{2,5} - \alpha_{3,7}) e_7 \\
[e_3, e_4] &= \alpha_{3,7} e_7
\end{aligned}$$

In this case, the Jacobi identity is satisfied automatically. Let $\lambda(e_i)$ as above. Then (I) is satisfied iff $\zeta_1 = \zeta_7 = 0,\ \zeta_3 = \alpha_{2,7},\ \zeta_5 = \alpha_{2,5} - 2\alpha_{3,7}$. The condition (II) is satisfied iff

$$\zeta_6 (2\alpha_{2,5} + \alpha_{3,7}) = 0.$$

We may take $\zeta_6 = 0$, hence this defines a (standard) LSA–structure on all 7 – dimensional filiform Lie algebras.

Proposition 5. *Let* $\mathfrak{g}$ *be a complex filiform nilpotent Lie algebra satisfying one of the following conditions:*

(1) $\mathfrak{g}$ *has abelian commutator algebra.*
(2) $\mathfrak{g}$ *is of dimension* $n < 10$ *.*
(3) $\mathfrak{g}$ *is the quotient of another filiform nilpotent Lie algebra of higher dimension.*

Then $\mathfrak{g}$ *admits an LSA–structure and* $\mu(\mathfrak{g}) = n$ *.*

Proof: It is known that $\mu(\mathfrak{g}) \geq n$ for filiform nilpotent Lie algebras of dimension n. To prove equality therefore it is enough to provide a faithful representation of dimension n. If $[\mathfrak{g}, \mathfrak{g}]$ is abelian, then there exists a basis $e_1, \dots, e_n$ such that the defining Lie brackets are as follows (see [BRA]):

$$[e_1, e_i] = e_{i+1}, \; i \geq 2$$
$$[e_2, e_i] = \alpha_{2,5} e_{i+2} + \cdots + \alpha_{2,n} e_n \;\; , \;\; i = 3, \dots, n-2$$

Here the Jacobi identity is satisfied automatically. Then $\mathfrak{g}$ admits a standard LSA–structure by setting $\zeta_i = \alpha_{2,n+3-i}$: In fact, the product is given by:

$$e_1.e_i = e_{i+1} \;\; , \;\; i \geq 2$$
$$e_2.e_i = \alpha_{2,5} e_{i+2} + \cdots + \alpha_{2,n} e_n$$

for $i = 2, \dots, n-2$. All other products $e_i.e_j$ are zero. This clearly satisfies $[e_i, e_j] = e_i.e_j - e_j.e_i$. We have to show $(e_i, e_j, e_k) = (e_j, e_i, e_k)$ for all $i \leq j \leq k$. This is clear for $i = j$ and $i \geq 2$. The only nontrivial case is $i = 1, j = 2$:

$$\begin{aligned}(e_1, e_2, e_k) &= e_1.(e_2.e_k) - (e_1.e_2).e_k = e_1.(\alpha_{2,5} e_{k+2} + \cdots + \alpha_{2,n} e_n) - e_3.e_k \\ &= \alpha_{2,5} e_{k+3} + \cdots + \alpha_{2,n} e_n = e_2.e_{k+1} = (e_2, e_1, e_k)\end{aligned}$$

The matrices $\lambda(e_i)$ are strictly lower–triangular. Its first and last column are zero. Hence the affine representation associated to this LSA–structure has a faithful subrepresentation of dimension n (see [BU2]), hence $\mu(\mathfrak{g}) = n$. This proves (1) .

For the second assertion, note that all nilpotent Lie algebras of dimension $n < 7$ admit a nonsingular derivation and hence an LSA–structure by proposition 4 . Moreover the filiform Lie algebras of dimension $n < 7$ have abelian commutator algebra. The case $n = 7$ is done in example 1 .

Let $\mathfrak{g}$ be filiform of dimension 8 . Then the brackets are given by lemma 3 , with eight parameters $\alpha_{k,s}$. The Jacobi identity is equivalent to

$$\alpha_{4,8}(2\alpha_{2,5} + \alpha_{3,7}) = 0.$$

If $2\alpha_{2,5} + \alpha_{3,7}$ is nonzero, $\mathfrak{g}$ admits a standard LSA–structure by setting $\zeta_1 = \zeta_7 = 0,\ \zeta_3 = \alpha_{2,8},\ \zeta_5 = \alpha_{2,6} - 2\alpha_{3,8}$ and $\zeta_6 = \alpha_{2,5}(2\alpha_{2,5} - 5\alpha_{3,7})/(2\alpha_{2,5} + \alpha_{3,7})$.
If $2\alpha_{2,5} + \alpha_{3,7} = 0$, then the standard LSA–structure does not work always. But it is easy to check that we can find a LSA–structure defined by $\lambda(e_1) = \operatorname{ad}(e_1)$ and some strictly lower-triangular matrix $\lambda(e_2)$.

Let $\mathfrak{g}$ be filiform of dimension 9 . Then $\mathfrak{g}$ depends on 9 parameters $\alpha_{k,s}$. The Jacobi identity is equivalent to

$$\alpha_{4,9}(2\alpha_{2,5}+\alpha_{3,7})-3\alpha_{3,7}^2=0.$$

In case $2\alpha_{2,5}+\alpha_{3,7}\neq 0$ $\mathfrak{g}$ admits a standard LSA–structure. Otherwise the Jacobi identity implies $\alpha_{2,5}=\alpha_{3,7}=0$ and there are LSA–structures with $\lambda(e_1)=\operatorname{ad}(e_1)$ and some strictly lower-triangular matrix $\lambda(e_2)$. Again the associated affine represenation has a faithful subrepresentation of dimension n such that $\mu(\mathfrak{g})=n$ for $n<10$.

For the third assertion let $\mathfrak{h}$ and $\mathfrak{g}$ be filiform Lie algebras with $\dim\mathfrak{h}>\dim\mathfrak{g}$ and

$$0\longrightarrow\mathfrak{a}\longrightarrow\mathfrak{h}\longrightarrow\mathfrak{g}\longrightarrow 0$$

be a short exact sequence. We may assume that $\dim\mathfrak{h}=\dim\mathfrak{g}+1=n+1$ and $\mathfrak{h}=\operatorname{span}\{e_1,\ldots,e_{n+1}\}$ with $[e_1,e_i]=e_{i+1}$. Then $\mathfrak{a}\simeq\mathfrak{z}(\mathfrak{h})=\operatorname{span}\{e_{n+1}\}$ and the adjoint representation of $\mathfrak{h}$ restricted to $\mathfrak{g}\simeq\mathfrak{h}/\mathfrak{z}(\mathfrak{h})$ is faithful. This defines a faithful $\mathfrak{g}$ – module of dimension $n+1$. It is obvious that $M:=\operatorname{span}\{e_1,e_3,\ldots,e_{n+1}\}$ is a faithful submodule of dimension n , hence $\mu(\mathfrak{g})=n$. It can be shown that M is isomorphic to a module N such that $Z^1(\mathfrak{g},N)$ possesses a nonsingular 1 – cocycle. Hence we obtain an LSA–structure on $\mathfrak{g}$. $\square$

4. A general bound for nilpotent Lie algebras

In the general case of a nilpotent Lie algebra of nilpotency class k , there is the bound $\mu(\mathfrak{g})<n^k+1$ given in [REE]. This seems to be a very rough bound, in particular for $k=n-1$. One can improve this bound:

Proposition 6. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n and nilpotency class k . Then $\mu(\mathfrak{g})\le\nu(n,k)$ with $k<n$. Here $\nu(n,k):=\sum_{j=0}^{k}\binom{n-j}{k-j}p(j)$ and $p(j)$ is the number of partitions of j .*

Proof: One can construct a faithful representation $\varrho:\mathfrak{g}\to\mathfrak{gl}(V)$, such that $\varrho(X)$ is nilpotent for all $X\in\mathfrak{g}$ as follows, see [COG]:
Let $\mathfrak{g}^{(1)}=\mathfrak{g}$ and $\mathfrak{g}^{(i+1)}=[\mathfrak{g},\mathfrak{g}^{(i)}]$. Since $\mathfrak{g}$ is k -step nilpotent, $\mathfrak{g}^{(k+1)}=0$. Choose a basis $x_1,\ldots,x_n$ of $\mathfrak{g}$ such that the first n_1 elements span $\mathfrak{g}^{(k)}$, the first n_2 elements span $\mathfrak{g}^{(k-1)}$ and so on. We will take V as a quotient of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. By the Poincaré-Birkhoff-Witt Theorem the ordered monomials

$$x^\alpha=x_1^{\alpha_1}\ldots x_n^{\alpha_n},\quad \alpha=(\alpha_1,\ldots\alpha_n)\in\mathbb{Z}_+^n$$

form a basis for $U(\mathfrak{g})$. Let $t=\sum_\alpha c_\alpha x^\alpha$ be an element of $U(\mathfrak{g})$ (with only finitely many nonzero c_α). Define an order function as follows:

$$\begin{aligned}\operatorname{ord}(x_j)&=\max\{m:\ x_j\in\mathfrak{g}^{(m)}\} &\qquad \operatorname{ord}(x^\alpha)&=\textstyle\sum_{j=1}^n\alpha_j\operatorname{ord}(x_j)\\ \operatorname{ord}(t)&=\min\{\operatorname{ord}(x^\alpha)\ :\ c_\alpha\neq 0\} &\qquad \operatorname{ord}(1_{U(\mathfrak{g})})&=0,\ \operatorname{ord}(0)=\infty\end{aligned}$$

Let $U^m(\mathfrak{g}) = \{t \in U(\mathfrak{g}) \ : \ \mathrm{ord}(t) \geq m\}$. One can show that it is an ideal of $U(\mathfrak{g})$ having finite codimension. Define $V = U(\mathfrak{g})/U^m(\mathfrak{g})$. Choose a basis $\{t_1, \dots, t_l\}$ of V such that $t_1, \dots, t_{l_1}$ span $U^{m-1}(\mathfrak{g})/U^m(\mathfrak{g})$, $t_1, \dots, t_{l_2}$ span $U^{m-2}(\mathfrak{g})/U^m(\mathfrak{g})$ and so on. Then it is easy to check that the desired representation of $\mathfrak{g}$ is obtained by setting $\varrho(x)(t_j) = xt_j \pmod{U^m(\mathfrak{g})}$. If $m > k$ then $\varrho(x) \cdot 1_{U(\mathfrak{g})} = x \neq 0$ for all $x \in \mathfrak{g}$, so that ϱ is faithful.

Now we will construct a bound for $\dim V$: Choose m minimal, i.e., $m = k+1$. Let $\mathcal{B} = \{x^\alpha \mid \mathrm{ord}(x^\alpha) \leq k\}$ be a basis for V as above. Then $x_1, \dots, x_{n_1}$ have order k , $x_{n_1+1}, \dots, x_{n_2}$ have order $k-1$ and so on. Hence

$$\#\mathcal{B} = \#\{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n \mid \sum_{j=1}^{k} (k-j+1)(\alpha_{n_{j-1}+1} + \cdots + \alpha_{n_j}) \leq k\}$$

with $n_0 = 0$. On the other hand, $\dim \mathfrak{g}^{(k)} \geq 1$, $\dim \mathfrak{g}^{(k-1)} \geq 2$ and so on.
We can choose the x_i such that $\mathrm{ord}(x_1) = k$, $\mathrm{ord}(x_2) \geq k-1$, $\mathrm{ord}(x_3) \geq k-2, \dots, \mathrm{ord}(x_k) = \cdots = \mathrm{ord}(x_n) \geq 1$. If actually $\mathrm{ord}(x_i) = k+1-i$ for $i = 1, \dots, k$ and $\mathrm{ord}(x_{k+1}) = \cdots = \mathrm{ord}(x_n) = 1$, then $\#\mathcal{B}$ will be maximal, i.e. $\#\mathcal{B} \leq \nu(n,k)$, where

$$\nu(n,k) = \#\{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n \mid (\sum_{j=1}^{k} (k-j+1)\alpha_j) + \alpha_{k+1} + \cdots + \alpha_n \leq k\}.$$

Using the generating function $(1/(1-x))^{r+1} = \sum_{k \geq 0} \binom{r+k}{k} x^k$ for $|x| < 1$ we obtain

$$\#\{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n \mid \sum_{j=1}^{n} \alpha_j \leq k\} = \#\{(\alpha_0, \dots, \alpha_n) \in \mathbb{Z}_+^{n+1} \mid \sum_{j=0}^{n} \alpha_j = k\} = \binom{n+k}{k}.$$

Since $p(k) = \#\{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n \mid k\alpha_1 + (k-1)\alpha_2 + \cdots + \alpha_k = k\}$ we have

$$\nu(n,k) = \sum_{j=0}^{k} \binom{n-j}{k-j} p(j).$$

□

Example 2.

(a) Let $\mathfrak{g} = \mathrm{span}\{x_1, \dots, x_6\}$ with Lie brackets

$$[x_2, x_6] = -x_1,\ [x_3, x_6] = -x_2,\ [x_4, x_5] = -x_1,\ [x_5, x_6] = -x_3.$$

This is a 4–step nilpotent Lie algebra of dimension 6 . We have $\mathrm{ord}(x_1) = 4, \mathrm{ord}(x_2) = 3, \mathrm{ord}(x_3) = 2, \mathrm{ord}(x_i) = 1$ for $i = 4,5,6$. The proposition yields a faithful $\mathfrak{g}$ – module V with $\dim V = \#\mathcal{B} = \nu(6,4) = 51$. Here $n^k + 1 = 1297$.

(b) Let $\mathfrak{g} = < x_1, \ldots, x_6 \mid [x_6, x_i] = x_{i-1},\ i = 2, \ldots, 6 >$. This is a filiform Lie algebra of dimension 6 . We obtain a faithful $\mathfrak{g}$ – module V with $\dim V = \#\mathcal{B} = \nu(6,5) = 45.$ Here $n^k + 1 = 7777$. But in fact, $\mu(\mathfrak{g}) = 6$, see proposition 5 .

To estimate $\nu(n,k)$ we introduce the following notations:

$$f(n) := \frac{\sqrt{3}}{2\pi^2}\exp(\pi\sqrt{2n/3}),\ \ \alpha := \sqrt{2/\pi}F_\infty(\frac{1}{2}) \sim 2.762872,\ \ k_n := [(n+3)/2],$$

$$F_k(q) := \prod_{j=1}^{k}(1-q^j)^{-1} \text{ for } |q| < 1$$

Lemma 5. *The following holds for $\nu(n,k)$:*

(1) $\nu(n+1,k) = \nu(n,k) + \nu(n,k-1)$ *for* $1 < k \le n$
(2) $\nu(n,k) < \binom{n}{k}F_k(\frac{k}{n})$ *for* $1 < k < n$. *One has* $\nu(n,k) \sim \binom{n}{k}F_\infty(\frac{k}{n})$ *if* $k,n \to \infty$ *with* $k/n \le 1-\delta$ *for some fixed* $\delta > 0$.
(3) $\nu(n,k) \le \nu(n,k_n) < \frac{\alpha}{\sqrt{n}}2^n$ *for fixed* $n > 1$ *and all* $1 \le k \le n$.
(4) $\nu(n,n-1) < f(n)$.

Proof: Formula (1) follows by induction using $\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j}$. For (2) , let $p_k(j)$ be the number of those partitions of j in which each term in the partition does not exceed k . Then $\sum_{j=0}^k p(j)q^j < \sum_{j=0}^\infty p_k(j)q^j = \prod_{j=1}^k(1-q^j)^{-1}$ for $|q| < 1$. Using this and $\binom{n-j}{k-j} \le \binom{n}{k}(\frac{k}{n})^j$ we obtain

$$\nu(n,k) = \sum_{j=0}^k \binom{n-j}{k-j}p(j) < \binom{n}{k}\sum_{j=0}^k q^j p(j) < \binom{n}{k}\prod_{j=1}^k (1-q^j)^{-1} = \binom{n}{k}F_k(q)$$

with $q = \frac{k}{n}$. This proves (2) .
One can show that for fixed n , $\nu(n,k)$ becomes maximal for $k = k_n$. Asymptotically $\nu(n,k_n) \sim F_\infty(\frac{1}{2})\binom{n}{k_n}$ and $\binom{n}{k_n} \sim 2^n/\sqrt{\pi n/2}$. Then it is not difficult to see that $\nu(n,k_n) < F_\infty(\frac{1}{2})2^n/\sqrt{\pi n/2} = \frac{\alpha}{\sqrt{n}}2^n$.
There is a convergent series for the partition function (see [RAD]). By estimating the terms we derive $p(n) < f(n+1) - 2f(n) + f(n-1)\ \forall n > 6$. Using this, it follows by induction that $\nu(n,n) < f(n+1) - f(n)\ \forall n$. Here $\nu(n,n) = p(0) + p(1) + \ldots + p(n)$. Then $\nu(n,n-1) < f(n)$ again by induction: For small n , it is true and $\nu(n+1,n) = \nu(n,n) + \nu(n,n-1) < f(n+1) - f(n) + \nu(n,n-1) < f(n+1)$. This proves (4) . □

The lemma shows that the bound $\dim V \le \nu(n,k)$ for $\mu(\mathfrak{g})$ is much better than n^k+1 , especially if k is not small with respect to n . However, the real size of $\mu(\mathfrak{g})$ might be much smaller than $\nu(n,k)$. Note that $k = 1$ corresponds to the abelian case. By part (3) of the lemma we know that we may bound $\mu(\mathfrak{g})$ independently of k as follows:

Corollary 1. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra of dimension* n *. Then*

$$\mu(\mathfrak{g}) < \frac{\alpha}{\sqrt{n}} 2^n.$$

For $n = k - 1$ we can improve proposition 6 :

Proposition 7. *Let* $\mathfrak{g}$ *be a filiform nilpotent Lie algebra of dimension* n. *Then*

$$\mu(\mathfrak{g}) < 1 + \sum_{j=0}^{n-2} p(j) < 1 + f(n-1) - f(n-2).$$

Proof: Using the construction of proposition 6 with $x_1 = e_n, x_2 = e_{n-1}, \dots, x_n = e_1$ we obtain a faithful module V with basis $\mathcal{B} = \{e_n^{\alpha_n} \cdots e_1^{\alpha_1} \mid \sum_{j=2}^n (j-1)a_j + \alpha_1 \leq n-1\}$ for $\mathfrak{g} =< e_1, \dots, e_n >$ and $\dim V = \nu(n, n-1)$. Here $\operatorname{ord}(e_i) = i - 1$, $i = 2, \dots, n$ and $\operatorname{ord}(e_1) = 1$. The elements e_i of $\mathfrak{g}$ act on V by $e_i e_j = [e_i, e_j] + e_j e_i$ for $i < j$ and $e_j e_i$ is element of V for $j \geq i$. Let U be the submodule of V generated by e_1 . U has a basis of all monomials $e_n^{\alpha_n} \cdots e_1^{\alpha_1}$ with $\alpha_1 \neq 0$, hence $\dim U = \nu(n-1, n-2)$. The factor module V/U is a faithful $\mathfrak{g}$ – module of dimension $\nu(n, n-1) - \nu(n-1, n-2) = \nu(n-1, n-1)$. Its basis $\tilde{\mathcal{B}}$ contains the monomials $e_n^{\alpha_n} \cdots e_2^{\alpha_2}$ of maximal order, i.e., with $\sum_{j=2}^n (j-1)a_j = n-1$. These are $p(n-1)$ monomials. We may omit these monomials from $\tilde{\mathcal{B}}$, except for e_n in order to preserve faithfulness. Then we obtain a faithful module of dimension $\nu(n-1, n-1) - p(n-1) + 1 = 1 + \sum_{j=0}^{n-2} p(j)$. This equals $1 + \nu(n-2, n-2)$ which can be bounded by $1 + f(n-1) - f(n-2)$, see lemma 5 (4). □

References

[BRA] F. BRATZLAVSKY, Classification des algébres de Lie de dimension n, de classe n–1, dont l'idéal dérivé est commutativ, *Bull. Cl. Sci. Bruxelles* **60** (1974), 858-865.

[BAH] J. A. BAHTURIN, *Identities in Lie algebras*, VNU Science Press, Utrecht (1985).

[COG] L. J. CORWIN, F. P. GREENLEAF, *Representations of nilpotent Lie groups and their applications*, Cambridge university press, Cambridge (1990).

[BU1] D. BURDE, Left-invariant affine structures on reductive Lie groups, *J. of Algebra* **181** (1996), 884-902.

[BU2] D. BURDE, Affine structures on nilmanifolds, *J. Intern. Math.*, Vol. **7 (5)** (1996), 599-616.

[BU3] D. BURDE, Affine structures on filiform Lie algebras, *Preprint* (1997).

[RAD] H. RADEMACHER, Fourier expansions of modular forms and problems of partition, *Bull. Amer. Math. Soc.* **46** (1940), 59-73.

[REE] B. E. REED, Representations of solvable Lie algebras, *Mich. Math. J.* **16** (1969), 227-233.

[WEH] B. A. WEHRFRITZ, Faithful linear representations of certain free nilpotent groups, *Glasgow Math. J.* **37** (1995), 33-36.