Universität Wien
Prof. Dr. D. Burde

Homological Algebra
— Exercises —

No. 1-32 WS 2016/17

A *ring* is by convention here a ring with unit, not necessarily commutative.

Exercise 1. Find an example of a commutative ring R which is isomorphic to $R \times R$.

Exercise 2. Denote by $\mathbb{Z}/n$ the $\mathbb{Z}$-module of the integers modulo n. For two integers n and m, let $d = \gcd(n, m)$ be the greatest common divisor of n and m. Show that

$$\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m \cong \mathbb{Z}/d.$$

Exercise 3. Show that the infinite direct sum of rings cannot be a ring.

Exercise 4. Let $M = \prod_{i\in\mathbb{N}} \mathbb{Z}$ be the $\mathbb{Z}$-module of integral sequences, and $e^n = (0, \dots, 0, 1, 0, \dots)$ be the sequence with all entries equal to zero, except for the n-th entry, which is 1. Let $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathbb{Z})$. Show that $f(e^n) = 0$ for almost all $n \in \mathbb{N}$, and that if $f(e^n) = 0$ for all $n \in \mathbb{N}$, then $f = 0$.

Exercise 5. Show that the $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathbb{Z})$ is free, and is isomorphic to $\oplus_{i\in\mathbb{N}}\mathbb{Z}$.

Exercise 6. Use the previous exercises to show that the $\mathbb{Z}$-module M is not free, and conclude that the product of free modules need not be free.

Exercise 7. Show that the $\mathbb{Z}$-module $\mathbb{Q}$ is flat.

Exercise 8. Show that the $\mathbb{Z}$-module $\mathbb{Q}$ is not projective, and hence not free.

Exercise 9. Show that the $\mathbb{Z}$-modules $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Z}/2$ are both not flat.

Exercise 10. Let R be a PID and M be an R-module. Show that M is injective if and only if M is divisible. Conclude that the $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

Exercise 11. Show that the direct product cannot be a coproduct in the category of groups.

Exercise 12. Show that the category of cyclic groups does not have all binary products, and hence is not an abelian category.

Exercise 13. Let p be a prime and $\mathbb{Z}_{(p)} = \{\frac{n}{m} \in \mathbb{Q} \mid p \nmid m\}$ the ring of p-local numbers. Let $\mathbb{Z}/p^\infty = \cup_{i\geq 0}\mathbb{Z}/p^i$ with the uniquely given embeddings $\mathbb{Z}/p_i \hookrightarrow \mathbb{Z}/p^{i+1}$. Show that there is a short exact sequence

$$0 \to \mathbb{Z}_{(p)} \to \mathbb{Q} \to Z/p^\infty \to 0$$

which does not split.

Exercise 14. Define the ring of p-adic integers by $\mathbb{Z}_p = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p^\infty)$. Show that there exists a unique ring monomorphism $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Z}_p$, and that $\mathbb{Z}_p$ is flat over $\mathbb{Z}_{(p)}$.

Exercise 15. Let M be a $\mathbb{Z}$-module, and $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathbb{Z})$ its dual $\mathbb{Z}$-module. Show that M need not be isomorphic with its bi-dual $(M^\vee)^\vee$. For which class of $\mathbb{Z}$-modules do we have $M \cong (M^\vee)^\vee$?

Exercise 16. Give an example of an Epi in the category of rings which is not a surjective map.

Exercise 17. Give an example of a Monic in the category of divisible abelian groups which is not an injective map.

Exercise 18. Show that the category of divisible abelian groups is additive, but not abelian.

Exercise 19. Show that the category of torsion-free abelian groups is additive, but not abelian.

Exercise 20. Show that the category of finitely generated abelian groups does not have non-trivial injective objects.

Exercise 21. Show that the category of R-modules has enough projective and injective objects.

Exercise 22. Verify that the functor $\mathrm{Hom}(-, \mathbb{Z})$ is not exact by applying it to the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

Exercise 23. Denote by $\mathbb{Z}_{(p)} = \{\frac{n}{q} \in \mathbb{Q} \mid p \nmid q\}$ the ring of p-local numbers, by $\mathbb{Z}_p$ the ring of p-adic integers, and by $\mathbb{Q}_p$ the field of p-adic numbers. Apply the Snake Lemma to show that

$$\mathbb{Z}_p/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Q}.$$

Exercise 24. Suppose we are given a commutative diagram of R-module homomorphisms

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which all colums are exact. Use the Snake Lemma to show that if any two of the rows are exact, then so is the third.

Exercise 25. Show that

$$\mathrm{Tor}_k^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/gcd(n,m) & \text{for } k = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 26. Let A be a torsion abelian group. Show that

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}).$$

Exercise 27. Let A be a finite abelian group. Show that

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}/m) \cong \frac{\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})}{m\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})}.$$

Exercise 28. Let A and B be a finite abelian groups. Show that

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, B) \cong A \otimes_{\mathbb{Z}} B.$$

Exercise 29. Let $R = \mathbb{Z}[t]/(t^n - 1)$. Show that

$$\mathrm{Tor}^R_k(\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ \mathbb{Z}/n & \text{for } k \text{ odd}, \\ 0 & \text{for } k \text{ even}. \end{cases}$$

Exercise 30. Show that

$$\mathrm{Ext}^k_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/gcd(n,m) & \text{for } k = 0, 1, \\ 0 & \text{otherwise}. \end{cases}$$

Exercise 31. Let G be the trivial group, and A be a G-module. Compute the cohomology $H^n(G, A)$.

Exercise 32. Let L/K be a finite Galois extension. Show that

$$H^1(\mathrm{Gal}(L/K), L^\times) = 0.$$

Due: Wednesday, January 11, 2017