Left-invariant affine structures on reductive Lie groups

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We describe left-invariant affine structures (that is, left-invariant flat torsion-free affine connections ∇) on reductive linear Lie groups G. They correspond bijectively to LSA-structures on the Lie algebra $\mathfrak{g}$ of G. Here LSA stands for left-symmetric algebra, see [BUR], [SE2]. If $\mathfrak{g}$ has trivial or one- dimensional center $\mathfrak{z}$ then the affine representation $\alpha = \lambda \oplus 1$ of $\mathfrak{g}$, induced by any LSA-structure $\mathfrak{g}_\lambda$ on $\mathfrak{g}$ is *radiant*, i.e., the radiance obstruction $c_\alpha \in H^1(\mathfrak{g}, \mathfrak{g}_\lambda)$ vanishes. If $\dim \mathfrak{z} = 1$ we prove that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$, where $\mathfrak{s}$ is split simple, admits LSA-structures if and only if $\mathfrak{s}$ is of type A_ℓ, that is $\mathfrak{g} = \mathfrak{gl}_n$. Here we have the associative LSA-structure given by ordinary matrix multiplication corresponding to the bi-invariant affine structure on $GL(n)$, which was believed to be essentially the only possible LSA-structure on $\mathfrak{gl}_n$. We exhibit interesting LSA-structures different from the associative one. They arise as certain deformations of the matrix algebra. Then we classify all LSA-structures on $\mathfrak{gl}_n$ using a result of [BAU]. For $n = 2$ we compute all structures explicitely over the complex numbers.

1 Introduction

Let M denote an n-dimensional manifold (connected and without boundary). An *affine atlas* Φ on M is a covering of M by coordinate charts such that each coordinate change between overlapping charts in Φ is *locally affine*, i.e., extends to an affine automorphism $x \mapsto Ax + b$, $A \in \mathbf{GL}_n(\mathbb{R})$, of some n-dimensional real vector space E. A maximal affine atlas is an *affine structure* on M, and M together with an affine structure is called an *affine manifold*. An affine structure determines a differentiable structure and affine manifolds are *flat* – there is a natural correspondence between affine structures on M and *flat torsionfree affine connections* ∇ on M. Such an affine connection is a connection in the tangent bundle with zero torsion and zero curvature:

$$
\begin{aligned}
&(1) \qquad & T_{X,Y} &= \nabla_X Y - \nabla_Y X - [X,Y] &= 0 \\
&(2) \qquad & R_{X,Y} &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} &= 0
\end{aligned}
$$

Subclasses of affine manifolds are *Riemannian-flat* and *Lorentz-flat* manifolds. A fundamental problem is the question of existence of affine structures. A closed surface admits affine structures if and only if its Euler characteristic vanishes ([BEZ] and [MI1]). In higher dimensions there are only certain obstructions known ([SMI]).

Denote by $\mathbf{Aff}(E)$ the group of affine automorphisms,

$$\mathbf{Aff}(E) = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in \mathbf{GL}(E),\ b \in E \right\}$$

where the affine action is given by $\left(\begin{smallmatrix} A & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} Ax+b \\ 1 \end{smallmatrix}\right)$.

Let M be an affine manifold. Its universal covering $\widetilde{M}$ inherits a unique affine structure for which the covering projection $\widetilde{M} \to M$ is an affine immersion. The group π of deck-transformations acts on $\widetilde{M}$ by affine automorphisms. There exists an affine immersion $D : \widetilde{M} \to E$, called the *developing map* (see [FGH]). It is unique up to composition with an affine automorphism of E. Hence for every $p \in \pi$ there is a unique $\alpha(p) \in \mathbf{Aff}(E)$ such that $D \circ p = \alpha(p) \circ D$. The resulting homomorphism $\alpha : \pi \to \mathbf{Aff}(E)$ is called the *affine holonomy representation* and $\alpha(\pi)$ the *affine holonomy group*. α decomposes into a linear part λ and a translational part u. Then λ is a linear representation turning E into a π-module E_λ and u is a crossed homomorphism for λ, i.e., an 1-cocycle in $Z^1(\pi, E_\lambda)$: $u(pq) = u(p) + \lambda(p)u(q)$. $x \in E$ is a fixed point for α if and only if $u \in B^1(\pi, E_\lambda)$, i.e., $u(p) = x - \lambda(p)x$. The *radiance obstruction* of α is the cohomology class

$$c_\alpha = [u] \in H^1(\pi, E_\lambda).$$

For the affine manifold, the radiance obstruction c_M is the radiance obstruction of its affine holonomy representation α. If $c_M = 0$ then M is called *radiant*. Being radiant has quite a lot of consequences for M, see [GH1].

If D is a diffeomorphism, i.e., if $\widetilde{M}$ is affinely diffeomorph to E, then M is called *complete*. This happens if and only if ∇ is geodesically complete, see [AUM]. Compactness does not imply completeness.

Many examples of affine manifolds come from *left-invariant affine structures* on Lie groups. If G is a Lie group, an affine structure is called *left-invariant* if for each $g \in G$ the left-multiplication by g, $L_g : G \to G$ is an automorphism of the affine structure. (Hence the affine connection ∇ is left-invariant under left-translation as well.) Suppose G is simply connected. Let $D : G \to E$ be the developing map and $\alpha(g)$ be the unique affine automorphism of E such that $D \circ L_g = \alpha(g) \circ D$. Then $\alpha : G \to \mathbf{Aff}(E)$ is an affine representation.

Now it is not difficult to see ([FGH]) that G admits a complete left-invariant structure if and only if G acts *simply transitively* on E as affine transformations. In this case G must be solvable ([AUS]). If G has a left-invariant affine structure and Γ is a discrete subgroup of G, then the homogeneous space $\Gamma \backslash G$ of right cosets inherits an affine structure. If G is nilpotent, then $\Gamma \backslash G$ is called an *affine nilmanifold*.

In this context there is the following important question, also posed by Milnor ([MI2]) in the studies of fundamental groups of complete affine manifolds:

(3) *Which Lie groups admit left-invariant affine structures ?*

This question is particularly difficult for nilpotent Lie groups. There was much evidence that every nilpotent Lie group admits left-invariant affine structures (see [BGR]). Milnor conjectured this to be true even for solvable Lie groups ([MI2]). Recently, however, there

were counterexamples discovered ([BGR] and [BEN]). There are nilmanifolds which are not affine. We will show in a forthcoming paper that the class of nilpotent Lie groups of dimension $n \geq 10$ *not* admitting any left-invariant affine structure is rather large. The problem of classifying left-invariant affine structures on nilpotent Lie groups (see [KIM]) still seems to be hopeless.
If G is semisimple then G admits no left-invariant affine structures ([HE2], [BUR]). It is a natural question to ask what happens in the case of a *reductive* Lie group G. We may attempt then to give a classification of all left-invariant affine structures on G. In the general case we still have plenty of left-invariant affine structures ([HE1]). If G is a reductive linear Lie group with *one-dimensional* center and $[G,G]$ is simple, however, we are able to prove that the existence of left-invariant affine structures on G implies that G must be $\mathbf{GL}(n)$ itself. It possesses the unique (up to isomorphism) bi-invariant affine structure. By studying certain deformations of this structure we obtain interesting families of left-invariant affine structures on $\mathbf{GL}(n)$. In fact, using a result of [BAU], it follows that they exhaust all possible left-invariant affine structures on $\mathbf{GL}(n)$ for $n > 2$.

2 Left-invariant affine structures and LSA-structures

Let G be a finite-dimensional connected Lie group with Lie algebra $\mathfrak{g}$. We may assume that G is simply connected (otherwise consider $\widetilde{G}$). The following lemma is well known (see [SE2]):

Lemma 1 *There is a one-to-one correspondence between left-invariant affine structures on G and LSA-structures on $\mathfrak{g}$. Under this bijection, bi-invariant affine structures correspond to associative LSA-structures.*

Suppose G admits a left-invariant flat torsionfree affine connection ∇ on G. Since the connection is left-invariant, for any two left-invariant vector fields $X, Y \in \mathfrak{g}$, the covariant derivative $\nabla_X Y \in \mathfrak{g}$ is left-invariant. It follows that covariant differentiation $(X,Y) \mapsto \nabla_X Y$ defines a bilinear multiplication $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted by $(X,Y) \mapsto XY$ in short. Since ∇ is locally flat and torsionfree, we have by (1) and (2) of the introduction:

$$\text{(1)} \qquad [X,Y] = XY - YX$$

$$\text{(2)} \qquad [X,Y]Z = X(YZ) - Y(XZ)$$

We can rewrite (2) by using (1) as $(X,Y,Z) = (Y,X,Z)$ where (X,Y,Z) denotes the associator of the three elements X, Y, Z in $\mathfrak{g}$. Thus $(\mathfrak{g}, \cdot)$ is a *left-symmetric algebra* (or in short *LSA*) with product $x \cdot y = \nabla_X Y$, see [SE2], [BUR].
If we have any *LSA-structure* on $\mathfrak{g}$, i.e., a left-symmetric product $(x,y) \mapsto x \cdot y$ on $\mathfrak{g}$ satisfying $x \cdot y - y \cdot x = [x,y]$, then denote by $\lambda : x \mapsto \lambda(x)$ the left-regular representation on the LSA $(\mathfrak{g}, \cdot)$: $\lambda(x)y = x \cdot y$. It is a Lie algebra representation:

$$\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \qquad [\lambda(x), \lambda(y)] = \lambda([x,y]).$$

Denote the corresponding $\mathfrak{g}$ -module by $\mathfrak{g}_\lambda$. Furthermore, the identity map $\mathbf{1}:\ \mathfrak{g}\to\mathfrak{g}_\lambda$ is a 1 -cocycle in $Z^1(\mathfrak{g},\mathfrak{g}_\lambda)$:

$$\mathbf{1}([x,y]) = \mathbf{1}(x)\cdot y - \mathbf{1}(y)\cdot x$$

Let $\mathfrak{aff}(\mathfrak{g})$ be the Lie algebra of $\mathbf{Aff}(G)$, i.e., $\mathfrak{aff}(\mathfrak{g}) = \left\{ \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \mid A\in\mathfrak{gl}(\mathfrak{g}),\ b\in\mathfrak{g}\right\}$ which we identify with $\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g}$. Denote the linear part by $\ell(A,b)=A$ and the translational part by $t(A,b)=b$. Now we associate to the LSA $(\mathfrak{g},\cdot)$ the map

$$\alpha = \lambda\oplus\mathbf{1}:\ \mathfrak{g}\to\mathfrak{aff}(\mathfrak{g})$$

It is an affine representation of $\mathfrak{g}$. We have $\lambda=\ell\circ\alpha$ and $t\circ\alpha=\mathbf{1}$.
The *radiance obstruction* of α is the class $[\mathbf{1}]$ in $H^1(\mathfrak{g},\mathfrak{g}_\lambda)$, see [GH2]. For the proofs of the following proposition see [SE1], [BUR]. Let $\varrho(x)$ denote the right-multiplication by x in the LSA $(\mathfrak{g},\cdot)$:

Proposition 1

(1) *A left-invariant affine structure on G is complete if and only if all $\varrho(x)$ in the corresponding LSA are nilpotent endomorphisms.*
(2) *If G admits a complete left-invariant affine structure then G is solvable.*
(3) *If G is semisimple then G does not admit any left-invariant affine structure.*

The argument for the proof of (3) is roughly the following (see [BUR]): Let G be semisimple and $(\mathfrak{g},\cdot)$ be an LSA corresponding to a left-invariant affine structure on G . Then $\mathbf{1}\in Z^1(\mathfrak{g},\mathfrak{g}_\lambda)$ and by *Whitehead's Lemma*, $\mathbf{1}\in B^1(\mathfrak{g},\mathfrak{g}_\lambda)$, i.e., $\mathbf{1}(x)=x\cdot e=\varrho(e)x$ for some $e\in\mathfrak{g}_\lambda$. Then the LSA-property and $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ imply $\operatorname{tr}\lambda(x)=\operatorname{tr}\varrho(x)=0$ for all x and hence $\operatorname{tr}\mathbf{1}=\operatorname{tr}\varrho(e)=0$. Since the underlying field is of characteristic zero, we conclude that $\mathfrak{g}$ must be trivial which should be excluded.

3 LSA-structures on reductive Lie algebras

Let k be an algebraically closed field of characteristic zero. A Lie algebra $\mathfrak{g}$ is said to be *reductive* if its solvable radical $\mathfrak{r}(\mathfrak{g})$ coincides with the center $\mathfrak{z}=\mathfrak{z}(\mathfrak{g})$. Then the Lie algebra $\mathfrak{s}=[\mathfrak{g},\mathfrak{g}]$ is semisimple and we have

$$\mathfrak{g}=\mathfrak{s}\oplus\mathfrak{z}$$

A Lie algebra $\mathfrak{g}$ is reductive if and only if it admits a faithful completely reducible linear representation. A Lie group G is said to be reductive if its Lie algebra is reductive. Assume that $(\mathfrak{g},\cdot)$ is an LSA-structure on $\mathfrak{g}$. Since the first cohomology groups of a reductive Lie algebra do not vanish in general, we may have such structures. In fact, we know that there *are* LSA-structures on $\mathfrak{gl}_n(k)$, for example. The next question is whether the associated affine representation α is *radiant* or not. By a result of Milnor [MI2], one sufficient condition for an affine representation of $\mathfrak{g}$ to be radiant is that

the associated linear representation is completely reducible. However, the fact that $\mathfrak{g}$ is reductive does not imply that *any* finite-dimensional representation φ of $\mathfrak{g}$ is completely reducible. φ is completely reducible if and only if the center of $\mathfrak{g}$ is represented by semisimple endomorphisms, see [HUM]. However, it is true that α is radiant if $\mathfrak{z}$ is *one-dimensional.*
By saying $\mathfrak{s}$ is *split simple* we mean that $\mathfrak{s}$ is of one of the following types:

$$A_\ell,\ B_\ell,\ C_\ell,\ D_\ell,\ G_2,\ F_4,\ E_6,\ E_7,\ E_8$$

First we observe:

Lemma 2 *Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$ be a reductive Lie algebra with one-dimensional center, $\mathfrak{s}$ be split simple and $(\mathfrak{g}, \cdot)$ an LSA-structure on $\mathfrak{g}$. Then the algebra $(\mathfrak{g}, \cdot)$ is simple, i.e., has no proper two-sided ideals.*

Proof Any two-sided ideal $\mathfrak{a}$ in $(\mathfrak{g}, \cdot)$ is also a Lie ideal in $\mathfrak{g}$, since

$$[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{g} \cdot \mathfrak{a} - \mathfrak{a} \cdot \mathfrak{g} \subset \mathfrak{a}$$

The only proper ideals in $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$ are $\mathfrak{s}$ and $\mathfrak{z} = k$. However, both $(\mathfrak{a}, \cdot)$ and $(\mathfrak{g}/\mathfrak{a}, \cdot)$ inherit a natural LSA-structure from $(\mathfrak{g}, \cdot)$. Since $\mathfrak{s}$ and $\mathfrak{g}/\mathfrak{z}$ are semisimple it follows from Proposition 1 (3) that $\mathfrak{a}$ can neither be $\mathfrak{s}$ nor $\mathfrak{z}$. □

Suppose that $\mathfrak{g}$ is a *linear* Lie algebra. Given an LSA-structure $(\mathfrak{g}, \cdot)$, denote the $\mathfrak{g}$ - invariants of $\mathfrak{g}_\lambda$ by $(\mathfrak{g}_\lambda)^\mathfrak{g}$. We have $H^0(\mathfrak{g}, \mathfrak{g}_\lambda) = (\mathfrak{g}_\lambda)^\mathfrak{g}$. Since $\mathfrak{g}$ and $\mathfrak{g}_\lambda$ are identical as vector spaces, we may view an element $y \in \mathfrak{g}_\lambda$ also as an element of $\mathfrak{g}$. Our result is:

Theorem 1 *Let $(\mathfrak{g}, \cdot)$ be an LSA-structure on the reductive linear Lie algebra $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$. Then $(\mathfrak{g}_\lambda)^\mathfrak{g} \cap \mathfrak{s} = 0$.*

Corollary 1 *Let $(\mathfrak{g}, \cdot)$ be an LSA-structure on $\mathfrak{g}$. If $\dim \mathfrak{z} = 1$ then $H^0(\mathfrak{g}, \mathfrak{g}_\lambda) = 0$ and $H^1(\mathfrak{g}, \mathfrak{g}_\lambda) = 0$. Hence the associated affine representation of $\mathfrak{g}$ is radiant and the algebra $(\mathfrak{g}, \cdot)$ has a unique right-identity.*

Corollary 2 *Let $(\mathfrak{g}, \cdot)$ be an associative LSA-structure on $\mathfrak{g}$ where $\mathfrak{s}$ is simple. If $\dim \mathfrak{z} = 1$, then $(\mathfrak{g}, \cdot)$ is isomorphic to the matrix algebra $M_n(k)$ and $\mathfrak{g}$ is $\mathfrak{gl}_n(k)$.*

Proof of the Corollaries: Let $\mathfrak{z}$ be generated by z and $y \in (\mathfrak{g}_\lambda)^\mathfrak{g}$ be nonzero; hence by the Theorem $y = s + \gamma z \in \mathfrak{s} \oplus \mathfrak{z}$ where $s \in \mathfrak{s}$ and $\gamma \neq 0$. Then $0 = \varrho(y) = \varrho(s) + \gamma \varrho(z)$. Take the trace of both sides to obtain $\operatorname{tr} \varrho(z) = 0$ (note that $\operatorname{tr} \varrho(s) = 0$ for all $s \in \mathfrak{s}$ since $\operatorname{tr} \lambda([a, b]) = \operatorname{tr}([\lambda(a), \lambda(b)])$, $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ and $\operatorname{tr} \varrho(x) = \operatorname{tr} \operatorname{ad}(x) - \operatorname{tr} \lambda(x) = 0.$) Then $\operatorname{tr} \varrho(x) = 0$ for all $x \in \mathfrak{g}$ and as a consequence, all $\varrho(x)$ are nilpotent ($\operatorname{tr} \varrho(x)^2 = 0$ by $\varrho(x)^2 = \varrho(x^2) - [\lambda(x), \varrho(x)]$, and by the formulas (2.1) in [KIM], also $\operatorname{tr} \varrho(x)^n = 0$ for all n). Then by Proposition 1 , $\mathfrak{g}$ must be solvable. This is a contradiction. Thus $y = 0$, i.e., $(\mathfrak{g}_\lambda)^\mathfrak{g} = 0$, which is the first part of Corollary 1 .

The second statement of Corollary 1 follows immediately from the following fact:

Lemma 3 *Let* $\mathfrak{g}$ *be a reductive Lie algebra with* $\dim \mathfrak{z} = 1$ *and* M *be a finite-dimensional* $\mathfrak{g}$ *-module. Then* $H^0(\mathfrak{g}, M) = 0$ *is equivalent to* $H^1(\mathfrak{g}, M) = 0$.

Proof: The claim is true if $\mathfrak{g}$ is one-dimensional (see [BAR]). Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. The Hochschild-Serre spectral sequence gives the following exact sequence:

$$0 \longrightarrow H^1(\mathfrak{g}/\mathfrak{a}, M^{\mathfrak{a}}) \longrightarrow H^1(\mathfrak{g}, M) \longrightarrow H^1(\mathfrak{a}, M)^{\mathfrak{g}} \tag{4}$$

Assume $H^1(\mathfrak{g}, M) = 0$. Then $H^1(\mathfrak{g}/\mathfrak{s}, M^{\mathfrak{s}}) = 0$ by (4) with $\mathfrak{a} = \mathfrak{s}$. Since $\mathfrak{g}/\mathfrak{s}$ is one-dimensional, we have $M^{\mathfrak{g}} = (M^{\mathfrak{s}})^{\mathfrak{g}/\mathfrak{s}} = 0$.
To show the other direction, assume $H^0(\mathfrak{g}, M) = 0$. Let M be irreducible. Then the submodule $M^{\mathfrak{z}}$ is 0 or M. In the first case, $H^1(\mathfrak{z}, M) = 0$ and (4) gives $H^1(\mathfrak{g}, M) = 0$ with $\mathfrak{a} = \mathfrak{z}$. In the second case, $M^{\mathfrak{z}} = M$ is a $\mathfrak{g}/\mathfrak{z}$ -module and $H^1(\mathfrak{g}/\mathfrak{z}, M) = 0$ since $\mathfrak{s} = \mathfrak{g}/\mathfrak{z}$ is semisimple. The claim follows again by (4) with $\mathfrak{a} = \mathfrak{z}$.
If M is reducible, let N be a proper submodule. Then $N^{\mathfrak{g}} \le M^{\mathfrak{g}} = 0$. By induction on $\dim M$ we may assume $H^1(\mathfrak{g}, N) = 0$. The exact sequence $0 \to N \to M \to M/N \to 0$ induces the corresponding long exact sequence of H^0 and H^1 -groups. From this we derive $(M/N)^{\mathfrak{g}} = 0$. Again, by induction $H^1(\mathfrak{g}, M/N) = 0$. Looking at the H^1 -groups we obtain $H^1(\mathfrak{g}, M) = 0$. □

Now the last part of Corollary 1 is easy: $\mathbf{1}$ is in $Z^1(\mathfrak{g}, \mathfrak{g}_\lambda)$, hence also in $B^1(\mathfrak{g}, \mathfrak{g}_\lambda)$. That means $\varrho(e) = \mathbf{1}$ for some $e \in \mathfrak{g}_\lambda$. If e' is another right-identity, then $\varrho(e - e') = 0$, i.e., $e - e' \in (\mathfrak{g}_\lambda)^{\mathfrak{g}} = 0$.
If the LSA-structure is associative and $\dim \mathfrak{z} = 1$, then $(\mathfrak{g}, \cdot)$ posseses a *two-sided central identity*: If $\varrho(e) = \mathbf{1}$ then $0 = [\varrho(e), \varrho(x)] = \varrho([x, e])$ for all x. Since $(\mathfrak{g}_\lambda)^{\mathfrak{g}} = 0$ it follows $[x, e] = 0$ for all $x \in \mathfrak{g}$, hence $e \in \mathfrak{z}$ and $\lambda(e) = \varrho(e) = \mathbf{1}$. By Lemma 2 $(\mathfrak{g}, \cdot)$ is a simple associative algebra with unit, hence a matrix algebra by *Wedderburn's Theorem.* □

Proof of Theorem 1 *:* Consider the restriction $\mathbf{1}_{\mathfrak{s}}$ of the identity map $\mathbf{1} : \mathfrak{g} \to \mathfrak{g}_\lambda$ to $\mathfrak{s}$. Then $\mathbf{1}_{\mathfrak{s}} \in Z^1(\mathfrak{s}, \mathfrak{g}_\lambda)$. By Whitehead's Lemma, $\mathbf{1}_{\mathfrak{s}}$ is an one-coboundary, i.e., it exists an $e \in \mathfrak{g}_\lambda$ such that $x = \mathbf{1}_{\mathfrak{s}}(x) = \lambda(x)e$ for all $x \in \mathfrak{s}$. Assume that y is an element in $(\mathfrak{g}_\lambda)^{\mathfrak{g}} \cap \mathfrak{s}$. Then $y \in \mathfrak{s}$ and we obtain by the above (also using $\operatorname{ad}(y) = \lambda(y)$),

$$y = \lambda(y)e = [y, e] \tag{5}$$

That means, y and e generate a two-dimensional solvable subalgebra of $\mathfrak{g}$. By Lie's Theorem, y, e are upper triangular (relative to a suitable basis). Hence $y = [y, e]$ is strictly upper triangular, i.e., *nilpotent.* Then by the Morozow-Jacobson Theorem there exist $\overline{y}, h \in \mathfrak{g}$ such that

$$[y, \overline{y}] = h, \qquad [y, h] = 2y \tag{6}$$

We have the following Lemma:

Lemma 4 *Let* $(\mathfrak{g}, \cdot)$ *be an LSA with Lie algebra* $\mathfrak{g}$ *. If* $y \in (\mathfrak{g}_\lambda)^{\mathfrak{g}}$ *then* $\mathrm{ad}(y) = \lambda(y)$ *is a derivation of* $(\mathfrak{g}, \cdot)$ *, and in particular:*

$$(7) \qquad (\mathrm{ad}\, y)^3(v \cdot w) = \sum_{i=0}^{3} \binom{3}{i} (\mathrm{ad}\, y)^{3-i}(v) \cdot (\mathrm{ad}\, y)^i(w)$$

Proof: $y \in (\mathfrak{g}_\lambda)^{\mathfrak{g}}$ means $\varrho(y)v = v \cdot y = 0$ for all $v \in \mathfrak{g}_\lambda$ and $\mathrm{ad}(y) = \lambda(y) - \varrho(y) = \lambda(y)$. By the LSA-property (2) we have

$$y \cdot (v \cdot w) - (y \cdot v) \cdot w = v \cdot (y \cdot w) - (v \cdot y) \cdot w = v \cdot (y \cdot w)$$

Hence $\lambda(y)(v \cdot w) = \lambda(y)(v) \cdot w + v \cdot \lambda(y)(w)$ and the claim follows. □

We apply the Lemma as follows. By (6) we have $\mathrm{ad}\, y(\overline{y}) = h$, $(\mathrm{ad}\, y)^2(\overline{y}) = 2y$ and $(\mathrm{ad}\, y)^3(\overline{y}) = 0$. Using formula (7) we calculate:

$$\begin{aligned} (\mathrm{ad}\, y)^3(\overline{y} \cdot \overline{y}) &= 3(\mathrm{ad}\, y)^2(\overline{y}) \cdot (\mathrm{ad}\, y)(\overline{y}) + 3(\mathrm{ad}\, y)(\overline{y}) \cdot (\mathrm{ad}\, y)^2(\overline{y}) = 6(y \cdot h + h \cdot y) \\ &= 6[y, h] = 12y \end{aligned}$$

The following Lemma shows that the last equation implies $y = 0$. □

Lemma 5 *Suppose* $y \in \mathfrak{g}$ *is a nilpotent matrix and* $\alpha \neq 0$ *. Then* $(\mathrm{ad}\, y)^3(x) = \alpha y$ *for some* $x \in \mathfrak{g}$ *implies* $y = 0$.

Proof: By the Morozov-Jacobson Theorem, y can be embedded in an $\mathfrak{sl}_2(k) \subset \mathfrak{g}$. By Weyl's Theorem, $\mathfrak{g}$ is completely reducible as $\mathfrak{sl}_2(k)$ -module. Let $\mathfrak{v}$ be a complement, i.e.,

$$\mathfrak{g} \;=\; \mathfrak{sl}_2(k) \;\oplus\; \mathfrak{v}$$

Decompose $x = s + v$ and apply $(\mathrm{ad}\, y)^3$ on both sides. We have $(\mathrm{ad}\, y)^3(s) = 0$ since y is a nilpotent element in $\mathfrak{sl}_2(k)$. Hence $\alpha y = (\mathrm{ad}\, y)^3(x) = (\mathrm{ad}\, y)^3(v)$ is in $\mathfrak{sl}_2(k) \cap \mathfrak{v} = 0$. Since $\alpha \neq 0$ we have $y = 0$. □

Remark 1 There is an elementary proof of Lemma 5 . Using $(\mathrm{ad}\, y)(x) = yx - xy$ (matrix product) the above equation becomes $\alpha y = y^3x - 3y^2xy + 3yxy^2 - xy^3$. Assuming $y^{k+1} = 0 \neq y^k$ where $k > 1$, multiply this equation by y^{k-i} from the left and by y^{i-1} from the right for $0 < i < k$. We obtain k linear equations in the unknowns $x_i = y^{k+1-i}xy^{i+1}$ and $x_k = y^k$. The corresponding matrix has nonzero determinant $-\frac{1}{12}\alpha k(k+1)^2(k+2)$, $k > 1$. Hence, there is only the trivial solution, i.e., $y^k = 0$, contradiction. Then $k = 1, y = 0$.

Remark 2 The first part of Corollary 1 can also be proved as follows: As an $\mathfrak{s}$ -module, $\mathfrak{g}_\lambda$ is completely reducible. We show $\mathfrak{g}_\lambda^{\mathfrak{s}} = 0$ and hence also $\mathfrak{g}_\lambda^{\mathfrak{g}} = 0$.
The $\mathfrak{s}$ -module $\mathfrak{g}_\lambda$ has nonzero invariants if and only if the trivial module is a summand in its decomposition: $H^0(\mathfrak{s}, k) = k$. Assume $\mathfrak{g}_\lambda = \mathfrak{v} \oplus k$ for a complementary $\mathfrak{s}$ -module

$\mathfrak{v}$. Then $\mathfrak{s}$ acts trivially on k , and for $m = v + \alpha \in \mathfrak{v} \oplus k$ we have $x \cdot m = x \cdot v \in \mathfrak{v}$ for all $x \in \mathfrak{s}$ and $m \in \mathfrak{g}_\lambda$. Then $x \cdot z$ and $z \cdot x$ are in $\mathfrak{v}$ for all $x, z \in \mathfrak{s}$, hence also all commutators $[x, z]$. Since $\mathfrak{s}$ is spanned by those commutators, we have $\mathfrak{s} \subset \mathfrak{v}$. In fact, $\mathfrak{s} = \mathfrak{v}$ because of dimension reasons. This implies that $\mathfrak{s}$ admits an LSA-structure; a contradiction to Proposition 1 . Therefore, $\mathfrak{g}_\lambda$ does not have a summand k as an $\mathfrak{s}$ -module. □

We use Corollary 1 to show the following:

Theorem 2 *Let* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$ *be a reductive linear Lie algebra such that* $\dim \mathfrak{z} = 1$ *and* $\mathfrak{s}$ *is split simple. Then* $\mathfrak{g}$ *admits an LSA-structure if and only if* $\mathfrak{s}$ *is of type* A_ℓ *.*

Proof: First we show that if $\mathfrak{s}$ is *not* of typ $A_\ell,\ B_3,\ D_5,\ D_7$, then $\mathfrak{g}$ does not admit any LSA-structure. Secondly we exclude the cases where $\mathfrak{s}$ is of type $B_3,\ D_5,\ D_7$. For $\mathfrak{s}$ of type A_ℓ we already know that there exist LSA-structures.

Let $\dim \mathfrak{s} = n$. Since $\mathfrak{g}_\lambda$ is completely reducible as an $\mathfrak{s}$ -module and has no invariants, we know that

$$\mathfrak{g}_\lambda = \bigoplus_i V_i \quad \text{and} \quad \sum_i \dim V_i = n + 1 \tag{8}$$

where V_i are irreducible $\mathfrak{s}$ -modules with $2 \le \dim V_i \le \dim \mathfrak{g} = n + 1$ ($\mathfrak{g}_\lambda$ does not contain a trivial $\mathfrak{s}$ -module). On the other hand, there are not many irreducible $\mathfrak{s}$ -modules of small dimensions. Up to dimension n they are classified in [BUR]. Are there irreducible $\mathfrak{s}$ -modules of dimension $n+1$? The answer is given by

Lemma 6 *Let* $\mathfrak{s}$ *be of type* $A_\ell,\ B_\ell,\ C_\ell,\ D_\ell,\ G_2,\ F_4,\ E_6,\ E_7,\ E_8$ *and* V *be an irreducible* $\mathfrak{s}$ *-module. Define* $\delta_\ell = \delta_\ell(\mathfrak{s}) = \dim \mathfrak{s} + 1$ *. If* $\ell > 1$ *, then* $\dim V = \delta_\ell$ *is impossible.*

Proof: In dimension $\delta_\ell - 1$ we have always the adjoint module. Let $m_\ell(\mathfrak{s}) = m_\ell$ denote the minimal dimension of irreducible $\mathfrak{s}$ -modules with bigger dimension than $\dim \mathfrak{s}$. For $\ell > 8$ the values of m_ℓ and δ_ℓ are as follows:

Type	A_ℓ	B_ℓ	C_ℓ	D_ℓ
m_ℓ	$\binom{\ell+1}{3}$	$2\ell^2 + 3\ell$	$\binom{2\ell}{3} - \binom{2\ell}{1}$	$2\ell^2 + \ell - 1$
δ_ℓ	$(\ell+1)^2$	$2\ell^2 + \ell + 1$	$2\ell^2 + \ell + 1$	$2\ell^2 - \ell + 1$

Type	G_2	F_4	E_6	E_7	E_8
m_ℓ	27	273	351	912	3875
δ_ℓ	15	53	79	134	249

To see this, we may use the same method as in [BUR], Lemma 2.2.3 . The irreducible $\mathfrak{s}$ -modules are *highest weight modules* $L(\lambda)$. The Weyl group acts on the weights by conjugation and we may estimate the dimension of $L(\lambda)$ from below by the number of the weights of $L(\lambda)$ which is the sum of $|\mathcal{W}\nu|$ over the dominant weights $\nu \leq \lambda$. Besides we can use *Weyl's dimension formula.* The Lemma can also easily be deduced from the computations in [SAK], p.41f.
Denote by $\omega_1, \ldots, \omega_\ell$ the fundamental weights, then the following modules (for the types $A_\ell, \ldots D_\ell$ respectively) have dimension m_ℓ : $L(\omega_3)$, $L(2\omega_1)$, $L(\omega_3)$, $L(2\omega_1)$.
Since $m_\ell - \delta_\ell$ is always positive, Lemma 6 follows for $\ell > 8$. In the case $\ell \leq 8$ we may use the tables from [BMP] to verify the result. (Of course, $\mathfrak{sl}_2$ has irreducible representations in any dimension, so we must exclude $\ell = 1$).
Consider the decomposition (8) . If $\ell > 8$, we have the following possibilities for the modules V_i occuring in (8) (see [BUR]):
For type A_ℓ we have the modules $L(\omega_1), L(\omega_2), L(2\omega_1), L(\omega_1 + \omega_\ell)$ and their dual modules of dimension $\ell + 1, \ell(\ell+1)/2, (\ell+1)(\ell+2)/2, \ell^2 + 2\ell$.
For type B_ℓ we have $L(\omega_1), L(\omega_2)$ of dimension $2\ell + 1, \ell(2\ell+1)$.
For type C_ℓ we have $L(\omega_1), L(\omega_2), L(2\omega_1)$ of dimension $2\ell, 2\ell^2 - \ell - 1, \ell(2\ell+1)$.
For type D_ℓ we have $L(\omega_1), L(\omega_2)$ of dimension $2\ell, \ell(2\ell-1)$.
It is easy to see that the dimensions cannot add up to δ_ℓ , except in the case of A_ℓ .
For the exceptional types and the cases $\ell \leq 8$ we see the result from the following table. It lists the possible dimensions:

Type	$\dim L(\lambda)$	δ_λ	**Type**	$\dim L(\lambda)$	δ_λ
B_3	$7, 8, 21$	22	C_8	$16, 119, 136$	137
B_4	$9, 16, 36$	37	D_4	$8, 28$	29
B_5	$11, 32, 55$	56	D_5	$10, 16, 45$	46
B_6	$13, 64, 78$	79	D_6	$12, 32, 66$	67
B_7	$15, 105$	106	D_7	$14, 64, 91$	92
B_8	$17, 136$	137	D_8	$16, 120$	121
C_2	$4, 5, 10$	11	G_2	$7, 14$	15
C_3	$6, 14, 21$	22	F_4	$26, 52$	53
C_4	$8, 27, 36$	37	E_6	$27, 78$	79
C_5	$10, 44, 55$	56	E_7	$56, 133$	134
C_6	$12, 65, 78$	79	E_8	248	249
C_7	$14, 90, 105$	106			

Only in the cases B_3, D_5, D_7 the dimensions can add up to δ_ℓ , and hence these are the only types for $\mathfrak{s}$ (besides A_ℓ) where we might have LSA-structures for $\mathfrak{g}$.

For these three cases we can deduce from the table what $\mathfrak{g}_\lambda$ must be:

$$(9) \qquad \mathfrak{g}_\lambda = \quad L(\omega_1) \oplus L(\omega_1) \oplus L(\omega_3)$$
$$(10) \qquad \mathfrak{g}_\lambda = \quad L(\omega_1) \oplus L(\omega_1) \oplus L(\omega_1) \oplus L(\omega_5)$$
$$(11) \qquad \mathfrak{g}_\lambda = \quad L(\omega_1) \oplus L(\omega_1) \oplus L(\omega_7)$$

For the dimensions we have $22 = 7+7+8$, $46 = 10+10+10+16$ and $92 = 14+14+64$ respectively.
Let $G = S \oplus k$ be a simply connected reductive algebraic group with Lie algebra $\mathfrak{g} = \mathfrak{s} \oplus k$. Let V be the rational S -module corresponding to the $\mathfrak{s}$ -module $\mathfrak{g}_\lambda$. We may look at V as an algebraic S -variety and apply methods from invariant theory. The following Lemma is due to O. Baues [BAU]:

Lemma 7 *Suppose $\mathfrak{g} = \mathfrak{s} \oplus k$ admits an LSA-structure. Then the corresponding S -module V has an S -orbit of codimension 1 . If $W \subset V$ is a proper S -submodule, then S has an open orbit in W.*

In the above cases, S is an orthogonal group and the S -modules in $(9), (10), (11)$ do not have an S -orbit of codimension 1 . This may be seen from the fact that the natural module $W = L(\omega_1)$ does not have an open S -orbit where S is an orthogonal group (see the classification in [SAK], p. 147). Hence $\mathfrak{g}$ does not admit an LSA-structure in these cases. $\square$

Remark 3 In the general case of a reductive Lie algebra, Theorem 2 has no easy analogue: as an example, the reductive Lie algebra

$$\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^4$$

of dimension 25 admits LSA-structures (see [HE1]).

4 Left-invariant affine structures on $\mathbf{GL(n)}$

Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{A} = (\mathfrak{g}, \cdot)$ an LSA-structure on $\mathfrak{g}$. We describe a procedure to obtain *new* (in general non-isomorphic) LSA-structures from $\mathcal{A} = (\mathfrak{g}, \cdot)$. We call these structures τ *-deformations of* $\mathcal{A}$, although they are not deformations in the usual sense. We apply this to the Lie algebra $\mathfrak{gl}_n(k)$ and the canonical associative LSA-structure.

Define the *associative kernel* of $\mathcal{A}$ by

$$k(\mathcal{A}) := \{a \in \mathcal{A} \mid [\lambda(b), \varrho(a)] = 0 \quad \text{for all } b \in \mathcal{A}\}$$

This is an associative subalgebra of $\mathcal{A}$ containing the center of $\mathfrak{g}$ by the identity

$$[\lambda(b), \varrho(a)] = [\operatorname{ad} a, \lambda(b)] + \lambda([b, a])$$

Denote by $\mathrm{End}_*(\mathfrak{g})$ the set $\{\tau \in \mathrm{End}(\mathfrak{g}) \mid (\mathbf{1}-\tau)^{-1}$ exists and $\tau(\mathcal{A}) \subset k(\mathcal{A})\}$. Then we have (see [HE1]):

Lemma 8 *Let* $\mathcal{A} = (\mathfrak{g}, \cdot)$ *be an LSA-structure on* $\mathfrak{g}$ *and* $\tau \in \mathrm{End}_*(\mathfrak{g})$ *with* $\phi = (\mathbf{1}-\tau)^{-1}$ *. Then* $\lambda_\tau(a) := \phi \circ (\lambda(a) - \varrho(\tau(a))) \circ \phi^{-1}$ *defines an LSA-structure on* $\mathfrak{g}$ *. We call* $\mathcal{A}_\tau$ *the* τ *-deformation of* $\mathcal{A}$ *.*

In general, $\mathcal{A}$ is not isomorphic to the deformation algebras $\mathcal{A}_\tau$. This happens however, if $\tau(\mathcal{A}) = \mathfrak{z}(\mathfrak{g})$. Let $\mathcal{A}$ be the matrix algebra $M_n(k)$ with Lie algebra $\mathfrak{gl}_n(k)$, and define τ by $\tau|_{\mathfrak{sl}_n} = 0$ and $\tau(z) \in \mathfrak{sl}_n(k)$ arbitrary, where z generates the center of $\mathfrak{g}$. Then $\tau^2 = 0$ and $(\mathbf{1}-\tau)(\mathbf{1}+\tau) = \mathbf{1}$. Hence $\tau \in \mathrm{End}_*(\mathfrak{g})$. Since also $k(\mathcal{A}) = \mathcal{A}$ we can apply the Lemma to obtain the deformation algebras $\mathcal{A}_\tau$. Note that these algebras do not have a two-sided central identity except in the case $\tau = 0$: If $\overline{\varrho}(z) = \mathbf{1}$ in $\mathcal{A}_\tau$, then $\lambda(z) - \varrho(\tau(z)) = \mathbf{1}$, i.e., $\varrho(\tau(z)) = 0$. Since $(\mathfrak{g}_\lambda)^{\mathfrak{g}} = 0$ it follows $\tau(z) = 0$ and then $\tau = 0$.
As we will see later, the τ -deformations exhaust all possible LSA-structures on $\mathfrak{gl}_n(k)$ for $n > 2$.

By explicit calculations now we classify the left-invariant affine structures on $\mathbf{GL}_2(\mathbb{C})$, i.e., the LSA-structures on $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C})$. Let $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ be the canonical $\mathfrak{sl}_2$ -basis for $\mathfrak{g}$. The LSA-structures are described by the endomorphisms $\lambda(x),\ \lambda(y),\ \lambda(h),\ \lambda(z)$ via $\lambda(a)b = a \cdot b$.

Theorem 3 *Let* $(\mathfrak{g}, \cdot)$ *be an LSA-structure on* $\mathfrak{g}$ *. Then it is isomorphic to* $\mathcal{A}_1,\ \mathcal{A}_{2,\alpha}$ *or* $\mathcal{A}_3$ *defined by the matrices* $\lambda(x),\ \lambda(y),\ \lambda(h),\ \lambda(z)$ *as follows:*

$$(i) \quad \begin{pmatrix} 0 & 1/2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & 1 \\ -1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & -1/2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1/2 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & -1/2 & 0 & 1 \end{pmatrix}$$

$$(ii) \quad \begin{pmatrix} 0 & 0 & -1 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & \beta/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma \\ -\gamma/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \alpha & \beta\gamma \\ 0 & 0 & 1 & -\alpha \end{pmatrix} \quad \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \beta\gamma & -\alpha\beta\gamma \\ 0 & 0 & -\alpha & 1+\alpha^2 \end{pmatrix}$$

$$(iii) \quad \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & -1/4 & 0 & 0 \\ 3 & 3/4 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\beta = 1+\alpha,\ \gamma = 1-\alpha$ *. Two LSA's* $\mathcal{A}_{2,\alpha}$ *and* $\mathcal{A}_{2,\tilde{\alpha}}$ *are isomorphic if and only if* $\alpha^2 = \tilde{\alpha}^2$ *. They are associative if and only if* $\alpha = 0$ *. In this case,* $\mathcal{A}_{2,0}$ *coincides with the matrix algebra* $M_2(\mathbb{C})$ *.*

Proof: Let $(\mathfrak{g}, \lambda)$ be an LSA-structure on $\mathfrak{g}$. By Corollary 1 there is a unique $e \in \mathfrak{g}_\lambda$ such that $\varrho(e) = \mathbf{1}$. Let $e = (e_1, e_2, e_3, e_4)$, i.e., $e = e_1x + e_2y + e_3h + e_4z$. The center of $\mathfrak{g}$ is generated by z . Two LSA-structures $(\mathfrak{g},\lambda)$ and $(\mathfrak{g},\mu)$ are isomorphic if and only if there is a $\psi\in\operatorname{Aut}(\mathfrak{g})$ such that $\mu(x)=\psi\circ\lambda(\psi^{-1}(x))\circ\psi^{-1}$. The Lie algebra automorphisms of $\mathfrak{g}$ are $\psi_A:\ X\mapsto AXA^{-1}$ with $A=\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)$, $\Delta=\alpha\delta-\gamma\beta\neq 0$ and $\psi_t:\ u\mapsto s+t\cdot z$ where $u=s+z,\ s\in\mathfrak{sl}_2$. For $A=\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)$ we look at the generators of $\mathbf{GL}(2)$, i.e., $\left(\begin{smallmatrix}1&\beta\\ 0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}\alpha&0\\ 0&\delta\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&1\\ 1&0\end{smallmatrix}\right)$. Let $\psi_1=\psi_{\left(\begin{smallmatrix}1&\beta\\ 0&1\end{smallmatrix}\right)}$, $\psi_2=\psi_{\left(\begin{smallmatrix}\alpha&0\\ 0&\delta\end{smallmatrix}\right)}$, $\psi_3=\psi_{\left(\begin{smallmatrix}0&1\\ 1&0\end{smallmatrix}\right)}$ and $\psi_4=\psi_t$. It is easy to see that

$$
\begin{array}{lll}
(a) & \psi_1\,(e_1,e_2,e_3,e_4)= & (e_1-\beta^2e_2-2\beta e_3,\ e_2,\ \beta e_2+e_3,\ e_4)\\
(b) & \psi_2\,(e_1,e_2,e_3,e_4)= & (\alpha e_1/\delta,\ \delta e_2/\alpha,\ e_3,\ e_4)\\
(c) & \psi_3\,(e_1,e_2,e_3,e_4)= & (e_2,\ e_1,\ -e_3,\ e_4)\\
(d) & \psi_4\,(e_1,e_2,e_3,e_4)= & (e_1,\ e_2,\ e_3,\ te_4)
\end{array}
$$

Lemma 9 *We may assume that the right-identity is* $e=x+z$ *or* $e=\alpha h+z$ *or* $e=z$ *.*

Proof: If $(\mathfrak{g},\lambda)$ and $(\mathfrak{g},\tilde\lambda)$ are isomorphic LSA's, then $\varrho(e)=\mathbf{1}$ implies $\tilde\varrho(\psi(e))=\psi\circ\varrho(e)\circ\psi^{-1}=\mathbf{1}$, i.e., the LSA $(\mathfrak{g},\tilde\lambda)$ has right-identity $\psi(e)$. First we may assume $e_2=0$. Otherwise let $\beta\in\mathbb{C}$ be a root of $\beta^2e_2+2\beta e_3-e_1=0$ and apply $(a),(c)$: $(\psi_3\circ\psi_1)(e_1,e_2,e_3,e_4)=(e_2,0,-\beta e_2-e_3,e_4)$.
Case 1: $e_1=0$. If $e_3=0$ then $\psi_t(e)=z$ with $t=1/e_4$ (note that $e\neq 0$). If $e_3\neq 0$ then it follows $e_4\neq 0$, otherwise $0=\operatorname{tr}\varrho(e_3h)=\operatorname{tr}\varrho(e)=4$, contradiction. Then $\psi_t(e)=e_3h+z$ with $t=1/e_4$.
Case 2: $e_1\neq 0$. We may assume $e_3=0$, otherwise $\psi_1(e_1,0,e_3,e_4)=(0,0,e_3,e_4)$ with $\beta=e_1/2e_3$ and we are back to case 1. Then $(\psi_t\circ\psi_2)(e_1,0,0,e_4)=(1,0,0,1)$ with $\delta/\alpha=e_1$ and $t=1/e_4$. Here again $e_4\neq 0$ by the above argument. Hence $e=x+z$. □

The LSA-product is given by 64 structure constants via $\lambda(x),\lambda(y),\lambda(h),\lambda(z)$. The condition $[a,b]=a\cdot b-b\cdot a$ determines 24 structure constants by linear equations. The LSA-property (2) is equivalent to quadratic equations in the structure constants. In general, they are quite difficult to solve. The existence of a non-central right-identity, however, simplifies the matter considerably. We have

$$(12)\qquad [\lambda(z),\operatorname{ad}e]\ =[\lambda(z),\lambda(e)]\ =\ \lambda([z,e])\ =\ 0$$

I. Algebras with $\varrho(e)=x+z$ *:*

Using (12) we have $[\lambda(z),\operatorname{ad}x]=0$ and $\varrho(x)+\varrho(z)=\mathbf{1}$. Also $\operatorname{tr}\varrho(s)=0$ for all $s\in\mathfrak{sl}_2$. This determines another 25 structure constants by *linear* equations. The remaining LSA-structure equations then are almost trivial. It is easy to see that they have a unique solution, which is given by the algebra $\mathcal{A}_1$ under (i) of Theorem 3 .

II. Algebras with $\varrho(e)=\alpha h+z,\ \alpha\in\mathbb{C}$ *:*

Assume first that $\alpha\neq 0$. Then $\alpha\varrho(h)+\varrho(z)=\mathbf{1}$ and $[\lambda(z),\operatorname{ad}h]=0$ determine 26 structure constants. It is easy to solve the remaining equations and to obtain the algebra $\mathcal{A}_{2,\alpha}$. It is associative if and only if $\alpha = 0$ (which we may include here as well) and the algebra is precisely $M_n(\mathbb{C})$. It is clear that two such algebras $\mathcal{A}_{2,\alpha}$ and $\mathcal{A}_{2,\tilde{\alpha}}$ are isomorphic if and only if $\alpha^2 = \tilde{\alpha}^2$: If they are isomorphic then the characteristic polynomials of $\lambda(z)$ and $\tilde{\lambda}(z)$ must be equal. This implies $\alpha^2 = \tilde{\alpha}^2$. On the other hand $\mathcal{A}_{2,\alpha}$ is isomorphic to $\mathcal{A}_{2,-\alpha}$ by ψ_3 .

III. Algebras with central right-identity $\varrho(e) = z$ *:*

Since $\lambda(z) = \mathbf{1}$, $\mathfrak{g}_\lambda$ is completely reducible as $\mathfrak{g}$ -module and $\lambda(h)$ is semisimple. Because $H^0(\mathfrak{g}, \mathfrak{g}_\lambda) = 0$, we have only two possibilities for $\mathfrak{g}_\lambda$. In the first case, $\mathfrak{g}_\lambda$ is *irreducible*, and in the second case, $\mathfrak{g}_\lambda = V \oplus V$, where V (as an $\mathfrak{sl}_2$ -module) is isomorphic to the 2 -dimensional natural representation of $\mathfrak{sl}_2$.

Lemma 10 *As matrices,* $\lambda(h)$ *is similar to* $\operatorname{diag}(3, 1, -1, -3)$ *or to* $\operatorname{diag}(1, -1, 1, -1)$ *and* $\lambda(x), \lambda(y)$ *are nilpotent.*

Proof: If $\mathfrak{g}_\lambda$ is irreducible, it is (as an $\mathfrak{sl}_2$ -module) a highest weight module with basis v_i such that $\lambda(h)v_i = (3 - 2i)v_i$, $\lambda(x)v_i = (4 - i)v_{i-1}$ and $\lambda(y)v_i = (i + 1)v_{i+1}$ for $i = 0, 1, 2, 3$. With respect to this basis, $\lambda(h) = \operatorname{diag}(3, 1, -1, -3)$ and $\lambda(x), \lambda(y)$ are nilpotent. Note that this basis does not satisfy the LSA-condition (1) . In the second case, choose a basis according to $V \oplus V$, where V is a highest weight module for $\mathfrak{sl}_2$. □

Let $\lambda(x) = (a_{ij}), \lambda(y) = (b_{ij}), \lambda(h) = (c_{ij})$ with $i, j = 1, \ldots, 4$. Using $\lambda(u) - \varrho(u) = \operatorname{ad} u$ we obtain:

$$\lambda(y) = \left(\begin{array}{c|c} Y_1 & Y_3 \\ \hline Y_2 & Y_4 \end{array}\right), \quad \lambda(h) = \left(\begin{array}{c|c} H_1 & H_3 \\ \hline H_2 & H_4 \end{array}\right), \quad \lambda(z) = \mathbf{1}, \quad \text{where}$$

$Y_1 = \begin{pmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{pmatrix}$, $Y_2 = \begin{pmatrix} a_{23}-1 & b_{32} \\ a_{42} & b_{42} \end{pmatrix}$, $Y_3 = \begin{pmatrix} b_{13} & 0 \\ b_{23} & 1 \end{pmatrix}$, $Y_4 = \begin{pmatrix} b_{33} & 0 \\ b_{43} & 0 \end{pmatrix}$, $H_1 = \begin{pmatrix} a_{33}+2 & b_{13} \\ a_{23} & b_{23}-2 \end{pmatrix}$,
$H_2 = \begin{pmatrix} a_{33} & b_{33} \\ a_{43} & b_{43} \end{pmatrix}$, $H_3 = \begin{pmatrix} c_{13} & 0 \\ c_{23} & 0 \end{pmatrix}$, $H_4 = \begin{pmatrix} c_{33} & 1 \\ c_{43} & 0 \end{pmatrix}$.

Since the trace of $\lambda(x), \lambda(y), \lambda(h)$ is zero, we have $a_{33} = -a_{11} - a_{22}$, $b_{33} = -a_{12} - b_{22}$ and $c_{33} = -a_{13} - b_{23}$. We simplify H_3 by applying $\psi_{\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)}$ or $\psi_{\left(\begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix}\right)}$. This respects $\lambda(z) = \mathbf{1}$ and it is not difficult to see that we can assume $H_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $H_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Case 1: $\mathfrak{g}_\lambda$ *is irreducible.*

Case a: $H_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The variables of H_1, H_4 satisfy the following LSA-equations:

$$(a) \qquad 2a_{13}^2 + a_{13}b_{23} + a_{23}b_{13} - c_{43} = 0$$
$$(b) \qquad 2b_{23}^2 + a_{13}b_{23} + a_{23}b_{13} - c_{43} = 0$$
$$(c) \qquad a_{23}(a_{13} + b_{23} - 2) = 0$$
$$(d) \qquad b_{13}(a_{13} + b_{23} + 2) = 0$$

From the fact that the characteristic polynomial of $\lambda(h)$ is $(t-3)(t-1)(t+1)(t+3)$ we obtain:

$$(e) \qquad a_{13}^2 + a_{13}b_{23} + b_{23}^2 + a_{23}b_{13} + 2(a_{13} - b_{23}) + c_{43} - 6 = 0$$

It follows that $c_{43} = 1, 3$ or 9 . We obtain the following solutions:
$H_1 = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$, $H_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $H_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $H_4 = \begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix}$ or $H_1 = \begin{pmatrix} 3 & 0 \\ a_{23} & -1 \end{pmatrix}$, $H_4 = \begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}$ or $H_1 = \begin{pmatrix} 1 & b_{13} \\ 0 & -3 \end{pmatrix}$, $H_4 = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$.

Then the remaining LSA-equations are very simple: The first solution is not possible, and all other cases are isomorphic. We may also normalize b_{13} to 1 . That means we may take

$$H_1 = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$$

and we obtain the LSA $\mathcal{A}_3$. Note that $\mathcal{A}_3$ is not associative.

Case b: $H_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

This case can be reduced to *Case a* by applying $\psi_{\left(\begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix}\right)}$ and $\psi_{\left(\begin{smallmatrix} 1 & \beta \\ 1 & \delta \end{smallmatrix}\right)}$.

Case 2: $\mathfrak{g}_\lambda = V \oplus V$.

The characteristic polynomial of $\lambda(h)$ now is $(t-1)^2(t+1)^2$ and equation (e) becomes:

$$(e) \qquad a_{13}^2 + a_{13}b_{23} + b_{23}^2 + a_{23}b_{13} + 2(a_{13} - b_{23}) + c_{43} + 2 = 0$$

Case a: $H_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The equations $(a), \dots, (e)$ have solutions
$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $H_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $H_1 = \begin{pmatrix} a_{13}+2 & 0 \\ 0 & a_{13}-2 \end{pmatrix}$, $H_4 = \begin{pmatrix} -2a_{13} & 1 \\ -1 & 0 \end{pmatrix}$
The first solution leads to the matrix algebra $M_2(\mathbb{C})$, and the second one is contradictory.

Case b: $H_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

After a short calculation we obtain a contradiction. □

The Theorem shows that there is one infinite family of non-isomorphic LSA-structures on $\mathfrak{gl}_2(\mathbb{C})$ with non-central right-identity. In fact, all those structures can be obtained as τ *-deformations* of the matrix algebra structure $M_2(\mathbb{C})$: Define τ by $\tau(z) = x$ and zero on $\mathfrak{sl}_2(\mathbb{C})$. Then $\mathcal{A}_\tau = \mathcal{A}_1$ of Theorem 3 below. If τ is defined by $\tau(z) = \alpha h$, we obtain precisely $\mathcal{A}_{2,\alpha}$. Other choices of $\tau(z)$ yield isomorphic algebras.

As for the algebras with two-sided central identity, there are precisely *two* non-isomorphic ones, $\mathcal{A}_3$ and $M_2(\mathbb{C})$.
In the general case the following holds:

Theorem 4 *The* τ *-deformations of the matrix algebra* $\mathcal{A}$ *exhaust all possible LSA-structures on* $\mathfrak{gl}_n(k)$ *for* $n > 2$. *Their isomorphism classes are parametrized by the conjugacy classes of elements* $X \in \mathfrak{gl}_n(k)$ *with* $\operatorname{tr}(X) = n$. *There is exactly one LSA-structure with a two-sided central identity – the matrix algebra structure.*

Proof: This is a consequence of a classification Theorem in [BAU] to be published elsewhere. Here we only present a typical case: Let $\tau(z) = \alpha h$ where h is an element of the Cartan subalgebra of $\mathfrak{gl}_n(k)$. We obtain a family of LSA-structures with right-identities $e_\alpha = \alpha h + z$. We will determine the isomorphism classes of this family.
Let $\phi = (\mathbf{1} - \tau)^{-1} = (\mathbf{1} + \tau)$. We have $\varrho_\tau(\alpha h + z) = \phi \circ (\varrho(\alpha h + z) - \varrho(\alpha h) + (\lambda(\alpha h) - \lambda(\alpha h + z) \circ \tau)) = \phi(\varrho(z) - \lambda(z) \circ \tau) = \phi \circ (\mathbf{1} - \tau) = \mathbf{1}$. Denote by $(\mathfrak{g}, \alpha)$ and $(\mathfrak{g}, \tilde{\alpha})$ any two τ -deformation algebras. By the above calculation $\varrho(e_\alpha) = \tilde{\varrho}(e_{\tilde{\alpha}}) = \mathbf{1}$. Assume that both algebras are isomorphic. Then there is a $\psi \in \operatorname{Aut}(\mathfrak{g})$ such that $\tilde{\varrho}(\psi(e_\alpha)) = \psi \circ \varrho(e_\alpha) \circ \psi^{-1} = \mathbf{1}$, i.e., $\psi(e_\alpha)$ also is a right-identity for $(\mathfrak{g}, \tilde{\alpha})$. It follows

$$\psi(e_\alpha) = e_{\tilde{\alpha}}, \quad \text{that is} \quad \psi(\alpha h + z) = \tilde{\alpha} h + z$$

by Corollary 1 . This is only possible for $\tilde{\alpha}^2 = \alpha^2$: The Lie algebra automorphisms of $\mathfrak{gl}_n(k)$ are of the form $X \mapsto -X^t$, $X \mapsto AXA^{-1}$ and $s + z \mapsto s + tz$. Given the canonical $\mathfrak{sl}_n$ -basis, all $\alpha h + z$ are diagonal matrices. Hence conjugation acts as permutation of the eigenvalues and the result follows. □

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