Etale affine representations of Lie groups

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1 Introduction

Let G be a finite–dimensional connected Lie group with Lie algebra $\mathfrak{g}$. Denote by E a real vector space and by $\mathbf{Aff}(E)$ the group of affine automorphisms,

$$\mathbf{Aff}(E) = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in \mathbf{GL}(E),\ b \in E \right\}$$

Let $\mathfrak{aff}(E)$ be the Lie algebra of $\mathbf{Aff}(E)$. An affine representation $\alpha : G \to \mathbf{Aff}(E)$ of G is called *étale*, if there exists a $v \in E$ whose stabilizer G_v is discrete in G, and whose $G-$ orbit $G \cdot v$ is open in E. Its differential $\varrho : \mathfrak{g} \to \mathfrak{aff}(E)$ is a Lie algebra homomorphism such that the *evaluation map* $\mathrm{ev}_p : \mathfrak{g} \to E$, $x \mapsto \varrho(x)p = \theta(x)p + u(x)$ is an isomorphism for some $p \in E$, where $\theta : \mathfrak{g} \to \mathfrak{gl}(E)$ is a linear representation and u is the translational part of ϱ. Such a Lie algebra representation is called *étale* again. In that case it follows $\dim E = \dim G$. We are interested in the following question:

(1) *Which Lie groups admit étale affine representations ?*

Etale affine representations of a Lie group arise in the theory of affine manifolds and affine crystallographic groups, see [MIL]. Here the most difficult case is when G is nilpotent. If G is *reductive*, étale affine representations can be studied by methods of invariant theory of affine algebraic varieties, see [BAU], [BU2]. The following has been proved: A semisimple Lie group G does not admit *any* étale affine representation. If G is reductive such that its Lie algebra $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$ has $1-$ dimensional center $\mathfrak{z}$ and $\mathfrak{s}$ is simple, then G admits étale affine representations iff $\mathfrak{s}$ is of type A_ℓ, i.e., if G is $\mathbf{GL}(n)$. For $\mathbf{GL}(n)$ all such representations can be classified, see [BU2].
There is a canonical one-to-one correspondence between étale affine representations of G (up to conjugacy in $\mathbf{Aff}(G)$) and left-invariant affine structures on G (up to affine equivalence), see Definition 1. Given such a structure on G we can construct many examples of affine manifolds. If G has a left-invariant affine structure and Γ is a discrete subgroup of G, then the homogeneous space $\Gamma\backslash G$ of right cosets inherits an affine structure. If G is nilpotent, then $\Gamma\backslash G$ is called an *affine nilmanifold*. Any compact complete affine manifold

with nilpotent fundamental group already is an affine nilmanifold ([FGH]).
Left-invariant affine structures also play an important role in the study of affine crystallographic groups (in short *ACGs*), and of fundamental groups of affine manifolds, see [MIL]. A group $\Gamma \leq \mathbf{Aff}(E)$ is called ACG if it acts properly discontinuously on E with compact quotient. There is the following well-known conjecture by Auslander: *An ACG is virtually polycyclic.* This may be restated as follows: The fundamental group of a compact complete affine manifold is virtually polycyclic. The conjecture is still open, though Abels, Margulis and Soifer recently made some progress proving the conjecture up to dimension 6 (see [AMS]).
Milnor proved that a finitely generated torsionfree virtually polycyclic group Γ can be realized as a subgroup of $\mathbf{Aff}(E)$ acting properly discontinuously. Hence it is the fundamental group of a complete affine manifold. Auslander's conjecture is equivalent to the following:
A compact complete affine manifold is finitely covered by quotients of solvable Lie groups with complete left-invariant affine structures.

Milnor asked in this context ([MIL]):

(2) *Which Lie groups admit left-invariant affine structures ?*

Of course, this is equivalent to our question (1). As said before, this question is particularly difficult for nilpotent Lie groups. There was much evidence that *every* nilpotent Lie group admits left-invariant affine structures. Milnor conjectured this to be true even for solvable Lie groups ([MIL]). Recently, however, counterexamples were discovered ([BGR] and [BEN]). There are nilmanifolds which are not affine. The key step here is to find $n-$dimensional nilpotent Lie algebras having no faithful representations in dimension $n+1$, hence no affine representation which could arise from a left-invariant affine structure on the Lie group G. We will present some new examples here. They are, however, no counterexamples for the Auslander conjecture.
Left-invariant affine structures on G also correspond to *left-symmetric* algebra structures on $\mathfrak{g}$ (in short, LSA-structures, see Definition 2). Given a Lie algebra $\mathfrak{g}$ over a field of *arbitrary* characteristic, the question of existence of LSA-structures on $\mathfrak{g}$ makes sense and leads to interesting structures. In case $\mathfrak{g}$ is a classical simple Lie algebra over a field k of prime characteristic, LSA-structures on $\mathfrak{g}$ are closely related to the first cohomology groups $H^1(G_1, L(\lambda))$, where G_1 is the first Frobenius kernel of a simple algebraic group G with $\mathrm{Lie}(G) = \mathfrak{g}$ and $L(\lambda)$ is a highest weight module of dimension less or equal than $\dim G$. We have the following result (see [JAN], [BU1]):

Let G be a connected semisimple algebraic group of type $A_l\ (l \geq 1), B_l\ (l \geq 3), C_l (l \geq 2), D_l\ (l \geq 4), G_2, F_4, E_6, E_7, E_8$ over an algebraically closed field k of characteristic $p > 2$. Let $X_1(T)$ denote the set of restricted dominant *weights and let* $\mathfrak{g} = \mathrm{Lie}(G)$ *. Assume that*

(1) $p > 3$ *,* *if* G *is of type* G_2, F_4, E_6, A_1
(2) $p \not| \; l+1,$ *if* G *is of type* A_l
(3) $p \not| \; l$ *,* *if* G *is of type* C_l

Then $H^1(G_1, L(\lambda)) = 0$ *for all* $\lambda \in X_1(T)$ *with* $\dim L(\lambda) \leq \dim G$. *Furthermore, if* $\mathfrak{g}$ *admits an LSA–structure, then* $p \mid \dim \mathfrak{g}$.

It is not known in general whether $p \mid \dim \mathfrak{g}$ implies the existence of LSA-structures on such Lie algebras. However, it is true for $\mathfrak{sl}(2,k)$ and $\mathfrak{sl}(3,k)$. In the case of $\mathfrak{sl}(2,k)$, all LSA-structures have been classified ([BU1]). Note that it follows from the proof of the above result that semisimple Lie algebras over characteristic zero do not admit LSA–structures. Hence semisimple Lie groups do not admit étale affine representations.

2 Preliminaries

We consider *affine structures* on a connected Lie group G . Therefore we recall the following definition (see [MIL]):

Definition 1.

Let M denote an n -dimensional manifold. An *affine atlas* on M is a covering of M by coordinate charts such that each coordinate change between overlapping charts is *locally affine*, i.e., extends to an affine automorphism $x \mapsto Ax + b$, $A \in \mathbf{GL}_n(\mathbb{R})$, of some n -dimensional real vector space E. A maximal affine atlas is an *affine structure* on M , and M together with an affine structure is called an *affine manifold*.

Affine manifolds are *flat* – there is a natural correspondence between affine structures on M and *flat torsionfree affine connections* ∇ on M . Such an affine connection is a connection in the tangent bundle with zero torsion and zero curvature.
Subclasses of affine manifolds are *Riemannian-flat* and *Lorentz-flat* manifolds. Note that a manifold does not always admit an affine structure: A closed surface admits affine structures if and only if its Euler characteristic vanishes, i.e., if it is a torus. For higher dimensions ($n \geq 3$) it is in general difficult to decide whether the manifold admits affine structures or not (see [SMI] for more information).
Many examples of affine manifolds come from *left-invariant affine structures on Lie groups*: For a Lie group G , an affine structure on G is *left-invariant*, if for each $g \in G$ the left-multiplication by g , $L_g : G \to G$, is an automorphism of the affine structure. For G simply connected let $D : G \to E$ be the

developing map. Then there is for each $g \in G$ a unique affine automorphism $\alpha(g)$ of E, such that $\alpha(g) \circ D = D \circ L_g$. In that case $\alpha : G \to \mathbf{Aff}(E)$ is an affine representation.
It is not difficult to see ([FGH]) that G admits a complete left-invariant structure if and only if G acts *simply transitively* on E as affine transformations. By a result of Auslander, G then must be solvable ([AUS]).

Definition 2.

A *left-symmetric algebra structure* (or *LSA–structure* in short) *on* $\mathfrak{g}$ *over a field* k is a k–bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto x \cdot y$ satisfying the conditions $x \cdot y - y \cdot x = [x, y]$ and $(x, y, z) = (y, x, z)$ for all x, y, z, where $(x, y, z) = x \cdot (y \cdot z) - (x \cdot y) \cdot z$ denotes the associator of $x, y, z \in \mathfrak{g}$.

The main definitions given so far are quite related:

Lemma 1. *There is a canonical one-to-one correspondence between the following classes of objects (up to suitable equivalence):*

(a) $\{$*Etale affine representations of* $G\}$
(b) $\{$*Left-invariant affine structures on* $G\}$
(c) $\{$*Flat torsionfree left-invariant affine connections* ∇ *on* $G\}$
(d) $\{$*LSA–structures on* $\mathfrak{g}\}$

Proof. This is well known, see [BU3],[SEG],[KIM]. We will give some arguments in order to establish notations.
If we have any LSA–structure on $\mathfrak{g}$ with product $(x, y) \mapsto x{\cdot}y$, then denote by $\lambda : x \mapsto \lambda(x)$ the left-regular representation on the LSA $(\mathfrak{g}, \cdot) :$ $\lambda(x)y = x{\cdot}y$. It is a Lie algebra representation: $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $[\lambda(x), \lambda(y)] = \lambda([x, y])$. Denote the corresponding $\mathfrak{g}$-module by $\mathfrak{g}_\lambda$. Furthermore, the identity map $\mathbf{1} : \mathfrak{g} \to \mathfrak{g}_\lambda$ is a 1-cocycle in $Z^1(\mathfrak{g}, \mathfrak{g}_\lambda) :$ $\mathbf{1}([x, y]) = \mathbf{1}(x) \cdot y - \mathbf{1}(y) \cdot x$. Let $\mathfrak{aff}(\mathfrak{g})$ be the Lie algebra of $\mathbf{Aff}(G)$, i.e.,

$$\mathfrak{aff}(\mathfrak{g}) = \left\{ \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{gl}(\mathfrak{g}),\ b \in \mathfrak{g} \right\}$$

which we identify with $\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}$. Denote the linear part by $\ell(A, b) = A$ and the translational part by $t(A, b) = b$. Now we associate to the LSA $(\mathfrak{g}, \cdot)$ the map $\alpha = \lambda \oplus \mathbf{1} :$ $\mathfrak{g} \to \mathfrak{aff}(\mathfrak{g})$. This is an affine representation of $\mathfrak{g}$. We have

$\lambda = \ell \circ \alpha$ and $t \circ \alpha = \mathbf{1}$. The corresponding affine representation of G is étale, see [SEG].

3 Affine representations of reductive Lie groups

Let k be an algebraically closed field of characteristic zero. A Lie algebra $\mathfrak{g}$ is said to be *reductive* if its solvable radical $\mathfrak{r}(\mathfrak{g})$ coincides with the center $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$. Then the Lie algebra $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ is semisimple and we have $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$. A Lie group G is said to be reductive if its Lie algebra is reductive. Assume that $(\mathfrak{g}, \cdot)$ is an LSA-structure on $\mathfrak{g}$. The first cohomology groups of a reductive Lie algebra do not vanish in general. However, if the center is one–dimensional and the $\mathfrak{g}$ – module is $\mathfrak{g}_\lambda$ arising from an étale affine representation of G , then we are able to prove (see [BU2]):

Proposition 1. *Let* $(\mathfrak{g}, \cdot)$ *be an LSA-structure on* $\mathfrak{g}$ *. If* $\dim \mathfrak{z} = 1$ *then* $H^0(\mathfrak{g}, \mathfrak{g}_\lambda) = 0$ *and* $H^1(\mathfrak{g}, \mathfrak{g}_\lambda) = 0$.

Proposition 2. *Let* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$ *be a reductive Lie algebra such that* $\dim \mathfrak{z} = 1$ *and* $\mathfrak{s}$ *is of type* $A_\ell,\ B_\ell,\ C_\ell,\ D_\ell,\ G_2,\ F_4,\ E_6,\ E_7,\ E_8$*. Then* $\mathfrak{g}$ *admits an LSA-structure if and only if* $\mathfrak{s}$ *is of type* A_ℓ .

Here is a brief outline of the proof to Proposition 2 : Let $\dim \mathfrak{s} = n$. The $\mathfrak{g}$ – module $\mathfrak{g}_\lambda$ is completely reducible as an $\mathfrak{s}$ -module and has *no invariants* by Proposition 1 , i.e., the trivial module k is not a summand in the decomposition of $\mathfrak{g}_\lambda$. Hence we know that $\mathfrak{g}_\lambda = \oplus_i V_i$ and $\sum_i \dim V_i = n+1$, where V_i are irreducible $\mathfrak{s}$ -modules with $2 \le \dim V_i \le \dim \mathfrak{g} = n+1$. On the other hand, there are not many irreducible $\mathfrak{s}$ -modules of dimension smaller or equal to $n+1$. It is possible to classify them. For a given type of $\mathfrak{s}$ the dimensions of these modules have to add up to $\dim \mathfrak{g}_\lambda$. However, in most cases this is possible only if $\mathfrak{s}$ is of type A_l . This argument only fails in case of type $B_3,\ D_5,\ D_7$, where the modules are

$\mathfrak{g}_\lambda = L(\omega_1) \oplus L(\omega_1) \oplus L(\omega_3)$ for B_3 ,
$\mathfrak{g}_\lambda = L(\omega_1) \oplus L(\omega_1) \oplus L(\omega_1) \oplus L(\omega_5)$ for D_5 ,
$\mathfrak{g}_\lambda = L(\omega_1) \oplus L(\omega_1) \oplus L(\omega_7)$ for D_7.

Here $\omega_1, \dots, \omega_\ell$ denote the fundamental weights and $L(\omega_i)$ the heighest weight module to ω_i . The dimensions satisfy $22 = 7+7+8$, $46 = 10+10+10+16$ and $92 = 14+14+64$ respectively.
To prove the result in these cases, we use invariant theory: Let $\varrho : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g})$ be an étale affine representation arising from an LSA–structure. Let S be the simply connected semisimple algebraic group with Lie algebra $\mathfrak{s}$. The linear part of ϱ is the differential of a rational representation $\rho : S \to \mathbf{Aff}(V)$.

Thus we may regard V as an algebraic S – variety. If the center of $\mathfrak{g}$ is one–dimensional, we know that V is isomorphic to a *linear* S – variety. Since ϱ is étale, we have $\dim V = \dim S + 1$ and V has an S – orbit of codimension 1. However, it is easy to see that the above modules (where S is an orthogonal group) do *not* have an S – orbit of codimension 1.

If the center of $\mathfrak{g}$ is higher-dimensional, then the situation becomes more complicated (see [HEL], [BU2]).

As mentioned before, in case of $\mathbf{GL}(n)$ we can classify all étale affine representations, i.e., all LSA-structures on $\mathfrak{gl}(n)$.
Let $\mathcal{A} = (\mathfrak{g}, \cdot)$ be an LSA–structure on $\mathfrak{g}$. Denote by $\mathrm{End}_*(\mathfrak{g})$ the set $\{\tau \in \mathrm{End}(\mathfrak{g}) \mid (\mathbf{1} - \tau)^{-1}$ exists and $\tau(\mathcal{A}) \subset k(\mathcal{A})\}$ where

$$k(\mathcal{A}) := \{a \in \mathcal{A} \mid [\lambda(b), \varrho(a)] = 0 \quad \forall\, b \in \mathcal{A}\}.$$

Here λ and ϱ denote left and right multiplication in $\mathcal{A}$. Let $\tau \in \mathrm{End}_*(\mathfrak{g})$ with $\phi = (\mathbf{1} - \tau)^{-1}$.
Then $\lambda_\tau(a) := \phi \circ (\lambda(a) - \varrho(\tau(a))) \circ \phi^{-1}$ defines an LSA-structure on $\mathfrak{g}$. We call $\mathcal{A}_\tau$ the τ *-deformation* of $\mathcal{A}$. The result is ([BAU],[BU2]):

Proposition 3. *The τ -deformations of the full matrix algebra exhaust all possible LSA-structures on $\mathfrak{gl}_n(k)$ for $n > 2$. Their isomorphism classes are parametrized by the conjugacy classes of elements $X \in \mathfrak{gl}_n(k)$ with $\mathrm{tr}\,(X) = n$. In case of $\mathfrak{gl}(2,k)$ we have one more isomorphism class.*

4 Affine representations of nilpotent Lie groups

Milnor conjectured in [MIL] that every nilpotent Lie group G admits étale affine representations, i.e., its Lie algebra $\mathfrak{g}$ admits LSA-structures. Indeed, many classes of nilpotent Lie algebras do admit LSA-structures (see [BU3]):

Proposition 4. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of characteristic zero satisfying one of the following conditions:*

(1) $\dim \mathfrak{g} < 8$.
(2) $\mathfrak{g}$ *is p – step nilpotent with $p < 4$.*
(3) $\mathfrak{g}$ *is $\mathbb{Z}$ – graded.*
(4) $\mathfrak{g}$ *possesses a nonsingular derivation.*
(5) $\mathfrak{g}$ *is filiform nilpotent and a quotient of a higher–dimensional filiform nilpotent Lie algebra.*
(6) $\mathfrak{g}$ *possesses a nonsingular 1 – cocycle in $Z^1(\mathfrak{g}, \mathfrak{g}_\theta)$, where $\theta : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a representation.*

Then $\mathfrak{g}$ admits an LSA–structure.

However, there are nilpotent Lie algebras without any LSA-structure. To construct such examples we use

Lemma 2. *If* $\mathfrak{g}$ *admits an LSA–structure then* $\mathfrak{g}$ *has a faithful representation of dimension* $\dim \mathfrak{g} + 1$.

Proof. The LSA–structure on $\mathfrak{g}$ induces a faithful affine representation $\alpha : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g})$, called the *affine holonomy representation.* If $\dim \mathfrak{g} = n$ then $\mathfrak{aff}(\mathfrak{g}) \subset \mathfrak{gl}(n+1)$ and we obtain a faithful linear representation of dimension $n + 1$.

Definition 3.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k . Define

$$\mu(\mathfrak{g}, k) := \min \{\dim_k M \mid M \text{ is a faithful } \mathfrak{g}\text{–module}\}$$

By Ado's Theorem (and Iwasawa's in prime characteristic) we know that μ is integer valued. It seems that there is not much known about μ in the literature. We list a few properties proved in [BU4]:

Proposition 5. *Let* $\mathfrak{g}$ *be a Lie algebra of dimension* $n \geq 2$ *over* $\mathbb{C}$.

(1) *If* $\mathfrak{g}$ *is abelian then* $\mu(\mathfrak{g}) = \lceil 2\sqrt{n-1} \rceil$.
(2) *If* $\mathfrak{g}$ *has trivial center then* $\mu(\mathfrak{g}) \leq n$.
(3) *If* $\mathfrak{g}$ *is a Heisenberg Lie algebra* $\mathfrak{h}_{2m+1}$ *of dimension* $2m + 1$, *then* $\mu(\mathfrak{g}) = m + 2$.
(4) *If* $\mathfrak{g}$ *is solvable then* $\mu(\mathfrak{g}) < 2^n$.
(5) *If* $\mathfrak{g}$ *is filiform nilpotent with abelian commutator algebra then* $\mu(\mathfrak{g}) = n$.
(6) *If* $\mathfrak{g}$ *is filiform nilpotent then* $n \leq \mu(\mathfrak{g}) < (\sqrt{3}/12) \exp(\pi \sqrt{2n/3})$.
(7) *If* $\mathfrak{g}$ *admits an LSA–structure then* $\mu(\mathfrak{g}) \leq n + 1$.
(8) *If* $\mathfrak{g}$ *is a quotient of a filiform nilpotent Lie algebra* $\mathfrak{g}'$ *with* $\dim \mathfrak{g}' > \dim \mathfrak{g} = n$ *then* $\mu(\mathfrak{g}) = n$.
(9) *If* $\mathfrak{g}$ *is filiform nilpotent of dimension* $n < 10$ *then* $\mu(\mathfrak{g}) = n$.

The key step for the construction of the counterexamples to the Milnor conjecture is to determine Lie algebras with $\mu(\mathfrak{g}) > \dim \mathfrak{g} + 1$. In the following we will construct filiform Lie algebras in dimensions $10, 11$ with that property. These algebras have no extension by any filiform Lie algebra of higher dimension.
Let $\mathfrak{g}$ be a p–step nilpotent Lie algebra and let $\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$. The series $\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \ldots \supset \mathfrak{g}^{p-1} \supset \mathfrak{g}^p = 0$ is called *lower central series.* Recall that a p–step nilpotent Lie algebra of dimension n is called *filiform nilpotent* if $p = n - 1$.

Definition 4.

Let $L = L(n)$ be the Lie algebra generated by $e_0, \dots, e_n$ with Lie brackets $[e_0, e_i] = e_{i+1}$ for $i = 1, 2, \dots, n-1$ and the other brackets zero. L is called the *standard graded filiform of dimension* $n+1$.

Consider the affine algebraic variety of all Lie algebra structures in dimension n over $\mathbb{C}$. In particular, we have the subvariety of nilpotent filiform Lie algebra structures. The following result is due to Vergne ([VER]):

Proposition 6. *Every filiform nilpotent Lie algebra of dimension* $n+1 \geq 8$ *is isomorphic to an infinitesimal deformation of the standard graded* $(n+1)$ – *dimensional filiform* L . *More precisely it is isomorphic to an algebra* $(L)_\psi$ *where* ψ *is an integrable* 2 –*cocycle whose cohomology class lies in*

$$\begin{array}{ll} F_1H^2(L,L) & \text{if} \quad n \equiv 0(2) \\ F_1H^2(L,L) \; + \; < \psi_{\frac{n-1}{2},n} > & \text{if} \quad n \equiv 1(2) \end{array}$$

Here the algebra $\mathfrak{g}_\psi = (L)_\psi$ is defined by the bracket $[a,b]_\psi = [a,b]_L + \psi(a,b)$. The fact that ψ is integrable means that this bracket satisfies the Jacobi identity, i.e., $\psi(a,\psi(b,c)) + \psi(b,\psi(c,a)) + \psi(c,\psi(a,b)) = 0$. For the definition of $F_1H^2(L,L)$ see [HAK]. Here we determine a canonical basis for this space (see [BU3]):

Proposition 7. *Define canonical* 2 – *cocycles* $\psi_{k,s}$ *by* $\psi_{k,s}(e_i, e_{i+1}) = \delta_{ik} e_s$ *for pairs* (k,s) *with* $1 \leq k \leq n-1$ *and* $2k \leq s \leq n$. *The cohomology classes of the cocycles* $\psi_{k,s}$ *with* $1 \leq k \leq [n/2]-1,\ 2k+2 \leq s \leq n$ *form a basis of* $F_1H^2(L,L)$. *This space has dimension* $\frac{(n-2)^2}{4}$ *if* n *is even, and dimension* $\frac{(n-3)(n-1)}{4}$ *if* n *is odd. The following formula holds:*
$\psi_{k,s}(e_i,e_j) = (-1)^k \binom{j-k-1}{k-i} (\mathrm{ad} e_0)^{i+j-2k-1} e_s$ *for* $1 \leq i < k < j-1 \leq n-1$.
In case $i > k$, $\psi_{k,s}(e_i,e_j) = 0$ *and* $\psi_{k,s}(e_k,e_j) = e_{s+j-k-1}$ *for* $k < j$.

4.1 Filiform Lie algebras in dimension 10

Let $L = L(9) =< e_0, e_1, \dots, e_9 >$ be the standard graded filiform Lie algebra of dimension 10. According to Proposition 6 every filiform nilpotent Lie algebra of dimension 10 is isomorphic to $\mathfrak{g}_\psi = (L)_\psi$ for some $\psi \in F_1H^2(L,L) + < \psi_{4,9} >$. In terms of the basis of this cohomology space we may write

$$\begin{aligned} \psi \;=\; & \alpha_1\psi_{1,4} + \alpha_2\psi_{1,5} + \ldots + \alpha_6\psi_{1,9} \\ & + \alpha_7\psi_{2,6} + \ldots + \alpha_{10}\psi_{2,9} \\ & + \alpha_{11}\psi_{3,8} + \alpha_{12}\psi_{3,9} \\ & + \alpha_{13}\psi_{4,9} \end{aligned}$$

The cocycle ψ is integrable if and only if $[a,b]_\psi = [a,b]_L + \psi(a,b)$ satisfies the Jacobi identity. This is equivalent to the following equations:

$$\alpha_{13}(2\alpha_3+\alpha_9)-\alpha_{12}(2\alpha_1+\alpha_7)-3\alpha_{11}(\alpha_2+\alpha_8)+7\alpha_7\alpha_8 = 0 \quad (1)$$

$$\alpha_{11}(2\alpha_1+\alpha_7)-3\alpha_7^2 = 0 \quad (2)$$

$$\alpha_{13}(2\alpha_1-\alpha_7-\alpha_{11}) = 0 \quad (3)$$

Using these simple conditions we obtain the following classes of filiform Lie algebras $\mathfrak{g}_\psi$ with bracket $[a,b]_\psi$:

Case A: $2\alpha_1+\alpha_7 \neq 0$:

Class $(A1)$: $\alpha_1 \neq 0,\ \alpha_7 = -\alpha_1,\ \alpha_{11} = 3\alpha_1$.
Class $(A2)$: $\alpha_1 \neq 0,\ \alpha_{11} = \alpha_7 = \alpha_1$.
Class $(A3)$: $\alpha_1 \neq 0,\ \alpha_7^2 \neq \alpha_1^2,\ \alpha_{11} = 3\alpha_7^2/(2\alpha_1+\alpha_7)$.

Case B: $2\alpha_1+\alpha_7 = 0$:

Class $(B1)$: $\alpha_{13} = \alpha_7 = \alpha_1 = 0,\ \alpha_{11}(\alpha_2+\alpha_8) = 0$.
Class $(B2)$: $\alpha_{13} \neq 0,\ \alpha_{11} = \alpha_7 = \alpha_1 = 0,\ \alpha_9 = -2\alpha_3$.

In case A, α_{12} is uniquely determined by equation (1) . We want to know the minimal dimension of faithful modules for these classes of Lie algebras. The result is:

Proposition 7. *If* $\mathfrak{g}_\psi$ *is a filiform Lie algebra of class* $A3, B1, B2$ *then* $\mu(\mathfrak{g}_\psi) = 10$ *; if* $\mathfrak{g}_\psi$ *is of class* $A1$ *satisfying the additional condition* $3\alpha_2 + \alpha_8 = 0$ *, or is of class* $A2$ *, then* $\mu(\mathfrak{g}_\psi) = 10$ *or* 11.

The class excluded above indeed provides counterexamples to Milnor's conjecture:

Proposition 8. *Let* $\mathfrak{g}_\psi = \mathfrak{g}(\alpha_1, \dots, \alpha_{13})$ *be a Lie algebra of class* $A1$ *, satisfying* $3\alpha_2+\alpha_8 \neq 0$. *Then* $12 \leq \mu(\mathfrak{g}_\psi) \leq 22$.

The proof is given in [BU3]. The rough idea is as follows: Let $\mathfrak{g}_\psi$ be a filiform nilpotent Lie algebra of dimension 10 . Suppose there is *any* faithful module M of dimension $m < 12$. By Lemma 3.2. in [BEN] we may assume that M is nilpotent and is of dimension 11. For such modules we construct a *combinatorical type*, thereby classifying such modules. Note that the faithfulness is a strong condition which excludes many types of modules. For each type we check the conditions for M to be a faithful nilpotent module of dimension $m < 12$. This means certain equations in the α_i . The crucial equation is $3\alpha_2+\alpha_8 = 0$. On the other hand, we construct a faithful module of dimension 22 for all filiform Lie algebras of dimension 10 .

Remark 1. Let G be the connected simply connected Lie group with filiform nilpotent Lie algebra as in Proposition 8 . Then G does not admit an étale affine representation. There is the question whether the Lie groups corresponding to the other classes (see Proposition 7) *do* admit such representations. We have not checked this in general. However for class $A3$ the answer is positive.

4.2 Filiform Lie algebras in dimension 11

Let $L = L(10) =< e_0, e_1, \dots, e_{10} >$ be the standard graded filiform Lie algebra of dimension 11. Then every filiform nilpotent Lie algebra of dimension 11 is isomorphic to $\mathfrak{g}_\psi = (L)_\psi$ for some $\psi \in F_1 H^2(L, L)$. In terms of the basis of this cohomology space we may write

$$\begin{aligned} \psi \;=\; & \alpha_1\psi_{1,4} + \alpha_2\psi_{1,5} + \ldots + \alpha_7\psi_{1,10} \\ & + \alpha_8\psi_{2,6} + \alpha_9\psi_{2,7} \ldots + \alpha_{12}\psi_{2,10} \\ & + \alpha_{13}\psi_{3,8} + \ldots + \alpha_{15}\psi_{3,10} \\ & + \alpha_{16}\psi_{4,10} \end{aligned}$$

The integrability of ψ is determined by four equations. We are interested here in the case $\alpha_1 \neq 0$. We have the following result, using the same methods as above (see also [BGR]):

Proposition 9. *Let* $\mathfrak{g}_\psi$ *be a filiform nilpotent Lie algebra of dimension* 11 *satisfying* $\alpha_1 \neq 0$ *. Then* $\mu(\mathfrak{g}_\psi) \leq 12$ *if and only if* $\alpha_8 = 0$ *or* $10\alpha_8 = \alpha_1$ *or* $5\alpha_8^2 = 2\alpha_1^2$ *or* $4\alpha_1^2 - 4\alpha_1\alpha_8 + 3\alpha_8^2 = 0$.

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