# LEFT-SYMMETRIC STRUCTURES ON SIMPLE MODULAR LIE ALGEBRAS

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## 0. INTRODUCTION

Let $V$ be a vector space over an arbitrary field $k$ . Consider a bilinear, distributive product $V \times V \to V$ , denoted $(x,y) \mapsto x.y$ , which gives $\mathcal{A} = (V, .)$ the structure of a nonassociative algebra over $k$ .
Then $\mathcal{A}$ is said to be a *left-symmetric algebra*, or *Koszul-Vinberg* algebra, if

$$x.(y.z) - (x.y).z = y.(x.z) - (y.x).z \tag{0.1}$$

for all $x, y, z$ in $\mathcal{A}$ . If $\mathcal{A}$ is a left-symmetric algebra, then the operation

$$[x, y] = x.y - y.x \tag{0.2}$$

is skew-symmetric and satisfies the Jacobi identity. Thus every left-symmetric algebra has an underlying Lie algebra structure. Conversely, if $\mathfrak{g}$ is a Lie algebra over $k$ , then a left-symmetric operation satisfying (0.1), (0.2) on the vector space of $\mathfrak{g}$ will be called a compatible left-symmetric algebra structure on $\mathfrak{g}$ , or a *left-symmetric structure on* $\mathfrak{g}$ , in short.
Left-symmetric structures on Lie algebras arise in the theory of affine manifolds (see below for further explanation, $k = \mathbb{K}$ being the field of real or complex numbers ). One asks whether the Lie algebra of a simply connected Lie group over $\mathbb{K}$ admits left-symmetric structures. In fact, the problem of finding those Lie algebras over $\mathbb{K}$ which admit left-symmetric structures is still unsolved (cf. Auslander [2], Kim [23], Medina [27], Milnor [26]). It is conjectured that every nilpotent Lie algebra admits leftsymmetric structures over $\mathbb{K}$ . Furthermore one is interested in the classification of left-symmetric structures over $\mathbb{K}$ . This has been done so far only in dimensions 2, 3 and 4 (cf. Kuiper [24], Fried and Goldman [12] and Kim [23]), but in the last case only for nilpotent Lie algebras.

1. *It will be assumed in the following that all Lie algebras and Lie modules are finite-dimensional over* $k$ *.*
2. *The purpose of this paper is to investigate left-symmetric structures on simple Lie algebras over an arbitrary field* $k$ *.*

It is known that every Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ with $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ does not admit any left-symmetric structure (Helmstetter [14], p.31). Furthermore, if $\mathfrak{g}$ is semisimple over a field $k$ of characteristic 0, it follows from Whitehead's lemma (for the first Lie algebra cohomology with coefficients in Lie modules) that $\mathfrak{g}$ does not admit any left-symmetric structure (see Proposition 1.2.8.) .

The assumption $\mathrm{char}(k) = 0$ , however, is essential for the validity of these results. If $k$ is a modular field, then Whitehead's lemma is no longer true, and there are even classical simple Lie algebras which admit left-symmetric structures. Consider the following example: Let $k$ be a field of characteristic 3.

Let $\mathfrak{g} := kx \oplus ky \oplus kz = \mathfrak{sl}(2,k)$ and $[x,y] = z,\ [z,x] = 2x,\ [z,y] = -2y$ .

It is easy to see, that the following product defines left-symmetric structures on $\mathfrak{g}$ (depending on $\gamma \in k^{\times}$ ) :

$$\begin{array}{lllllllll} x.x & = & 0 & y.x & = & (1-\gamma^{-1})z & z.x & = & (\gamma-1)x \\ x.y & = & -(1+\gamma^{-1})z & y.y & = & 0 & z.y & = & (\gamma+1)y \\ x.z & = & \gamma x & y.z & = & \gamma y & z.z & = & \gamma z \end{array}$$

There are further left-symmetric structures on $\mathfrak{sl}(2,k)$ (they are classified in [7], see Remark 2.1.2.), but only in characteristic 3. In fact, $\mathfrak{sl}(2,k)$ admits left-symmetric structures if and only if $\mathrm{char}(k) = 3$ .

We consider now left-symmetric structures on finite-dimensional simple modular Lie algebras. The known finite-dimensional simple Lie algebras over $k$ are of two types: *classical type* (analogues over $k$ of finite-dimensional simple Lie algebras over $\mathbb{C}$ ) and *Cartan type* (finite-dimensional analogues over $k$ of the infinite Lie algebras of Cartan [8] over $\mathbb{C}$ , see [22], [35]).
Let $k$ be algebraically closed of characteristic $p > 7$ . Then Block and Wilson [5] proved, that every finite-dimensional *restricted* Lie algebra over $k$ is of classical or Cartan type. It is conjectured that every nonclassical finite-dimensional simple Lie algebra over $k$ is of Cartan type. Recently H. Strade ([29], [30], [31]) made much progress towards the general solution.

Let $\mathfrak{g}$ be the Lie algebra of a connected semisimple algebraic group $G$ of type $A_l\ (l \geq 1), B_l\ (l \geq 3), C_l\ (l \geq 2), D_l\ (l \geq 4), G_2, F_4, E_6, E_7, E_8$ . In this paper we give the proof of the following main theorem:

**THEOREM 2.2.2.** *Let $\mathfrak{g}$ be a classical Lie algebra of the above type and assume*

$$\begin{array}{llll} (a) & p > 2, & \text{if } \mathfrak{g} \text{ is of type} & A_l, B_l, C_l, D_l, E_7 \\ (b) & p > 3, & \text{if } \mathfrak{g} \text{ is of type} & G_2, F_4, E_6 \\ (c) & p \not| \ l+1, & \text{if } \mathfrak{g} \text{ is of type} & A_l \end{array}$$

*If $p \not| \dim \mathfrak{g}$ then $\mathfrak{g}$ does not admit any left-symmetric structure.*

The proof is based on the computation of the algebraic group cohomology $H^1(G_1, M)$ vanishing for certain $G_1$-modules $M$ of small dimension ( $G_1$ denotes the first Frobenius kernel of $G$ and $k$ is algebraically closed of characteristic $p > 0$ ). The results used here are due to J.C. Jantzen [18].
More can be proved for restricted structures (cf. Proposition 2.3.5.).

In section 3, we show that the result of Theorem 2.2.2. cannot be extended to nonrestricted simple Lie algebras $\mathfrak{g}$ of Cartan type: This follows from the existence of certain left-symmetric structures (so-called *adjoint* structures, see Definition 1.2.4.) on $\mathfrak{g}$, which are in a one-to-one correspondence with nonsingular derivations of $\mathfrak{g}$.
The class of simple Lie algebras over $k$ possessing *nonsingular* derivations does not contain the restricted Lie algebras, since any restrictable Lie algebra which possesses a nonsingular derivation is nilpotent (cf. Winter [36], Cor. 4, p. 140. In characteristic 0 this has been proved by Jacobson [17].)
In the nonrestricted case, however, there are for all $p > 0$ *simple* Lie algebras over any field of characteristic $p > 0$ possessing nonsingular derivations, e.g. a simple Lie algebra $\mathcal{L}(G, \delta, f)$ of R. Block [3] of dimension $p^n - 1$.
Consequently an $\mathcal{L}(G, \delta, f)$ admits adjoint left-symmetric structures for all characteristics $p > 0$.
The automorphism group of the left-symmetric algebras corresponding to the adjoint structure on $\mathfrak{g}$ (induced by a nonsingular derivation $D$) can be described as the subgroup of $\mathrm{Aut}(\mathfrak{g})$ which consists of the Lie automorphisms $\varphi$ with $D\varphi = r\varphi D$, where $r^{\dim \mathfrak{g}} = 1$ (cf. Proposition 1.2.5.). Thus it may be possible to realize interesting finite groups as automorphism groups of left-symmetric algebras.
Finally, some additional results are stated for $p = 2$ ( cf. Proposition 3.2.2. and Example 2.3.6. ).

We give a brief description of the ties between left-symmetric structures and affine manifolds.
Let $\mathbb{K}$ be the field of real or complex numbers. Left-symmetric algebras were first introduced in the theory of convex homogeneous cones. E.B. Vinberg [34] established a one-to-one correspondence between all convex homogeneous cones and so-called compact left-symmetric algebras (Koecher used semisimple Jordan algebras for selfadjoint homogeneous cones). There is a large literature on left-symmetric algebras (see [7], [13], [14], [23], [27], [34] and the references cited there).
Let $M = M^n$ be a manifold with coordinate atlas such that the coordinate changes are locally affine. Such a structure is called *affine structure* on $M$, and $M$ is called affine manifold. $M$ is smooth. There is a natural correspondence between affine structures on $M$ and flat torsionfree affine connections on $M$.
In the context of affine manifolds it is natural to ask which Lie groups $G$ admit complete left-invariant affine structures. Let $\mathfrak{g}$ be the Lie algebra of left-invariant vector fields on $G$:
*There is an isomorphism between the categories of left-symmetric structures on* $\mathfrak{g}$ *and simply connected Lie groups* $G$ *with left-invariant structures* (Goldman [13]).
Under this isomorphism the associative structures correspond to bi-invariant affine structures.
Let $\nabla$ be the flat torsionfree affine connection on $G$ corresponding to a given left-invariant affine structure. Since the connection is left-invariant, for any two left-invariant vector fields $X, Y \in \mathfrak{g}$, the covariant derivative $\nabla_X Y \in \mathfrak{g}$ is left-invariant. It follows that covariant differentiation $(X, Y) \mapsto \nabla_X Y$ defines a bilinear multiplication $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(X, Y) \mapsto XY$ in short. Since $\nabla$ is locally flat, the condition that $\nabla$ has

zero torsion is $XY - YX = [X,Y]$ and the condition that $\nabla$ has zero curvature is $X(YZ) - Y(XZ) = (XY - YX)Z$ , which is equivalent to the left-symmetric property (0.1). This yields the left-symmetric structure on $\mathfrak{g}$ . Conversely one associates to a left-symmetric structure on $\mathfrak{g}$ a left-invariant affine structure on $G$ (cf. [23], [27]).

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# 1. PRELIMINARIES

## 1.1. Ordinary and restricted Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra over $k$ with universal enveloping algebra $U(\mathfrak{g})$ and $V$ be a $\mathfrak{g}$-module (also regarded as left $U(\mathfrak{g})$-module).
The ordinary Lie algebra cohomology with coefficients in $\mathfrak{g}$-modules $V$ usually is defined by means of the chain complex $\mathrm{Hom}_k(\Lambda^\bullet\mathfrak{g}, V)$ , where the $n$ -cochains are interpreted as $k$ -linear alternating $n$ -multilinear functions $f : \mathfrak{g} \times \ldots \times \mathfrak{g} \to V$ . The coboundary operator $d_n$ satisfies $d_n \circ d_{n-1} = 0$ and thus it makes sense to define the $n$ -th cohomology group of $\mathfrak{g}$ by $H^n(\mathfrak{g}, V) := \ker d_n / \operatorname{im} d_{n-1}$ .
The spaces $\ker d_n$ and $\operatorname{im} d_{n-1}$ are called the space of $n$ -cocycles and $n$ -coboundaries respectively. From the explicit definition of $d_n$ (cf. [16], p.94) it follows that the 1-cocycle condition is

$$f([x,y]) = x.f(y) - y.f(x) \quad \forall\, x, y \in \mathfrak{g} \tag{1}$$

while the 1-coboundary condition is that

$$f(x) = x.e \quad \text{for some} \quad e \in V \ , \tag{2}$$

where the dot denotes the module product.

Lie algebra cohomology can be handled without cocycles and coboundaries as follows: One starts with a projective resolution of $k$ , regarded as a trivial $U(\mathfrak{g})$-module

$$0 \longleftarrow k \xleftarrow{\varepsilon} X_0 \xleftarrow{\partial_1} X_1 \xleftarrow{\partial_2} \ldots$$

With this resolution and with a given $\mathfrak{g}$ -module $V$ one may associate the additive groups $V_i = \mathrm{Hom}_{U(\mathfrak{g})}(X_i, V)$ . Composition with the maps $\partial_i$ gives rise to a complex

$$V_0 \xrightarrow{\delta_0} V_1 \xrightarrow{\delta_1} V_2 \xrightarrow{\delta_2} \ldots \quad , \text{ i.e. } \delta_n \circ \delta_{n-1} = 0 \text{ for all } n \ .$$

It follows from general principles (cf. [9], [10]) that the groups associated to this complex are independent of the particular projective resolution chosen.
The groups $H^n(\mathfrak{g}, V)$ defined in terms of cocycles and coboundaries can be identified with those obtained by this latter process from a particular free resolution of $k$ as $U(\mathfrak{g})$-module. Whenever one has an associative algebra $U$ over $k$ and an augmentation $\varepsilon$ :

$U \to k$ which is a homomorphism of $U$-modules one can use the above procedure (taking $X_0 := U$) to define cohomology groups $H^n(U, V)$ with coefficients in left $U$-modules. One has $H^n(\mathfrak{g}, V) = H^n(U(\mathfrak{g}), V)$ and $H^n(\mathfrak{g}, V) = Ext^n_{\mathfrak{g}}(k, V)$ .

Now let $k$ be a field of characteristic $p > 0$ and $(\mathfrak{g}, [p])$ a *restricted* Lie algebra with restricted universal enveloping algebra $u(\mathfrak{g})$ (cf. Strade, Farnsteiner [32] for definitions). One defines the *restricted cohomology groups* of $(\mathfrak{g}, [p])$ with coefficients in a restricted $\mathfrak{g}$-module $M$ by means of

$$H^n_*(\mathfrak{g}, M) := H^n(u(\mathfrak{g}), M) \ .$$

Interpretations of $H^n_*(\mathfrak{g}, M)$ have been first given by Hochschild. The $H^n_*(\mathfrak{g}, M)$ can be interpreted as the extension groups of the trivial $\mathfrak{g}$-module $k$ and of $M$ in the category of all restricted $\mathfrak{g}$-modules, cf. Jantzen [20].
The study of restricted Lie algebra cohomology has proved more useful than the ordinary one : Since the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of an algebraic group $G$ is restricted, methods from representation theory of algebraic groups are available. In particular, one can use these techniques to compute the Hochschild cohomology groups of algebraic groups with coefficients in rational $G$-modules (see [18], [19] and section 1.3.):
The representation theory of $G_1$ , the first Frobenius kernel of $G$ , is equivalent to that of $\mathrm{Lie}(G)$ as a restricted Lie algebra ([19], p.150), i.e.,

(1) The Hochschild cohomology groups $H^n(G_1, M)$ of a $G_1$-module $M$ and the restricted cohomology of the corresponding $\mathfrak{g}$-module coincide.

If $\mathfrak{g}$ is restricted , the ordinary cohomology is trivial for nonrestricted simple $\mathfrak{g}$-modules:

LEMMA 1.1.1 ([11], p. 131). *Let $\mathfrak{g}$ be a restricted Lie algebra and suppose that an irreducible $\mathfrak{g}$-module $V$ is not restricted. Then $H^\bullet(\mathfrak{g}, V)$ is trivial.*

In order to prove Theorem 2.2.2. it is necessary to compute $H^1(\mathfrak{g}, V)$ for certain left-symmetric $\mathfrak{g}$-modules, which are defined in 1.2. . We need the following (cf. [19], I.9.19):

(2) The first Lie algebra cohomology $H^1(\mathfrak{g}, E)$ coincides with the first restricted Lie algebra cohomology $H^1_*(\mathfrak{g}, E)$ for simple restricted modules (except for the trivial module in case $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ ).

### 1.2. Left-symmetric $\mathfrak{g}$ -modules

Let $(x, y) \mapsto x.y$ be a left-symmetric structure on $\mathfrak{g}$ over $k$ . Denote by $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ , $x \mapsto \lambda_x$ (resp. $\varrho$, $x \mapsto \varrho_x$ ) the operation of left- (resp. right-) multiplication, where $\lambda_x(y) = x.y$ and $\varrho_x(y) = y.x$ .
One has $\lambda_x - \varrho_x = \mathrm{ad}\, x \quad \forall\, x \in \mathfrak{g}$ by (0.2). Thus (0.1) is equivalent to

$$\lambda_{[x,y]} = [\lambda_x, \lambda_y] \quad \forall\, x, y \in \mathfrak{g} \tag{1.2.1}$$

i.e. $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a representation of $\mathfrak{g}$ .

Denote by $M_\lambda$ the corresponding $\mathfrak{g}$-module (the module product is given by the left-symmetric product); $M_\lambda$ and $\mathfrak{g}$ are identical as vector spaces.
The identity map $I : \mathfrak{g} \to M_\lambda$ defines a 1-cocycle of the Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module $M_\lambda$ :

$$I(x).y - I(y).x = I([x,y]) \ , \quad \text{i.e.} \quad I \in Z^1(\mathfrak{g}, M)$$

DEFINITION 1.2.1. Let $M$ be a $\mathfrak{g}$-module structure on $\mathfrak{g}$ such that the identity map $I : \mathfrak{g} \to M$ is in $Z^1(\mathfrak{g}, M)$ . Then $M$ is called a *left-symmetric* $\mathfrak{g}$-module.

Left-symmetric structures on $\mathfrak{g}$ correspond bijectively to left-symmetric $\mathfrak{g}$-modules $M_\lambda$. But left-symmetry of $\mathfrak{g}$-modules is not an invariant under $\mathfrak{g}$-module isomorphisms. More precisely the following holds:

LEMMA 1.2.2. *Let $M_\lambda$ and $M_\theta$ be $\mathfrak{g}$-module structures on $\mathfrak{g}$ and $\psi : M_\lambda \to M_\theta$ be a $\mathfrak{g}$-module isomorphism. Then one has:*

$$M_\lambda \text{ is left-symmetric} \iff \psi \in Z^1(\mathfrak{g}, M_\theta) \tag{1.2.2}$$

*Proof:* One has the implication

$$\phi \in Z^1(\mathfrak{g}, M_\lambda) \Longrightarrow \psi\phi \in Z^1(\mathfrak{g}, M_\theta) \quad \forall\, \phi \in \mathrm{End}(\mathfrak{g}) \tag{1.2.3}$$

since $\psi$ is a $\mathfrak{g}$-module homomorphism, and also

$$\phi \in Z^1(\mathfrak{g}, M_\theta) \Longrightarrow \psi^{-1}\phi \in Z^1(\mathfrak{g}, M_\lambda) \quad \forall\, \phi \in \mathrm{End}(\mathfrak{g}) \tag{1.2.4}$$

If $M_\lambda$ is left-symmetric, i.e. $I \in Z^1(\mathfrak{g}, M_\lambda)$ , then (1.2.3) implies $\psi \in Z^1(\mathfrak{g}, M_\theta)$ . Conversely, using (1.2.4), $\psi \in Z^1(\mathfrak{g}, M_\theta)$ implies $I = \psi^{-1}\psi \in Z^1(\mathfrak{g}, M_\lambda)$ , hence $M_\lambda$ is left-symmetric. □

If $\mathfrak{g}$ possesses nonsingular 1-cocycles in $M_\theta$ , this allows one to construct left-symmetric structures on $\mathfrak{g}$ :

COROLLARY 1.2.3. *Let $M_\theta$ be a $\mathfrak{g}$-module structure on $\mathfrak{g}$ and assume that there exists an invertible $\psi \in Z^1(\mathfrak{g}, M_\theta)$ . Define $\lambda_x$ by*

$$\lambda_x = \psi^{-1}\theta_x\,\psi \quad \forall\, x \in \mathfrak{g} \tag{1.2.5}$$

*Then $M_\lambda$ is a left-symmetric $\mathfrak{g}$-module and $\psi : M_\lambda \to M_\theta$ is a $\mathfrak{g}$-module isomorphism.*

*Proof:* Condition (1.2.5) is equivalent to $\psi(x.y) = x_*\psi(y) \quad \forall\, x, y \in \mathfrak{g}$ , where $\lambda_x(y) = x.y$ and $\theta_x(y) = x_*y$ . Hence $\psi$ is a $\mathfrak{g}$-module isomorphism.
But $\psi \in Z^1(\mathfrak{g}, M_\theta)$ implies $I \in Z^1(\mathfrak{g}, M_\lambda)$ by (1.2.4), hence $M_\lambda$ is left-symmetric by (1.2.2). □

Consider the special case where $M_\theta = M_{\mathrm{ad}}$ :

DEFINITION 1.2.4. A left-symmetric structure on $\mathfrak{g}$ is called *adjoint* structure, if the corresponding left-symmetric module $M_\lambda$ is isomorphic to $M_{\text{ad}}$, the module of the adjoint representation of $\mathfrak{g}$.

Note that the module $M_{\text{ad}}$ itself is not left-symmetric (assuming that $\mathfrak{g}$ is non-abelian), since $[x,y] = \lambda_x(y) - \lambda_y(x) = [x,y] - [y,x] = 2[x,y]$ is impossible. One has

PROPOSITION 1.2.5. *A Lie algebra $\mathfrak{g}$ admits adjoint structures $\mathcal{A}_D$ if and only if $\mathfrak{g}$ possesses a nonsingular derivation $D$. In this case the left-symmetric structure on $\mathfrak{g}$ is given by*

$$x.y = D^{-1}([x, D(y)]) \quad \forall\, x, y \in \mathfrak{g}, \tag{1.2.6}$$

*$D : M_\lambda \to M_{\text{ad}}$ is a $\mathfrak{g}$-module isomorphism and $\lambda_x(y) = x.y$.*
*If $\mathfrak{g}$ is simple then the automorphism group of the left-symmetric algebra $\mathcal{A}_D$ is the subgroup of $Aut(\mathfrak{g})$ consisting of the Lie automorphisms $\varphi$ with $\varphi D = rD\varphi$, where $r^{\dim \mathfrak{g}} = 1$.*

*Proof:* Adjoint left-symmetric structures (with $\mathfrak{g}$-module $M_\lambda$) correspond to $\mathfrak{g}$-module isomorphisms $\psi : M_\lambda \to M_{\text{ad}}$.
Since $M_\lambda$ is left-symmetric, Lemma 1.2.2. implies $\psi \in Z^1(\mathfrak{g}, M_{\text{ad}})$. The cocycle condition for $\psi$ is $\psi([a,b]) = [a, \psi(b)] - [b, \psi(a)]$, thus one has $Z^1(\mathfrak{g}, M_{\text{ad}}) = \text{Der}(\mathfrak{g})$ and $\psi$ is an invertible derivation of $\mathfrak{g}$.
Conversely, by Corollary 1.2.3., any nonsingular derivation $D$ gives rise to an adjoint left-symmetric structure (take $\theta = \text{ad}$) and (1.2.5) implies (1.2.6).
Let $(\mathcal{A}, \lambda)$ be a left-symmetric algebra with underlying Lie algebra $\mathfrak{g}$ (and vector space $V$ of dimension $n$). By definition

$$\begin{aligned}
(i)&\quad \text{Aut}(\mathcal{A}) = \{\varphi \in GL(V) \mid \varphi\lambda_x\varphi^{-1} = \lambda_{\varphi(x)} \;\; \forall\, x \in V\}\\
(ii)&\quad \text{Aut}(\mathfrak{g}) = \{\varphi \in GL(V) \mid \varphi \,\text{ad}\, x\, \varphi^{-1} = \text{ad}\, \varphi(x) \;\; \forall\, x \in V\}\\
(iii)&\quad \text{Aut}(\mathcal{A}) \text{ is a subgroup of } \text{Aut}(\mathfrak{g}).
\end{aligned}$$

Since $[x,y] = x.y - y.x$, $\varphi \in \text{Aut}(\mathcal{A})$ satisfies $\varphi([x,y]) = \varphi(x.y - y.x) = \varphi(x).\varphi(y) - \varphi(y).\varphi(x) = [\varphi(x), \varphi(y)]$. So *(iii)* follows.
Now let $\psi \in \text{Aut}\,(\mathcal{A}_D)$, i.e. $\psi\lambda_x\psi^{-1} = \lambda_{\psi(x)}$ $\forall x$, where $\lambda_x = D^{-1}\text{ad}\, x\, D$ and $\text{ad}\, x = \psi^{-1}\circ \text{ad}\, \psi(x) \circ \psi$ by *(iii)* and *(ii)*. These equations combined yield

$$F \circ \text{ad}\, x = \text{ad}\, x \circ F \quad \forall\, x \in \mathfrak{g}\,; \qquad F := \psi^{-1} D \psi D^{-1}.$$

Since $\mathfrak{g}$ is simple, the adjoint representation $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is irreducible and thus Schur's lemma implies that $F = r \cdot \text{id}_{|V}$. But $\det(F) = 1$ and so $r^n = 1, \psi D = rD\psi$. $\square$

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over a field $k$ of characteristic $p > 0$.

DEFINITION 1.2.6. A left-symmetric structure on $(\mathfrak{g}, [p])$ is called *restricted*, if the corresponding $\mathfrak{g}$-module $M_\lambda$ is a restricted module, i.e., $\lambda_{x^{[p]}} = \lambda_x^p \quad \forall\, x \in \mathfrak{g}$.

Restricted structures remain restricted under left-symmetric algebra isomorphisms, if $\mathfrak{g}$ has trivial center (see [7]).
The next results relate left-symmetric structures on $\mathfrak{g}$ to the first Lie algebra cohomology:

LEMMA 1.2.7. *Let $\mathfrak{g}$ be any nonzero Lie algebra over a field of characteristic $p \geq 0$ such that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ . Assume that $\mathfrak{g}$ admits a left-symmetric structure $M_\lambda$ .*

*Then*

(1) $\operatorname{tr} \lambda_x = 0$ *and* $\operatorname{tr} \varrho_x^n = 0 \quad \forall\, x \in \mathfrak{g}$ *and* $n \in \mathbb{N}$ .
(2) $\varrho_x$ *is nilpotent* $\forall\, x \in \mathfrak{g}$ , *if* $p = 0$ *or* $p > \dim \mathfrak{g}$ .
(3) *If there is an* $e \in \mathfrak{g}$ *with* $\lambda_e = \operatorname{id}$ *or* $\varrho_e = \operatorname{id}$ *then* $p > 0$ *and* $p \mid \dim \mathfrak{g}$ .
(4) *If* $\mathfrak{g}$ *is semisimple (i.e., the only abelian ideal* $\mathfrak{h}$ *in* $\mathfrak{g}$ *is* $\mathfrak{h} = 0$ *) then there is no* $e \in \mathfrak{g}$ *with* $\lambda_e = \operatorname{id}$ .

*Proof:* $\operatorname{tr} \lambda_{[x,y]} = \operatorname{tr}([\lambda_x, \lambda_y]) = 0$ and $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ imply $\operatorname{tr} \lambda_x = 0 \quad \forall\, x \in \mathfrak{g}$ . Similarly $\varrho_{[x,y]} = \lambda_{[x,y]} - \operatorname{ad}\,[x,y] = [\lambda_x, \lambda_y] - [\operatorname{ad} x, \operatorname{ad} y]$ gives $\operatorname{tr} \varrho_x = 0 \quad \forall\, x.$
(0.1) and (0.2) imply

$$\varrho_{y.z} - \varrho_z \circ \varrho_y = [\lambda_y, \varrho_z] \quad \forall\, y, z \in \mathfrak{g}. \tag{1.2.7}$$

Putting $y = z = x$ one has $\varrho_x^2 = \varrho_{x.x} - [\lambda_x, \varrho_x]$ and hence $\operatorname{tr} \varrho_x^2 = \operatorname{tr} \varrho_{x.x} = 0 \quad \forall\, x \in \mathfrak{g}.$ By induction (use $\operatorname{tr}(A[B,C]) = \operatorname{tr}([A,B]C)$ for endomorphisms $A, B, C$ ) one obtains $\operatorname{tr} \varrho_x^n = 0 \;\; \forall\, x \in \mathfrak{g}$ . This proves (1).
Let $A \in M_n(k)$ be a matrix with $\operatorname{tr}(A^k) = 0$ for $k = 1, \dots, n$ . Then $A$ is nilpotent (cf. [16] p.43) if $p = 0$ . This is not true in general for $p > 0$ . (Take $A = I_p$ ):
The coefficients of the characteristic polynomial $\sum_{j=0}^{n} (-1)^j \omega_j(A) t^{n-j}$ of $A$ satisfy the following identity:

$$\sum_{i=1}^{j} (-1)^{i+1} \operatorname{tr}(A^i) \omega_{j-i}(A) = j \cdot \omega_j(A)$$

where $\omega_0(A) = 1$ and $\omega_{n+j}(A) = 0 \;\; \forall\, j \in \mathbb{N}.$ If $p > n$ this implies $\omega_j(A) = 0 \;\; \forall\, j \geq 1$ and hence $A$ has characteristic polynomial $t^n$ .
(2) follows with $A = \varrho_x$ and (1). By (1) one has $0 = \operatorname{tr} \lambda_e = \operatorname{tr} \operatorname{id}_{|\mathfrak{g}} = \dim \mathfrak{g}$ or $0 = \operatorname{tr} \varrho_e = \dim \mathfrak{g}$. So (3) follows.
Assume that there is an $e \in \mathfrak{g}$ with $\lambda_e = \operatorname{id}_{|\mathfrak{g}}$ . Then $\mathfrak{h} = \operatorname{Ker}(\lambda)$ is a Lie ideal, since $\lambda$ is a representation of $\mathfrak{g}$ . One has $[\mathfrak{h},\mathfrak{h}] = 0$ and thus $\mathfrak{h} = 0$ , since $\mathfrak{g}$ is semisimple; $\lambda_{[e,x]} = [\lambda_e, \lambda_x] = [\operatorname{id}, \lambda_x] = 0$ implies $[e, \mathfrak{g}] \subset \mathfrak{h} = 0$ , thus $e \in Z(\mathfrak{g})$ . Likewise the center is zero, hence $\lambda_e = 0$ which is a contradiction. □

PROPOSITION 1.2.8. *Let $\mathfrak{g}$ be a Lie algebra over $k$ such that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ and assume that $\mathfrak{g}$ admits a left-symmetric structure $M_\lambda$ . Then $H^1(\mathfrak{g}, M_\lambda) = 0$ implies $p > 0$ and $p \mid \dim \mathfrak{g}$ .*

*Proof:* Assume $H^1(\mathfrak{g}, M_\lambda) = 0$ . Since $I \in Z^1(\mathfrak{g}, M_\lambda)$ , the identity map is a 1-coboundary (see 1.1. (1) and (2)), i.e. there is an $e \in \mathfrak{g}$ such that $\varrho_e = \operatorname{id}_{|\mathfrak{g}}$ .
By Lemma 1.2.7.(3), one has $p > 0$ and $p \mid \dim \mathfrak{g}$ . □

## 1.3. $H^1(G_1, M)$ for simple $G_1$-modules $M$

Let $G$ be a connected semisimple algebraic group over an algebraically closed field $k$ of prime characteristic $p > 0$. Assume that $G$ is almost simple and simple connected. Let $G_1$ be the first Frobenius kernel of $G$ and $T$ a maximal torus in $G$.
Denote the *group of characters* of $T$ by $X(T)$ and the *root system* by $R \subset X(T)$. Choose a set of *positive* roots $R^+$ in $R$. Denote the *simple* roots by $\alpha_1, \ldots, \alpha_n$ and the *fundamental weights* by $\omega_1, \ldots, \omega_n$, where $n$ is the rank of $T$. Let $X(T)_+$ be the set of *dominant* weights and $X_1(T)$ the set of *restricted* dominant weights, i.e., the set of all $\sum_{i=1}^n r_i\omega_i$ with $r_i \in \mathbb{Z}$ and $0 \leq r_i < p$ for all $i$.
Consider on $X(T)$ the usual order relation where $\lambda \leq \mu$ holds if and only if there are integers $r_i \geq 0$ with $\mu - \lambda = \sum_{i=1}^n r_i\alpha_i$.
Let $B \supset T$ be the *Borel subgroup* of $G$ corresponding to the negative roots and let $U$ be the *unipotent radical* of $B$. Each $\lambda \in X(T)$ defines a one dimensional module $k_\lambda$ of $B$ via the isomorphism $B/U \simeq T$. The induced module $H^0(\lambda) := \operatorname{ind}_B^G k_\lambda$ is nonzero if and only if $\lambda \in X(T)_+$. In this case the socle $L(\lambda)$ of $H^0(\lambda)$ is the *simple* $G$*-module with highest weight* $\lambda$.
(It also can be constructed as the unique simple image of the Weyl module $V(\lambda)$.)
The simple $G$-modules (up to isomorphism) correspond bijectively to the $L(\lambda)$ with $\lambda \in X(T)_+$ (cf. [19], p.197). Now the $L(\lambda)$ with $\lambda \in X_1(T)$ remain simple under restriction to $G_1$. One has

(1) Any simple $G_1$-module is isomorphic to exactly one $L(\lambda)$ with $\lambda \in X_1(T)$.

To compute the $H^1(G_1, L(\lambda))$ for $\lambda \in X_1(T)$, Jantzen [18] uses the isomorphism $H^1(G_1, H^0(\lambda))^{(-1)} \simeq \operatorname{ind}_B^G(H^1(B_1, k_\lambda)^{(-1)})$ (cf. [19], $II.12.2(2)$ ) to get first information on the cohomology of the $H^0(\lambda)$ and then on the cohomology of the $L(\lambda)$ by looking at the obvious long exact sequence.
Let $\tilde{\alpha}$ be the largest root in $R$. The Weyl module $V(\tilde{\alpha})$ is just $\mathfrak{g}$ with the adjoint representation and its submodules are the ideals. Thus $L(\tilde{\alpha}) = M_{\mathrm{ad}}$, if $\mathfrak{g}$ is simple. If $R$ has two root length, let $\alpha_0$ be the largest short root.
The following propositions can be easily derived from [18] (cf. Prop. 6.2.,6.4.,6.5.,6.6.,6.7.).

PROPOSITION 1.3.1. *Let*

| | | | |
|---|---|---|---|
| $(a)$ | $p > 2,$ | *if* $\mathfrak{g}$ *is of type* | $A_l, B_l, C_l, D_l, E_7$ |
| $(b)$ | $p > 3,$ | *if* $\mathfrak{g}$ *is of type* | $G_2, F_4, E_6$ |
| $(c)$ | $p \not| l+1,$ | *if* $\mathfrak{g}$ *is of type* | $A_l$ |
| $(d)$ | $p \not| l,$ | *if* $\mathfrak{g}$ *is of type* | $C_l$ |

*and assume that* $\lambda$ *is one of the following: zero, a minuscule weight,* $\alpha_0$ *or* $\tilde{\alpha}$.
*Then* $H^1(G_1, L(\lambda)) = 0$.

PROPOSITION 1.3.2. *Let* $R$ *be of type* $A_l$ *and suppose* $p > 2$. *The* $r$*-th symmetric power of the natural representation (resp. of its dual) of* $G = SL_{l+1}(k)$ *is isomorphic to* $H^0(r\omega_1)$ *(resp. to* $H^0(r\omega_l)$ *). It is irreducible for* $r < p$ *and one has:*

(a) $H^1(G_1, L(r\omega_1)) = 0$ *if* $R$ *is of type* $A_1,\ 0 \leq r < p$ *and* $r \neq p-2$
(b) $H^1(G_1, L(r\omega_1)) = H^1(G_1, L(r\omega_l)) = 0$ *if* $R$ *is of type* $A_l,\ l \geq 2$ *and* $0 \leq r < p$ .

Table I shows the possible $\lambda$ which occur in Proposition 1.3.1. (see [6] VI, p.232).

TABLE I

| type | minuscule weights | largest root $\tilde{\alpha}$ | largest short root $\alpha_0$ |
|---|---|---|---|
| $A_l,\ l \geq 1$ | $\omega_1, \omega_2, ..., \omega_l$ | $\omega_1 + \omega_l$ | – |
| $B_l,\ l \geq 3$ | $\omega_l$ | $\omega_2$ | $\omega_1$ |
| $C_l,\ l \geq 2$ | $\omega_1$ | $2\omega_1$ | $\omega_2$ |
| $D_l,\ l \geq 4$ | $\omega_1, \omega_{l-1}, \omega_l$ | $\omega_2$ | – |
| $G_2$ | – | $\omega_2$ | $\omega_1$ |
| $F_4$ | – | $\omega_1$ | $\omega_4$ |
| $E_6$ | $\omega_1, \omega_6$ | $\omega_2$ | – |
| $E_7$ | $\omega_7$ | $\omega_1$ | – |
| $E_8$ | – | $\omega_8$ | – |

## 2. CLASSICAL LIE ALGEBRAS

### 2.1. Left-symmetric structures on $\mathfrak{sl}(2,k)$

Let $k$ be a field of characteristic $p > 2$ and $\mathfrak{g} := \mathfrak{sl}(2,k)$ with standard basis $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $[x,y] = z,\ [z,x] = 2x,\ [z,y] = -2y$ .

PROPOSITION 2.1.1. *The classical simple Lie algebra* $\mathfrak{sl}(2,k)$ *admits left-symmetric structures if and only if* $p = 3$ .

*Proof:* If $p = 3$ then $\mathfrak{g}$ admits left-symmetric structures (see Introduction). Assume now, that $\mathfrak{g}$ admits a left-symmetric structure $M_\lambda$ and that $p > 3$ . Since every left-symmetric structure over $k$ can be regarded as defined over the algebraic closure of $k$ , it suffices to prove the following:

(1) Let $k$ be an algebraically closed field of characteristic $p > 3$ and $M$ be a 3-dimensional $\mathfrak{g}$-module. Then $H^1(\mathfrak{g}, M) = 0$ .

For $M = M_\lambda$ Proposition 1.2.8. now implies $p \mid \dim \mathfrak{g} = 3$ , which contradicts $p > 3$ .

The following result for $\mathfrak{g} := \mathfrak{sl}(2,k)$ is well known (cf.[11], Theorem 4) :
If $k$ is algebraically closed and $E$ is an *irreducible* $\mathfrak{g}$-module then

(2) $$H^1(\mathfrak{g}, E) \simeq \begin{cases} k \oplus k \ , & \text{if } \dim E = p-1 \\ 0 & \text{otherwise} \end{cases}$$

In order to prove (1) , consider a composition series for $M$ . All irreducible composition factors are of dimension less than 3 . Since $p-1>3$ they have trivial 1-cohomology by (2). Use the long exact sequence to obtain the result. □

REMARK 2.1.2. *Let $k$ be an algebraically closed field of characteristic 3 . Then it is possible to classify all left-symmetric structures on $\mathfrak{sl}(2,k)$ (see [7] ). Two important examples are the following structures which are defined by the matrices $\lambda_x$ , $\lambda_y$ and $\lambda_z$ as follows:*

$$(i) \quad \begin{pmatrix} 0 & u & 1 \\ 0 & w & -1 \\ 1 & w^2+1 & -w \end{pmatrix} \quad \begin{pmatrix} u & u-vw^2 & vw \\ w & v & 1 \\ w^2 & -vw & -u-v \end{pmatrix} \quad \begin{pmatrix} 0 & vw & -v \\ -1 & -1 & 0 \\ -w & -u-v & 1 \end{pmatrix}$$

$$(ii) \quad \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ 0 & -1-\gamma^{-1} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \\ 1-\gamma^{-1} & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \gamma-1 & 0 & 0 \\ 0 & \gamma+1 & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

*where $w := v-u$ and $u, v \in k$ , $\gamma \in k^\times$ .*

## 2.2. Proof of the main theorem

Let $G$ be a connected semisimple algebraic group of type $A_l$ $(l \geq 1), B_l$ $(l \geq 3), C_l$ $(l \geq 2), D_l$ $(l \geq 4), G_2, F_4, E_6, E_7, E_8$ over an algebraically closed field $k$ of characteristic $p>2$ . Assume that $\mathfrak{g} = \mathrm{Lie}(G)$ admits a left-symmetric structure $M_\lambda$ and suppose $p \not| \dim \mathfrak{g}$ . The last condition implies $p \not| \ l$ for $\mathfrak{g}$ of type $C_l$, since $\dim C_l = l(2l+1)$. Assume furthermore that

(i) $p > 3$ , if $G$ is of type $G_2, F_4, E_6$
(ii) $p \not| \ l+1$, if $G$ is of type $A_l$
(iii) $p \not| \ l$ , if $G$ is of type $C_l$

The main theorem has been proved for $\mathfrak{g}$ of type $A_1$ in Prop. 2.1.1. . The general case is treated similarly: *One shows that $H^1(\mathfrak{g}, E) = 0$ for all simple $\mathfrak{g}$-modules $E$ of dimension less than or equal to the dimension of $\mathfrak{g}$ which implies $H^1(\mathfrak{g}, M_\lambda) = 0$ .* This contradicts Prop. 1.2.8. and Theorem 2.2.2. is proven.
By Lemma 1.1.1. we may assume that $E$ is restricted. Thus it suffices to look at restricted cohomology only (see 1.1.(2)), i.e., to show that $H^1(G_1, L(\lambda)) = 0$ for all $\lambda \in X_1(T)$ with $\dim L(\lambda) \leq \dim G$ (see 1.1.(1) and 1.3.(1)). Thus Theorem 2.2.2. follows from

PROPOSITION 2.2.1. *Let $p>2$ and $G$ be of the above type such that $(i), (ii)$ and $(iii)$ hold. Then $H^1(G_1, L(\lambda)) = 0$ for all $\lambda \in X_1(T)$ with* $\dim L(\lambda) \leq \dim G$.

*Proof :* *Case 1:* Assume that $G$ is of type $A_l$ $(l \geq 2)$ and let $n := l+1$. One has $\dim G = n^2 - 1$ and $p \not| \ n$ . There are only a few $G_1$-modules $L(\lambda)$, $\lambda \in$

$X_1(T)$, (up to isomorphism) which are of dimension less than $n^2$. More precisely one has $\lambda \in \{0, \omega_i, 2\,\omega_1, 2\,\omega_{n-1}, \omega_1 + \omega_{n-1}\}$, $i$ subject to the condition $\binom{n}{i} < n^2$. This will be shown in the lemma below. According to Prop. 1.3.1. one has $H^1(G_1, L(\omega_i)) = H^1(G_1, L(\omega_1 + \omega_{n-1})) = 0$. Since $p > 2$, Prop. 1.3.2. implies $H^1(G_1, L(2\,\omega_1)) = H^1(G_1, L(2\,\omega_{n-1})) = 0$. It remains to prove

LEMMA 2.2.3. *Let $L(\lambda)$, $\lambda \in X_1(T)$, be a simple $G_1$-module for $G$ of type $A_l, l \geq 2$ and $p > 2$ , $p \not| n$ . Assume that* $\dim L(\lambda) < n^2$. *Then $L(\lambda)$ is isomorphic to one of the following $G_1$-modules:*

| $\lambda$ | $0$ | $\omega_1$ | $\omega_{n-1}$ | $\omega_2$ | $\omega_{n-2}$ | $2\,\omega_1$ | $2\,\omega_{n-1}$ |
|---|---|---|---|---|---|---|---|
| $L(\lambda)$ | $k$ | $V$ | $V^*$ | $\Lambda^2(V)$ | $\Lambda^2(V^*)$ | $S^2(V)$ | $S^2(V^*)$ |
| $\dim L(\lambda)$ | 1 | $n$ | $n$ | $\frac{n\,(n-1)}{2}$ | $\frac{n\,(n-1)}{2}$ | $\frac{n\,(n+1)}{2}$ | $\frac{n\,(n+1)}{2}$ |

| $l$ | $\forall l$ | 5 | 6 | 6 | 7 | 7 |
|---|---|---|---|---|---|---|
| $\lambda$ | $\omega_1 + \omega_{n-1}$ | $\omega_3$ | $\omega_3$ | $\omega_4$ | $\omega_3$ | $\omega_5$ |
| $L(\lambda)$ | $M_{\mathrm{ad}}$ | $\Lambda^3(V)$ | $\Lambda^3(V)$ | $\Lambda^3(V^*)$ | $\Lambda^3(V)$ | $\Lambda^3(V^*)$ |
| $\dim L(\lambda)$ | $n^2 - 1$ | 20 | 35 | 35 | 56 | 56 |

*Here $k$ denotes the trivial one-dimensional representation, $\Lambda^i(V) = L(\omega_i)$ the fundamental representation on the $i$-fold alternating power of the natural module $V = k^n$ of dimension $\binom{n}{i}$ , $S^2(V) = L(2\,\omega_1)$ the representation on the 2-fold symmetric power of the module $V$ and $L(\omega_1 + \omega_{n-1})$ the adjoint representation. One has $L(\omega_i)^* \simeq L(\omega_{n-i})$ where $L(\omega_i)^*$ denotes the $G_1$-module dual to $L(\omega_i)$ .*

*Proof:* Let $\mathcal{W}$ be the Weyl group of G and denote by $w_0$ the unique element of $\mathcal{W}$ of greatest length. Then for $\lambda \in X(T)_+$ one has $L(\lambda)^* \simeq L(-w_0\,\lambda)$ . For $G$ of type $A_l$ it follows $L(\omega_i)^* \simeq L(\omega_{n-i})$ and ${M_{\mathrm{ad}}}^* \simeq M_{\mathrm{ad}}$ .
Let $\lambda = \sum_{i=1}^{l} r_i \omega_i \ \in X_1(T), 0 \leq r_i < p$ be the highest weight of the simple $G_1$-module $L(\lambda)$ and assume

(1) $$\dim L(\lambda) < n^2.$$

$\mathcal{W}$ operates on the weights by conjugation and the $\mathcal{W}$-conjugates of a weight are weights. Denote the orbit of a weight $\nu$ under $\mathcal{W}$ by $\mathcal{W}\nu$ . One has to use that each dominant weight $\nu \leq \lambda$ is a weight of $L(\lambda)$ . (The fact is classical over $\mathbb{C}$ ; for fields of positive characteristic this has been shown in [33], $\lambda$ has to be a *restricted* highest weight.) Denote by $m$ the number of weights of $L(\lambda)$ . One has

(2) $$\dim L(\lambda) \geq m = \sum_{\nu \leq \lambda, \nu \text{ dominant}} \mid \mathcal{W}\nu \mid$$

Let $\nu = \sum_{i=1}^{l} m_i \, \omega_i$ be a dominant weight. The stabilizer of $\nu$ in $\mathcal{W}$ is generated by the simple reflections $s_i$ with $m_i = 0$. So $\text{stab}_{\mathcal{W}} \, \nu$ is a direct product of symmetric groups (for $G$ of type $A_l$ ) and it is easy to compute its order and hence to compute the order of $\mathcal{W}\nu$ . One has:

(3) If there is only one coordinate of $\nu$ different from zero (e.g. $m_j \neq 0, \; j < n$ ) then $\mid \mathcal{W}\nu \mid \geq n.$

(4) If there are at least two coordinates different from zero (e.g. $m_k \neq 0, m_j \neq 0$ , $1 \leq k < j < n$ ) then $\mid \mathcal{W}\nu \mid \geq n^2 - n$. Equality holds if $k = 1$ (or $k = n-2$ ) and $j = n-1$ and $\nu = m_1 \, \omega_1 + m_{n-1} \, \omega_{n-1}$.

If $m_j \neq 0$ then $\mid \mathcal{W}\nu \mid = \binom{n}{j} \geq n$ , and if $m_k \neq 0, m_j \neq 0$ then $\mid \mathcal{W}\nu \mid = \binom{n}{j} \cdot \binom{j}{k} \geq \binom{n}{2} \, 2 = n^2 - n$ since $j \geq 2$. So (3) and (4) follow.
We will show that $\lambda \in \{\omega_i, 2\,\omega_1, 2\,\omega_{n-1}, \omega_1 + \omega_{n-1}\}$, where $\binom{n}{i} < n^2$. In all other cases one obtains $\dim L(\lambda) \geq n^2$ by $(2), (3), (4)$.

*1. At least two coordinates of $\lambda$ are different from zero (e.g. $r_i$ and $r_j$ ):*

Assume first $\lambda = r_i \, \omega_i + r_j \, \omega_j$. If $\lambda$ is not in the root lattice then there is an $i$ with $\omega_i \leq \lambda$ , hence $\mid \mathcal{W}\lambda \mid \geq n^2 - n$ and $\mid \mathcal{W}\omega_i \mid \geq n$ by (3) and (4). This implies $\dim L(\lambda) \geq n^2$ by (2) contradicting (1) .
For $\lambda = \nu = \omega_1 + \omega_{n-1}$ , $L(\lambda)$ is the adjoint representation. Otherwise $\lambda \neq \omega_1 + \omega_{n-1}$ and $\nu \leq \lambda$ . It follows $\dim L(\lambda) \geq \mid \mathcal{W}\nu \mid + \mid \mathcal{W}\lambda \mid \geq 2(n^2 - n) \geq n^2$ which contradicts (1) . The same applies when $\lambda$ has more than two coordinates different from zero.

*2. One has $\lambda = r_i \omega_i$ for an $i$ :*

$\lambda$ lies in the root lattice. If $r_i = 1$, one obtains the fundamental representations of dimension $\binom{n}{i}$ . (1) is satisfied for $i = 1, 2, n-2, n-1$ for all $n$ and $(n, i) \in \{(6,3), (7,3), (7,4), (8,3), (8,5)\}$.
Assume now $r_i \geq 2$ :
Choose $\nu := \lambda - \alpha_i$ (note that $\alpha_i = 2\,\omega_1 - \omega_{i-1} - \omega_{i+1}$ ) and let $1 < i < n-1$. Then $\nu = \omega_{i-1} + (r_i - 2)\,\omega_i + \omega_{i+1}$ is dominant and one has $\mid \mathcal{W}\nu \mid \geq n^2 - n$ by (4) . Hence $\dim L(\lambda) \geq \mid \mathcal{W}\nu \mid + \mid \mathcal{W}\lambda \mid \geq n^2$ . It remains to look at the case $i = 1 (i = n-1$ is dual to it). For $r_1 = 2$ one has $\lambda = 2\,\omega_1$ (and $\lambda = 2\,\omega_{n-1}$ for $r_{n-1} = 2$ ). If $r_1 \geq 3$ then $\nu = (r_1 - 2)\omega_1 + \omega_2$ is dominant and as before $\dim L(\lambda) \geq n^2$ which contradicts (1) . □

*Case 2:* The proofs of Proposition 2.2.1. for $G$ of type $B_l, C_l, D_l, G_2, F_4, E_6, E_7$ or $E_8$ are very similar in each case, so we omit the proof of the following lemmas. In [25] all irreducible $G_1$-modules of dimension less than $\frac{1}{2}(\dim V)^2$ are classified, where $V$ denotes the natural module. The results are covered by our lemmas. Assume $p > 2$.

LEMMA 2.2.4. *Let* $L(\lambda),\ \lambda \in X_1(T)$, *be a simple* $G_1$*-module for* $G$ *of type* $B_l, l \geq 3$ *and* $\dim L(\lambda) \leq \frac{n(n-1)}{2}$ , *where* $n = 2l+1$. *Then* $L(\lambda)$ *is isomorphic to one of the following* $G_1$*-modules:*

| $l$ | $\forall l$ | $\forall l$ | $\forall l$ | $l = 3,4,5,6$ |
|---|---|---|---|---|
| $\lambda$ | $0$ | $\omega_1$ | $\omega_2$ | $\omega_l$ |
| $L(\lambda)$ | $k$ | $V$ | $M_{\mathrm{ad}}$ | $L(\omega_l)$ |
| $\dim L(\lambda)$ | $1$ | $n$ | $\frac{n\,(n-1)}{2}$ | $8, 16, 32, 64$ |

*$L(\omega_l)$ denotes the spin module of dimension $2^l$. One has $L(\omega_i)^* \simeq L(\omega_i)\ \forall i$.*

LEMMA 2.2.5. *Let* $L(\lambda), \lambda \in X_1(T)$, *be a simple* $G_1$*-module for* $G$ *of type* $C_l$, $l \geq 2$ , $p \not| \ l$ *and* $\dim L(\lambda) \leq \frac{n\,(n+1)}{2}$ , *where* $n = 2l$. *Then* $L(\lambda)$ *is isomorphic to one of the following* $G_1$*-modules:*

| $l$ | $\forall l$ | $\forall l$ | $p \not| \ l$ | $\forall l$ | $l = 3$ |
|---|---|---|---|---|---|
| $\lambda$ | $0$ | $\omega_1$ | $\omega_2$ | $2\,\omega_1$ | $\omega_3$ |
| $L(\lambda)$ | $k$ | $V$ | $\Lambda^2(V)$ | $M_{\mathrm{ad}}$ | $L(\omega_3)$ |
| $\dim L(\lambda)$ | $1$ | $n$ | $\frac{n\,(n-1)}{2} - 1$ | $\frac{n\,(n+1)}{2}$ | $14$ |

*One has $L(\omega_i)^* \simeq L(\omega_i)\ \forall i$.*

LEMMA 2.2.6. *Let* $L(\lambda),\ \lambda \in X_1(T)$, *be a simple* $G_1$*-module for* $G$ *of type* $D_l, l \geq 4$ *and* $\dim L(\lambda) \leq \frac{n(n-1)}{2}$ , *where* $n = 2l$. *Then* $L(\lambda)$ *is isomorphic to one of the following* $G_1$*-modules:*

| $l$ | $\forall l$ | $\forall l$ | $\forall l$ | $l = 3,4,5,6$ | $l = 3,4,5,6$ |
|---|---|---|---|---|---|
| $\lambda$ | $0$ | $\omega_1$ | $\omega_2$ | $\omega_l$ | $\omega_{l-1}$ |
| $L(\lambda)$ | $k$ | $V$ | $M_{\mathrm{ad}}$ | $L(\omega_l)$ | $L(\omega_{l-1})$ |
| $\dim L(\lambda)$ | $1$ | $n$ | $\frac{n\,(n-1)}{2}$ | $8, 16, 32, 64$ | $8, 16, 32, 64$ |

*Here $L(\omega_l)$ and $L(\omega_{l-1})$ are the spin modules of dimension $2^{l-1}$. One has $L(\omega_{l-1})^* \simeq L(\omega_l)$ and $L(\omega_i)^* \simeq L(\omega_i)$ for $i = 1, 2, .., l-2$.*

LEMMA 2.2.7. *Let $G$ be of type $G_2, F_4, E_6, E_7, E_8$ and $L(\lambda)$, $\lambda \in X_1(T)$, be a simple $G_1$-module with $\dim L(\lambda) \leq \dim G$. Then $L(\lambda)$ is isomorphic to one of the modules listed in table II:*

TABLE II

| type | $p$ | dim $G$ | $L(\lambda)$ | dim $L(\lambda)$ |
|---|---|---|---|---|
| $G_2$ | $p > 3$ | 14 | $k, L(\omega_1), L(\omega_2)$ | $1, 7, 14$ |
| $F_4$ | $p > 3$ | 52 | $k, L(\omega_4), L(\omega_1)$ | $1, 26, 52$ |
| $E_6$ | $p > 3$ | 78 | $k, L(\omega_1), L(\omega_6), L(\omega_2)$ | $1, 27, 27, 78$ |
| $E_7$ | $p > 2$ | 133 | $k, L(\omega_7), L(\omega_1)$ | $1, 56, 133$ |
| $E_8$ | $-$ | 248 | $k, L(\omega_8)$ | $1, 248$ |

*One has $L(\lambda)^* \simeq L(\lambda)$ except for $G$ of type $E_6$ where $L(\omega_6)^* \simeq L(\omega_1)$ and $L(\omega_2)^* \simeq L(\omega_2)$.*

All weights occuring in the lemmas can also be found in table I except for $\lambda = \omega_3$ and $G$ of type $C_3$. But in this case $H^0(\omega_3)$ is irreducible since $p > 2$. Hence $H^1(G_1, L(\omega_3)) = 0$ by 4.1. and 6.4. of [18]. Thus Proposition 2.2.1. follows from Proposition 1.3.1. and the main theorem is proven. □

## 2.3. Restricted structures

In Theorem 2.2.2. it remains open whether a classical simple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ admits left-symmetric structures in case $p \mid \dim G$. Let $G$ be of type

$$A_l\ (l \geq 2, l \neq 5, p \not| \ l+1),\ C_l\ (l \geq 2, p \not| \ l),\ D_l\ (l \geq 4),\ E_6, E_7, E_8. \qquad (2.3.1.)$$

In this section we answer the above question for *restricted* structures and for $G$ of type (2.3.1) by a simple argument: *If $p > 2$ then $\mathfrak{g}$ does not admit any restricted structure, see Prop.* 2.3.5. For $G$ of type $B_l\ (l \geq 3)$, $G_2$ and $F_4$ the argument fails. Example 2.3.6. shows that Proposition 2.3.5. is not valid for $p = 2$. Because $\mathfrak{g}$ is a restricted Lie algebra, any adjoint structure on $\mathfrak{g}$ is restricted.

LEMMA 2.3.1. *Let $G$ be a restrictable non-nilpotent Lie algebra over a field $k$. Then $\mathfrak{g}$ does not admit any adjoint structure.*

*Proof:* $\mathfrak{g}$ admits adjoint structures if and only if $\mathfrak{g}$ possesses a nonsingular derivation (see Prop. 1.2.5.). But in this case $\mathfrak{g}$ has to be nilpotent ([36], Cor.4). □

LEMMA 2.3.2. *Let $M_\lambda$ be a left-symmetric $\mathfrak{g}$-module such that $H^1(\mathfrak{g}, M_\lambda) = 0$. Then there is no decomposition like $M_\lambda = M \oplus k$, where $k$ denotes the trivial one-dimensional module.*

*Proof:* Assume that $M_\lambda = M \oplus k$ and $y \in M_\lambda$. $y$ can be written uniquely in the form $y = m + a$ where $m \in M, a \in k$. Let $\mathrm{ann}_\mathfrak{g}(y) := \{x \in \mathfrak{g} \mid x.y = 0\}$ be the annihilator of $y$ in $\mathfrak{g}$. $H^1(\mathfrak{g}, M_\lambda) = 0$ implies that the map $\psi : \mathfrak{g} \to M_\lambda$ defined by $x \mapsto x.y$ is bijective for some $y \neq 0$. One concludes that $\mathrm{ann}_\mathfrak{g}(y) = \ker \psi = 0$. Since $\mathfrak{g}$ operates trivially on $k$ one has $\mathrm{ann}_\mathfrak{g}(m) = \mathrm{ann}_\mathfrak{g}(y) = 0$. This contradicts the fact that the map $\mathfrak{g} \to M$, $x \mapsto x.m$ has non-trivial kernel (namely $\mathrm{ann}_\mathfrak{g}(m)$) since $\dim M < \dim M_\lambda = \dim \mathfrak{g}$. □

By a similar argument any $\mathfrak{g}$-module $N$ satisfying $\dim \mathrm{ann}_\mathfrak{g}(n) > \dim \mathfrak{g} - \dim N \quad \forall n \in N$ cannot be a direct summand of $M_\lambda$ in the situation of the above lemma.

LEMMA 2.3.3. *Let $M$ be a $\mathfrak{g}$-module structure on $\mathfrak{g}$ such that $H^1(\mathfrak{g}, E) = H^1(\mathfrak{g}, E^*) = 0$ for all composition factors $E$ of $M$. Then there is a decomposition like $M = N \oplus k \oplus \cdots \oplus k$ where all composition factors of $N$ are not isomorphic to $k$ ($M = N$ is permitted).*

*Proof:* Recall that $\mathrm{Ext}^1_\mathfrak{g}(k, M) \simeq H^1(\mathfrak{g}, M)$, $\mathrm{Ext}^1_\mathfrak{g}(M, k) \simeq \mathrm{Ext}^1_\mathfrak{g}(k, M^*) \simeq H^1(\mathfrak{g}, M^*)$ and that $\mathrm{Ext}^1_\mathfrak{g}(M, N)$ can be interpreted as the set of classes of equivalent extensions $0 \to N \to V \to M \to 0$. (The class of split extensions corresponds to the zero element in $\mathrm{Ext}^1_\mathfrak{g}(M, N)$. ) By assumption $\mathrm{Ext}^1_\mathfrak{g}(E, k) = \mathrm{Ext}^1_\mathfrak{g}(k, E) = 0$ for all composition factors $E$ of $M$. Hence $\mathrm{Ext}^1_\mathfrak{g}(M', k) = \mathrm{Ext}^1_\mathfrak{g}(k, M') = 0$ for all subquotients $M'$ of $M$ by the long exact sequence for the subquotients. If $k$ is not a composition factor, one has $M = N$ and the proof is finished. Otherwise there exist submodules $M' \supset M''$ of $M$ such that $M'/M'' \simeq k$. Since $\mathrm{Ext}^1_\mathfrak{g}(k, M'') = 0$ it follows $M' = M'' \oplus k$. Thus $k$ is a submodule of $M$. As before $\mathrm{Ext}^1_\mathfrak{g}(M/k, k) = 0$. One obtains $M = M/k \oplus k$ and the lemma follows by induction. □

The two preceding lemmas imply the following proposition:

PROPOSITION 2.3.4. *Let $M_\lambda$ be a left-symmetric $\mathfrak{g}$-module such that $H^1(\mathfrak{g}, M_\lambda) = 0$ and $H^1(\mathfrak{g}, E) = H^1(\mathfrak{g}, E^*) = 0$ for all composition factors of $M_\lambda$. Then none of the composition factors of $M_\lambda$ is isomorphic to $k$.*

PROPOSITION 2.3.5. *Let $G$ be of one of the types listed in* (2.3.1) *and $k$ a field of characteristic $p > 2$. Then $\mathfrak{g}$ does not admit any restricted structure.*

*Proof:* Assume that $M_\lambda$ is a restricted $\mathfrak{g}$-module:

*Case 1: $G$ is of type $A_l$*
The only $\mathfrak{g}$-modules which may occur as composition factors of $M_\lambda$ are listed in Lemma 2.2.3. (n=l+1). Since we may assume that $k$ is algebraically closed one has $H^1(\mathfrak{g}, E) = 0$

for all simple $\mathfrak{g}$-modules $E$ with $\dim E \leq \dim \mathfrak{g}$ and $H^1(\mathfrak{g}, M_\lambda) = 0$ by Prop. 2.2.1. and 1.1.(1),(2). . Thus we can exclude the trivial module $k$ as a composition factor of $M_\lambda$ by Prop. 2.3.4. . We can also exclude the adjoint module $M_{\text{ad}}$ by Lemma 2.3.1. . Hence all composition factors of $M_\lambda$ are isomorphic to $V, \Lambda^r(V), S^2(V)$ and their dual modules $(r \geq 2)$. However, the dimension of $M_\lambda$ is built up by the dimensions of these modules, i.e. $n^2 - 1 = \sum_i m_i$ , where $m_i \in \{n, \frac{n\,(n-1)}{2}, \binom{n}{r} < n^2\}$ .
If $n > 8$ one has $\binom{n}{r} \geq n^2$ for $r \geq 3$ and one obtains the equation $n^2 - 1 = \alpha n + \frac{\beta n\,(n-1)}{2} + \frac{\gamma n\,(n+1)}{2}$ where $n > 2$ and $\alpha, \beta, \gamma$ are nonnegative integers. Since this is equivalent to $2 = n\,(2n - 2\alpha - \beta(n-1) - \gamma(n+1))$ which has no integral solution for $n > 2$ one obtains a contradiction.
The case $n \leq 8$ is left to the interested reader (for $n = 6$ one has $\dim M_\lambda = 35$ and $\dim \Lambda^2(V) = 10$ , $\dim \Lambda^3(V) = 15$. )

*Case 2: $G$ is of type $B_l, C_l$ or $D_l$*
By the same procedure as in case 1, one determines the composition factors of $M_\lambda$ which may occur (cf. Lemma 2.2.4., 2.2.5., 2.2.6.). One excludes the trivial and the adjoint module by Prop. 2.2.1., 2.3.4. and Lemma 2.3.1. . The dimensions $m_i$ of the remaining composition factors can be read off table III. We omit for convenience some cases where $l$ is small (see table III). But it can be easily checked to be correct.

For $G$ of type $D_l$ one obtains $\frac{n\,(n-1)}{2} = \alpha n$ which implies $\frac{n-1}{2} \in \mathbb{N}$ , a contradiction since $n$ is even (see table III). For $G$ of type $B_l$ , however, $\frac{n-1}{2} \in \mathbb{N}$ is correct and the conclusion fails.
For $G$ of type $C_l$ one has $n\,(n+1) = 2\alpha n + \beta(n\,(n-1) - 1)$. Assume $n \geq 4$ . Then $2(n\,(n-1) - 1) > n\,(n+1)$ . This implies $\beta = 1$ and $2 = n(\alpha - 2)$ which is impossible.

TABLE III

| type | $n$ | $\dim M_\lambda$ | $m_i$ |
|---|---|---|---|
| $A_l, l \geq 8$ | $l+1$ | $n^2 - 1$ | $n, \frac{n\,(n-1)}{2}, \frac{n\,(n+1)}{2}$ |
| $B_l, l \geq 7$ | $2l+1$ | $\frac{n\,(n-1)}{2}$ | $n$ |
| $C_l, l \geq 4$ | $2l$ | $\frac{n\,(n+1)}{2}$ | $n, \frac{n\,(n-1)}{2} - 1$ |
| $D_l, l \geq 8$ | $2l$ | $\frac{n\,(n-1)}{2}$ | $n$ |

*Case 3: $G$ is of type $G_2, F_4, E_6, E_7, E_8$*
One has $\dim V = 7$ , $\dim G_2 = 14$ resp. $\dim V = 26$ , $\dim F_4 = 52$ . Thus the conclusion fails for $G$ of type $G_2$ and $F_4$ . However, it is obvious from table II, that one obtains the desired contradiction for $G$ of type $E_6, E_7, E_8$ . □

In the exceptional case $p = 2$ there are classical simple Lie algebras admitting restricted structures:

EXAMPLE 2.3.6. Let $\mathfrak{g} = \mathfrak{sl}(3,k)$ and $k$ be a field of characteristic $2$. Denote by $e_{ij}$ the matrix having $1$ in the $(i,j)$ position and $0$ elsewhere. $\mathfrak{g}$ has standard basis $a = e_{12}$ , $b = e_{13}$ , $c = e_{21}$ , $d = e_{23}$ , $f = e_{31}$ , $g = e_{32}$ , $h = e_{11} - e_{22}$ , $j = e_{22} - e_{33}$. $h$ and $j$ span a Cartan subalgebra of $\mathfrak{g}$. The Lie multiplication is given by

$$\begin{array}{lll} [a,j]=[b,g]=a & [a,d]=[b,h]=[b,j]=b & [c,j]=[d,f]=c \\ [b,c]=[d,h]=d & [c,g]=[f,h]=[f,j]=f & [a,f]=[g,h]=g \\ [a,c]=h & [d,g]=j & [b,f]=h+j \end{array}$$

and all other products zero.
$\mathfrak{g}$ is restricted by $a^{[2]} = \cdots = g^{[2]} = 0$ and $h^{[2]} = h$ , $j^{[2]} = j$ . One may check that the following product defines a restricted left-symmetric structure $M = M_\lambda$ on $\mathfrak{sl}(3,k)$ :

$$\begin{array}{lllllllll} \lambda_a & = & e_{18} & + & e_{24} & + & e_{65} & + & e_{73} \\ \lambda_b & = & e_{28} & + & e_{75} \\ \lambda_c & = & e_{37} & + & e_{42} \\ \lambda_d & = & e_{35} & + & e_{48} \\ \lambda_f & = & e_{57} & + & e_{82} \\ \lambda_g & = & e_{12} & + & e_{53} & + & e_{67} & + & e_{84} \\ \lambda_h & = & e_{22} & + & e_{33} & + & e_{44} & + & e_{77} \\ \lambda_j & = & e_{33} & + & e_{44} & + & e_{55} & + & e_{88} \end{array}$$

The space of invariants of $M$ is $M^{\mathfrak{g}} = ka \oplus kg$ and $M/M^{\mathfrak{g}} = R \oplus N$ where $R = \langle \bar{b}, \bar{d}, \bar{j} \rangle$ and $N = \langle \bar{h}, \bar{c}, \bar{f} \rangle$ are isomorphic to the natural module. Furthermore $H^1(\mathfrak{g}, M)$ is nontrivial.

The algebra $\mathfrak{sl}(3,k)$ admits left-symmetric structures if and only if $p = 2$. This is an immediate consequence of Theorem 2.2.2. and the above example.

## 3. NONRESTRICTED SIMPLE LIE ALGEBRAS

If $\mathfrak{g}$ is of classical type as in the main theorem there are only finitely many primes such that $\mathfrak{g}$ might admit left-symmetric structures, namely the primes dividing the dimension of $\mathfrak{g}$. It is by no means easy to determine left-symmetric structures on $\mathfrak{g}$ in this case.
The situation is different for nonrestricted simple Lie algebras of Cartan type. There are many more left-symmetric structures and some of them can be constructed explicitly. In view of Proposition 1.2.5. we investigate *adjoint* structures, induced by nonsingular derivations of $\mathfrak{g}$. This leads to the problem of determining the simple Lie algebras which possess nonsingular derivations. Necessarily one has $\operatorname{char} k > 0$ and $\mathfrak{g}$ nonrestrictable (see [32] for definition), c.f. Lemma 2.3.1. . The following example shows that simple Lie algebras may indeed possess nonsingular derivations :

Let $k$ be a field of characteristic $2$ and $\mathfrak{g} := kx \oplus ky \oplus kz = \mathfrak{so}(3,k)$ with

$[x,y]=z,\ [y,z]=x,\ [z,x]=y$ . $\mathfrak{g}$ is a nonrestricted simple Lie algebra (see [21]) and $\mathrm{Der}\,(\mathfrak{g})=\{D=(\alpha_{ij})\mid \alpha_{ij}=\alpha_{ji},\ \alpha_{11}+\alpha_{22}=\alpha_{33}\}$ . One has $\det(D)=\alpha_{22}\,({\alpha_{11}}^2+{\alpha_{12}}^2+{\alpha_{13}}^2)+\alpha_{11}\,({\alpha_{12}}^2+{\alpha_{22}}^2+{\alpha_{23}}^2)$ . The space of outer derivations consists of the matrices $\mathrm{diag}\,(\alpha_{11},\alpha_{22},\alpha_{11}+\alpha_{22})$ . We may choose the nonsingular derivation

$$D=\begin{pmatrix}1&0&0\\0&0&1\\0&1&1\end{pmatrix}$$

obtaining the following adjoint left-symmetric structure on $\mathfrak{g}$ (see 1.2.6):

$$\begin{array}{lll} x.x=0 & y.x=y & z.x=y+z\\ x.y=y+z & y.y=x & z.y=0\\ x.z=z & y.z=x & z.z=x \end{array}$$

Let now $k$ be an algebraically closed field of characteristic $p>7$ and $\mathfrak{g}$ be simple. Identify $\mathfrak{g}$ with $\mathrm{ad}\,(\mathfrak{g})\subseteq \mathrm{Der}\,(\mathfrak{g})$ . Now $\mathrm{Der}\,(\mathfrak{g})$ is restricted. Let $\mathfrak{h}\subseteq\mathfrak{g}$ be a Cartan subalgebra. Let $\bar{\mathfrak{h}}$ denote the restricted subalgebra of $\mathrm{Der}\,(\mathfrak{g})$ generated by $\mathfrak{h}$ . Let $\mathfrak{t}$ denote the (unique) maximal torus of $\bar{\mathfrak{h}}$ . Call $\dim\mathfrak{t}$ the *toral rank* of $\mathfrak{g}$ with respect to $\mathfrak{h}$ .

The first step in determining the simple Lie algebras which possess nonsingular derivations might be to consider simple Lie algebras of toral rank one. By a result of R. Wilson (cf. [4], Th. 1.7.1.) $\mathfrak{g}$ is simple over $k$ and has toral rank one if and only if a) $\mathfrak{g}\simeq\mathfrak{sl}\,(2,k)$ , b) $\mathfrak{g}\simeq W(1:\mathbf{n})$ or c) $\mathfrak{g}\simeq H(2:\mathbf{n}:\Phi)^{(2)}$ . (See [35] for definitions).

$W(1:\mathbf{n})$ , $\mathbf{n}=(n)$ , is restricted if and only if $n=1$ , i.e. the Witt algebra $W(1:\mathbf{1})$ . The simplest nonrestricted algebra of this type is $W(1:\mathbf{2})$ of dimension $p^2$ . It is easy to see that $W(1:\mathbf{2})$ does not possess nonsingular derivations (in the examples computed the characteristic polynomial of a derivation is a $p$-polynomial, i.e. it has the form $X^{p^n}+\beta_{n-1}\,X^{p^{n-1}}+\cdots+\beta_0\,X$ ). Thus it seems reasonable to investigate the algebras of type c) . An algebra of Cartan type $H(2:\mathbf{n}:\Phi)^{(2)}$ once again is isomorphic to an algebra of a certain type $(i)$ , $(ii)$ or $(iii)$ of dimension $p^n-2$ , $p^n-1$ or $p^n$ respectively (cf. [4], Th. 1.8.1.). It is possible to identify these algebras with the well-known algebras of R. Block and the Albert-Zassenhaus algebras ([4], Cor. 1.8.2. and Lemma 1.8.3.): The Block algebras $\mathcal{L}(G,\delta,f)$ with $\mid G\mid=p^n$ and $G=G_0$ are, for example, isomorphic to the algebras of type $(ii)$ of dimension $p^n-1$ . We are concerned with these algebras in the following section:

### 3.1. The algebra $\mathcal{L}(G,\delta,f)$ of R. Block

Let $k$ be a field of characteristic $p>0$ and $G$ be an elementary abelian $p$-group of order $p^n$ which is a direct sum of $m$ elementary abelian $p$-groups $G_0,G_1,...,G_m$ . For $0\le i\le m$ define $f:\ G\times G\to k$ such that $f_{|G_i}=f_i:\ G_i\times G_i\to k$ is a skew-symmetric nondegenerate biadditive form. Suppose that for each $i$ , there exist additive functions $g_i,h_i:\ G_i\to k$ such that $f_i(\alpha,\beta)=g_i(\alpha)h_i(\beta)-g_i(\beta)h_i(\alpha)$ . Choose $\delta_i\in G_i$ with $g_i(\delta_i)=0$ and set $\delta_0=0$ , $\delta=\delta_1+...+\delta_m$ .

Let $\mathcal{L}$ be a vector space over $k$ with basis $\{u_\alpha\}$ in one-to-one correspondence $u_\alpha \leftrightarrow \alpha$ with elements of $G \setminus \{0, -\delta\}$ and define a product in $\mathcal{L}$ by bilinearity and

$$[u_\alpha, u_\beta] = \sum_{i=0}^{m} f_i(\alpha_i, \beta_i)\, u_{\alpha+\beta-\delta_i} \tag{3.1.1}$$

where $\alpha_i$ and $\beta_i$ denote the $i$-th component of $\alpha$ and $\beta$, respectively. Then $\mathcal{L}$ is a Lie algebra over $k$ (cf. [3] Th.1), denoted by $\mathcal{L}(G, \delta, f)$. It is called an *algebra of Block*. Simplicity of $\mathcal{L}(G, \delta, f)$ follows from any of the following: a) $0 \neq G_1 \neq G$ ; b) $G = G_0$, $n > 1$ ; c) $G = G_1$, $n > 1, p > 2$. Furthermore the simple Lie algebra $\mathcal{L}(G, \delta, f)$ is restricted if and only if $G_0 = 0$ and $G_1, ..., G_m$ have order $p^2$ ([3], Th.8).
Note that $G$ may be regarded as a vector space over $\mathbb{F}_p$ of dimension $n$. We can represent the elements of $G$ as $n$-tuples $\alpha = (\alpha_1, ..., \alpha_n)$ with coordinates in $\mathbb{F}_p$.
For $\mathcal{L}(G_0, 0, f)$ of type b) it is easy to construct nonsingular derivations. If $[k : \mathbb{F}_p] \geq n > 1$ then invertible derivations of diagonal form (i.e., the matrix is of diagonal form) can be found. For $k = \mathbb{F}_p$ this construction fails, i.e. the specified derivations are singular.

Under the following restriction on $f$, however, it is also possible to construct nonsingular derivations over $\mathbb{F}_p$ :

$$f(\alpha, \beta) = 0 \iff \alpha \text{ and } \beta \text{ are linearly dependent over } \mathbb{F}_p \tag{3.1.2}$$

LEMMA 3.1.1. *Let $G$ be an elementary abelian group of order $p^n$ and let $S = G \setminus \{0\}$. Let $M$ be a vector space over $k$ with basis $\{u_\alpha \mid \alpha \in S\}$. Set $u_0 = 0$. Suppose that there is some function $f : S \times S \to k$ such that the product*

$$[u_\alpha, u_\beta] = f(\alpha, \beta)\, u_{\alpha+\beta} \tag{3.1.3}$$

*gives $M$ the structure of a Lie algebra.*
*(a) If $[k : \mathbb{F}_p] \geq n$ then $Der(M)$ contains invertible derivations over $k$.*
*(b) If $f$ satisfies (3.1.2) then $Der(M)$ contains invertible derivations over $\mathbb{F}_p$.*

*Proof:* We may assume $G = (\mathbb{Z}/p\mathbb{Z})^n$. Define a linear map $D \in \mathrm{End}\,(M)$ by $D(u_\alpha) = c_\alpha\, u_\alpha$ , $c_\alpha \in k$. If

$$c_\alpha + c_\beta = c_{\alpha+\beta} \quad \forall\ \alpha, \beta \in S \tag{3.1.4}$$

then it is immediate from (3.1.3) that $D \in \mathrm{Der}\,(M)$.
*Case* $(a)$: Let $\alpha_1, ..., \alpha_n \in k$ be linearly independent over $\mathbb{F}_p$ and set $e_i := (0, .., 1, 0, .., 0)$ . $\alpha \in S$ is representable as $\alpha = \sum_{i=1}^n r_i\, e_i$ for some $(r_1, ..., r_n) \in \mathbb{F}_p^n \setminus \{(0, ..., 0)\}$. Set $c_\alpha := \sum_{i=1}^n r_i\, \alpha_i \in k$ and define $D$ by $D(u_\alpha) = c_\alpha\, u_\alpha$. The distribution law in $k$ implies (3.1.4), thus $D \in \mathrm{Der}\,(M)$. The matrix of $D$ is of diagonal form containing precisely the $p^n - 1$ elements $\sum_{i=1}^n l_i\, \alpha_i$ (where $(l_1, ..., l_n)$ runs through the set $\mathbb{F}_p^n \setminus \{(0, ..., 0)\}$ ) on the diagonal. All diagonal elements $\sum_{i=1}^n l_i\, \alpha_i$

are different from zero, otherwise $\alpha_1, ..., \alpha_n$ would be linearly dependent over $\mathbb{F}_p$ . Thus $D$ is invertible.

*Case* $(b)$ : For $1 \leq i \leq m$ let $\sigma_i : G \to \mathbb{F}_p$ denote the projection onto the $i$-th coordinate. Define $D_i \in \mathrm{End}\,(M)$ by $D_i(u_\alpha) = \sigma_i(\alpha)u_\alpha$ . Since $\sigma_i(\alpha)+\sigma_i(\beta) = \sigma_i(\alpha+\beta)$ one has $D_i \in \mathrm{Der}\,(M)$ (see (3.1.4)).

*Assume first* $m = 2$
Let $S$ be ordered as follows: $S = \{(1,0), ..., (1,p-1), (2,0), ..., (2,p-1), ...., (p-1,0), ...(p-1,p-1)\}$ . Denote $u_{(i,j)}$ by $u_{ij}$ for $0 \leq i,j \leq p-1$ . The matrix of $D_2$ with respect to this basis is given by $D_2 = \mathrm{diag}\,(B, B_0, ..., B_0) \in M_{p^2-1}(\mathbb{F}_p)$ where the blocks are $B = \mathrm{diag}\,(1,2,...,p-1)$ and $B_0 = \mathrm{diag}\,(0,1,2,...,p-1)$ . Now consider the inner derivation $\mathrm{ad}\,u_{01}$ . One has $\mathrm{ad}\,u_{01}\,(u_{ij}) = f(\,(0,1),(i,j)\,)u_{i,j+1}$ by (3.1.3) ; $u_{i,j+1}$ is zero if and only if $(i,j) = (0,p-1)$ . For $i > 0$ $f(\,(0,1),(i,j)\,)$ is always different from zero since $\alpha = (0,1)$ and $\beta = (i,j)$ are linearly independent over $\mathbb{F}_p$ .
Define $D := \mathrm{ad}\,u_{01} + D_2$ ; $D$ is a derivation of $M$ with block matrix

$$D = \mathrm{diag}\,(A_0, A_1, ..., A_{p-1}) \in M_{p^2-1}(\mathbb{F}_p) \text{ where } A_0 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ * & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & p-1 \end{pmatrix} \text{ and}$$

$$A_i = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & a_{ip} \\ a_{i1} & 1 & 0 & \dots & 0 & 0 \\ 0 & a_{i2} & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p-2 & 0 \\ 0 & 0 & 0 & \dots & a_{i\,p\text{-}1} & p-1 \end{pmatrix}$$

for some $a_{ij} \in \{f(\,(0,1),(k,l)\,) \mid 0 \leq k,l \leq p-1\}$ , $i > 0$ . Since $\det(A_0)$ and $\det(A_i) = \prod_{k=1}^{p} a_{ik}$ have nonzero determinant in $\mathbb{F}_p$ it follows that $D$ is a nonsingular derivation of $M$ .
For $m > 2$ consider $D := \mathrm{ad}\,u_{(0,...,0,1)} + D_m$ ; $D$ is a block matrix containing several blocks 'of type $A_0$ or $A_i$ '. By the same argument $D$ is a nonsingular derivation of $M$ . □

The following result is an immediate consequence of the above lemma and Theorem 11 of Albert and Frank [1]:

PROPOSITION 3.1.2. *Let* $G$ *be an elementary abelian group of order* $p^n > p$ *and suppose that* $f : G \times G \to k$ *is a skew-symmetric biadditive functional satisfying* (3.1.2) *. Let* $\mathcal{L}_0$ *be a vector space over* $k$ *with basis* $\{u_\alpha \mid \alpha \in S\}$ *where* $S = G \setminus \{0\}$ *. Set* $u_0 = 0$ *. Then the product* (3.1.3) *gives* $\mathcal{L}_0$ *the structure of a simple Lie algebra (cf.* [1] *);* $\mathcal{L}_0$ *is a nonrestricted algebra* $\mathcal{L}(G_0, 0, f)$ *of Block possessing nonsingular derivations for every prime* $p > 0$ *and integer* $n > 1$ *.*

EXAMPLE 3.1.3. Take $p = 2, m = 2$ and $S = \{\alpha_1, \alpha_2, \alpha_3\} = \{(0,1),(1,0),(1,1)\}$ . Set $v_i = v_{\alpha_i}$ , $i = 1,2,3$ and $f(\alpha_i, \alpha_j) = 1$ for $i \neq j$ . The Lie product is given by

$[v_1, v_2] = v_3,\ [v_1, v_3] = v_2,\ [v_2, v_3] = v_1$ . One obtains the example given at the beginning of section 3 (i.e. $\mathcal{L}_0 = \mathfrak{so}\,(3, k)$ and the adjoint structure given there). More examples (for larger primes $p$ and different $m$ ) can be found in [7] .

Proposition 3.1.2. allows to construct adjoint left-symmetric structures on the simple Lie algebra $\mathcal{L}_0$ of characteristic $p$ for *every* prime $p > 0$ . This contrasts with the situation of Theorem 2.2.2. .

### 3.2. The Lie algebra $\mathfrak{so}$(n,k) of characteristic two

Let $I = \mathbb{N} \setminus \{1, 2, 4\}$ and $n \in I$ . The algebra $\mathfrak{so}\,(n, k)$ of characteristic 2 also provides an example of a simple Lie algebra which admits adjoint left-symmetric structures.

DEFINITION 3.2.1. A matrix $X = (x_{ij})$ is called *alternate*, if $x_{ii} = 0$ and $x_{ij} = -x_{ji}$ for all $i$ and $j$ . Define $\mathfrak{so}\,(n, k) = \{X \in M_n(k) \mid X \text{ is alternate}\}$ . This is a Lie algebra under the bracket operation $[X, Y] = XY - YX$ , being simple for *any* characteristic (see [21], p.470).

PROPOSITION 3.2.2. *Let $k$ be a field of characteristic 2 containing at least $n$ elements. Then $\mathfrak{so}\,(n, k)$ , $n \in I$ , is a simple nonrestricted Lie algebra possessing nonsingular derivations.*

*Proof:* Let $\bar{e}_{ij} := e_{ij} + e_{ji}$ , $1 \le i < j \le n$ , be a basis of $\mathfrak{g} = \mathfrak{so}\,(n, k)$ . One has

$$[\bar{e}_{ij}, \bar{e}_{kl}] = \delta_{ik}\bar{e}_{jl} + \delta_{il}\bar{e}_{kj} + \delta_{jk}\bar{e}_{il} + \delta_{jl}\bar{e}_{ik} \tag{3.2.1}$$

for all indices $i < j$ , $k < l$ . Thus $(\mathrm{ad}\,\bar{e}_{ij})^2$ is of diagonal form in contrast to $(\mathrm{ad}\,\bar{e}_{ij})$ . Hence $(\mathrm{ad}\,\bar{e}_{ij})^2 \notin \mathrm{ad}\,(\mathfrak{g})$ and $\mathfrak{g}$ is not restrictable. Define $D \in \mathrm{End}\,(\mathfrak{g})$ by $D(\bar{e}_{ij}) = \alpha_{ij}\bar{e}_{ij}$ , $\alpha_{ij} \in k$ . Then

(1) $$D \in \mathrm{Der}\,(\mathfrak{g}) \iff \alpha_{1j} + \alpha_{1l} = \alpha_{jl} \quad \forall \ 1 \le j < l \le n$$

This follows from (3.2.1) : $D$ is a derivation if and only if $\delta_{ik}(\alpha_{ij} + \alpha_{kl} + \alpha_{jl})\,\bar{e}_{jl} + \delta_{il}(\alpha_{ij} + \alpha_{kl} + \alpha_{kj})\,\bar{e}_{kj} + \delta_{jk}(\alpha_{ij} + \alpha_{kl} + \alpha_{il})\,\bar{e}_{il} + \delta_{jl}(\alpha_{ij} + \alpha_{kl} + \alpha_{ik})\,\bar{e}_{ik} = 0 \quad \forall \ i < j, k < l$ . The equation is non-trivial only in case that precisely two of the indices $i, j, k, l$ are equal. Because of the symmetry we may assume $i = k$ and $i < j, k < l, i < l$ . Then one has $\alpha_{ij} + \alpha_{il} + \alpha_{jl} = 0 \quad \forall \ 1 \le i < j < l \le n$ . For $i = 1$ one obtains $\alpha_{1j} + \alpha_{1l} = \alpha_{jl}$ and conversely this implies $\alpha_{ij} + \alpha_{il} + \alpha_{jl} = (\alpha_{1i} + \alpha_{1j}) + (\alpha_{1i} + \alpha_{1l}) + (\alpha_{1j} + \alpha_{1l}) = 0$ . Now choose $\alpha_{12}, ...\alpha_{1n} \in k$ such that $0, \alpha_{12}, ...\alpha_{1n}$ are pairwise distinct. This is possible since $\mid k \mid \ge n$ . It follows from (1) that $\det(D) = \prod_{1<i<j} (\alpha_{1i} + \alpha_{1j}) \cdot \prod_{1<k} \alpha_{1k}$ . Thus $D$ has nonzero determinant in $k$ . □

REMARK 3.2.3. It is not necessary to assume $\mid k \mid \ge n$ . However, the construction of a nonsingular derivation then becomes more difficult. It would be interesting to find general conditions which may guarantee the existence of nonsingular derivations of simple Lie algebras (or more generally of invertible 1-cocycles for $\mathfrak{g}$-module structures on $\mathfrak{g}$ , see Cor. 1.2.3.).

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| | |
|---|---|
| $\mathcal{A}$ | left-symmetric algebra |
| $\mathfrak{g}$ | Lie algebra |
| $\mathrm{End}\,(\mathfrak{g})$ | vector space endomorphisms |
| $\mathrm{Aut}\,(\mathfrak{g})$ | group of Lie algebra automorphisms |
| $\mathbb{K}$ | field of real or complex numbers |
| $\mathcal{L}(G,\delta,f)$ | algebra of R. Block |
| $\nabla$ | affine connection on a Lie group |
| $U(\mathfrak{g})$ | universal enveloping algebra of $\mathfrak{g}$ |
| $(\mathfrak{g},[p])$ | restricted Lie algebra of characteristic $p$ |
| $u(\mathfrak{g})$ | restricted universal enveloping algebra of $\mathfrak{g}$ |
| $M_\lambda$ | left-symmetric module |
| $M_{\mathrm{ad}}$ | adjoint module |
| $G$ | algebraic group (1.3.) |
| $T$ | maximal torus in $G$ |
| $X(T)$ | group of characters of $T$ |
| $R$ | root system |
| $R^+$ | set of positive roots |
| $\alpha_i$ | simple root |
| $\omega_i$ | fundamental weight |
| $X(T)_+$ | set of dominant weights |
| $X_1(T)$ | set of restricted dominant weights |
| $B$ | Borel subgroup of $G$ |
| $\mathrm{ind}_H^G\, M$ | $G$-module induced by an $H$-module $M$ |
| $H^0(\lambda)$ | $= \mathrm{ind}_B^G k_\lambda$ |
| $G_1$ | first Frobenius kernel of $G$ |
| $L(\lambda)$ | $G$-module of highest weight $\lambda$ |
| $\mathcal{W}$ | Weyl group of $G$ |

## ABSTRACT

This paper investigates left-symmetric structures on finite-dimensional simple Lie algebras $\mathfrak{g}$ over a field $k$ . If $k$ is of characteristic $0$ , then $\mathfrak{g}$ does not admit any left-symmetric structure. This is known in the theory of affine manifolds. In the modular case, however, such structures may exist. The main purpose of this paper is to show that classical simple Lie algebras of characteristic $p > 3$ admit left-symmetric structures only in case $p$ divides $\dim(\mathfrak{g})$ . The proof involves the computation of the first cohomology groups of classical Lie algebras for certain $\mathfrak{g}$-modules of small dimension. Here $\mathfrak{g}$ is regarded as the Lie algebra of a connected semisimple algebraic group over an algebraically closed field of characteristic $p > 0$ . Most of these computations are due to J.C. Jantzen.
For nonrestricted simple Lie algebras of Cartan type it is shown that many more left-symmetric structures can be found. One studies so-called adjoint structures, induced by nonsingular derivations of $\mathfrak{g}$ . The simple algebra $\mathcal{L}(G,\delta,f)$ of R. Block of dimension $p^n-1$ , for example, admits adjoint structures for every $p>0$ .
If $p=2$ , the results are more complicated.