DYNAMICS OF GENERAL RELATIVITY

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ABSTRACT. We give a brief introduction to the initial value problem in General Relativity with an emphasis on the vacuum Einstein flow in CMCSH gauge. We then review the local-existence result for this system and its application to the nonlinear stability problem of the Milne model following the original works by Andersson and Moncrief.

1. PRELUDE

Note that these notes contain material from the references [1, 2, 6], often without local citations.

1.1. **Initial data sets and their developments.** Consider a 4-dimensional Lorentzian manifold $(\overline{M}, \overline{g})$, where $\overline{g}$ has signature $(-1,1,1,1)$ that solves the Einstein equations

$$\overline{R}_{\mu\nu} = 0, \tag{1.1}$$

where $\overline{R}_{\mu\nu}$ is the Ricci tensor of $\overline{g}$. Let then M be a spacelike submanifold of $\overline{M}$ and we assume for now that $\overline{M} = \mathbb{R} \times M$. We express $\overline{g}$ with respect to local coordinates (x^α), where $x^0 = t$ is a time-coordinate (constant on M) and x^i are local coordinates on M by

$$\overline{g} = -N^2 dt^2 + g_{ab}(dx^a + X^a dt) \otimes (dx^b + X^b dt). \tag{1.2}$$

Here, N is called the *lapse function* and X the *shift vector field.* For an interpretation of these quantities, consider the future directed, time-like vector field $\mathbf{n}$ unit normal to M. The following decomposition holds

$$\partial_t = N\mathbf{n} + X \tag{1.3}$$

The second fundamental form[1] is then given by

$$k_{ab} := -\frac{1}{2N}\left(\partial_t g_{ab} - \nabla_a X_b - \nabla_b X_a\right), \tag{1.4}$$

then the Einstein equations induce the constraint equations on M, which read

$$\begin{aligned} R(g) - k_{ab}k^{ab} + (\mathrm{tr}_g k)^2 &= 0 \\ \nabla^a k_{ab} - \nabla_b(\mathrm{tr}_g k) &= 0. \end{aligned} \tag{1.5}$$

A detailed derivation of the constraint equations in this setting is given for instance in the textbook [5], p. 22-23.

[1] This can also be written as $k = -\frac{1}{2}\mathcal{L}_{\mathbf{n}} g$.

Forgetting about the origin of the system (1.5) we can consider this *initial data* independently of its ambient spacetime.

Definition 1.6. *An* initial data set *consists of a 3-dimensional Riemannian manifold* (M, g_0) *and a symmetric 2-tensor field on* M *denoted by* k_0 *such that* (g_0, k_0) *solve the system* (1.5)*.*

By construction, every initial data set arising from a spacetime solving Einstein's equations, is of the form above. Whether every initial data set in the above sense can be embedded into a suitable spacetime solving the Einstein equations is however a non-trivial problem.

It gives rise to the following definition.

Definition 1.7. *Given an initial data set* (M, g_0, k_0)*. A* development *of* (M, g_0, k_0) *is a 4-dimensional Lorentzian manifold* $(\overline{M}, \overline{g})$ *and an embedding*[2] $\imath : M \to \overline{M}$ *such that*

$$\begin{aligned} \imath^*\overline{g} &= g_0 \\ \imath^* K &= k_0, \end{aligned} \tag{1.8}$$

where $\imath^*$ *is the pull-back*[3] *with respect to* $\imath$ *and* K *is the second fundamental form of* $\imath(M) \subset \overline{M}$*.*

Moreover, if $\imath(M)$ *is a Cauchy surface*[4] *of* $\overline{M}$*, we say* $(\overline{M}, \overline{g})$ *is a* globally hyperbolic development*.*

Remark 1.9. *In general, if* $\overline{M}$ *has a Cauchy surface, then it is globally hyperbolic.*

We can formulate the above problem precisely: Does every initial data set have a globally hyperbolic development and if so, is it unique?

The answer will be given in the following section.

Before, let us recall a fundamental Lemma about the topology of developments.

Lemma 1.10. *Let* $(\overline{M}, \overline{g})$ *be a globally hyperbolic development of an initial data set* (M, g_0, k_0)*. Then*

$$\overline{M} \cong \mathbb{R} \times M. \quad (\cong \textit{homeomorphic}) \tag{1.11}$$

1.2. **The maximal globally hyperbolic development.** The following theorem answers the first part of the question.

Theorem 1.12. *Every initial data set has a globally hyperbolic development.*

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[2]An embedding $f : M \to \overline{M}$ is a smooth map such that its derivative is everywhere injective and moreover f is a homeomorphism onto its image.

[3]Considering the differential $D\imath : TM \to T\overline{M}$ the pull-back is defined by $\imath^*\overline{g}(X, Y) := \overline{g}(D\imath X, D\imath Y)$, for X, $Y \in TM$.

[4]A Cauchy surface is a subset of $\overline{M}$ that is met exactly once by every inextendible timelike curve.

Proof. The proof (page 156 of [6]) is based on the local existence result of Choquet-Bruhat [3], which casts the Einstein equations into a well-posed system of quasilinear wave equations using the *harmonic gauge condition.* □

It remains to clarify whether this development is unique. Since two different developments are two different manifolds the following definition is required to introduce a suitable notion to compare those.

Definition 1.13. *Let* $(\overline{M}, \overline{g}, \imath)$ *be a development of the initial data set* (M, g_0, k_0)*. This development is called a* maximal globally hyperbolic development (MGHD) *iff for every development* $(\overline{M}', \overline{g}', \imath')$ *of this initial data set the following holds.*

There exists a time orientation preserving diffeomorphism onto its image

$$\psi : \overline{M}' \to \overline{M} \tag{1.14}$$

such that

$$\begin{aligned} \psi^* \overline{g} &= \overline{g}' \\ \psi \circ \imath' &= \imath. \end{aligned} \tag{1.15}$$

The following theorem then answers the question of uniqueness.

Theorem 1.16 ([4])**.** *For every initial data set* (M, g_0, k_0) *there exists a MGHD, which is unique up to isometry.*

Moreover, the following two Corollaries clarify the structure of the MGHD.
First, an MGHD is the MGHD of each of its induced initial data sets.

Corollary 1.17. *Let* M' *be a Cauchy surface in the MGHD* $(\overline{M}, \overline{g})$ *and let* (M', g', k') *be the induced initial data set on* M'*. Then* $(\overline{M}, \overline{g})$ *is the MGHD of this initial data set.*

Secondly, two initial data sets in a spacetime give rise to isometric MGHDs.

Corollary 1.18. *Let* $(\overline{M}, \overline{g})$ *be a solution to the Einstein equations and let* M*,* M' *be two Cauchy surfaces in this spacetime. Then the MGHDs of the initial data sets induced on both* M *and* M' *are isometric.*

We conclude that the existence of the MGHD ensures that the local existence problem for the Einstein equations with initial on a spacelike Cauchy surface is geometrically well-understood.

However, if we intend to study the Einstein equations as a system of partial differential equations we need to fix a gauge condition to obtain well-posedness (see details below). Whether or not local existence in the sense of PDEs holds for a certain choice of gauge depends on the class of initial data sets and the respective gauge condition.

1.3. **Global structure of the MGHD.** So far, the existence of a MGHD ensures that every initial data set can be uniquely evolved to a "global" spacetime, the global geometry of which, nevertheless, is yet unknown.
Let us gather some questions about the MGHD that we wish to answer.

(1) Is the MGHD timelike- and null-geodesically complete[5]/incomplete (in either time direction)?
(2) Does curvature blow up towards the incomplete direction of spacetime? (related to strong cosmic censorship)
(3) What is the asymptotic behavior of the metric in both time directions?
(4) Does the MGHD contain trapped surfaces (related to the formation of black holes)?
(5) Is there a relation between the spatial topology of a spacetime and its long-time behavior (Closed universe recollapse)?

1.4. **The nonlinear stability problem.** Some of the above questions are combined in the so-called *Nonlinear stability problem.* This problem can be formulated as follows. Consider an *explicit* spacetime (such as Minkowski space) and consider an initial data set induced on a spacelike hypersurface of this spacetime (M, g_0, k_0). We then define the notion of *perturbation.*

Definition 1.19. *Let $\varepsilon > 0$. Say an ε-*perturbation *of the initial data set (M, g_0, k_0) is an initial data set (M, g_0', k_0') such that $\|g_0 - g_0'\|_\ell < \varepsilon$ and $\|k_0 - k_0'\|_{\ell-1} < \varepsilon$ for a chosen Sobolev-norm[6] $\|.\|_\ell$.*

Then we say a spacetime is nonlinearly stable if for small perturbations their MGHDs are asymptotically similar to those of the initial data corresponding to the background spacetime. More precisely below [7].

Definition 1.20. *A spacetime $(\overline{M}, \overline{g})$ is* nonlinearly stable *iff the following holds. Let $t : \overline{M} \to \mathbb{R}$ a time function on $\overline{M}$. For a given initial data set (M, g_0, k_0) in $(\overline{M}, \overline{g})$ there exists an $\varepsilon > 0$ such that for every ε-perturbation (M, g_0', k_0') there exists a time-function t' on $(\overline{M}', \overline{g}')$ (the MGHD of (M, g_0', k_0')) such that the following holds. Let now for $s \in \mathbb{R}$ consider the foliations[8] by pre-images $t^{-1}(s) \subset \overline{M}$ and $t'^{-1}(s) \subset \overline{M}'$ and let $(g(s), k(s))$ and $(g'(s), k'(s))$ correspond to the data induced on the leaves of the respective foliations. Then, considering both as s-dependent tensor fields on the manifold M*

$$\|g(s) - g'(s)\|_\ell + \|k(s) - k'(s)\|_{\ell-1} \to 0$$

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[5]A spacetime is complete in a time direction, iff the affine parameter of a timelike- or null-geodesic is defined on an interval $[0, \infty)$. Otherwise a geodesic is incomplete. If one incomplete geodesic (timelike or null) exists in a spacetime. This spacetime is incomplete.

[6]A Sobolev-norm of order $\ell \in \mathbb{N}$ on a Riemannian manifold (M, g) is given by $\|\phi\|_{H^\ell}^2 = \sum_{i \le \ell} \int_M |\nabla^i \phi|_g^2 \mu_g$, where μ_g and ∇ are the volume form and the covariant derivative corresponding to g.

[7]Each nonlinear stability result is formulated in the context of the formalism of the respective problem. For details we refer to the individual results in the list below. This definition tries to gives a general notion.

[8]A foliation of an 4-manifold $\overline{M}$ is a set of 3-dimensional submanifolds M_t depending on the parameter $t \in I$ for I some interval, such that $\overline{M} = \bigcup_{t \in I} M_t$ and each point $p \in \overline{M}$ has a neighborhood admitting a coordinate system $(U; (x^0, x^1, x^2, x^3))$ such that on $M_t \cap U \neq \emptyset$ we have $x^0 = \mathsf{const}$.

as $s \to \infty$.

We give a list of several milestone-results for nonlinear stability.

- Stability of Minkowski space (Christodoulou-Klainerman, 1993)
- —— in harmonic gauge (Lindblad-Rodnianski, 2005)
- —— (Zipser-Bieri, 2009)
- Stability of the Milne model (Andersson-Moncrief, 2011)
- Stability of cosmological models for $\Lambda > 0$ (Friedrich 1986)
- —–in higher dimensions (Anderson 2005)
- —– in full generality (Ringström 2008)
- —– Stability of the above solutions in the set of solutions to Einstein-matter systems (many works)
- —– First work on stable singularity formation in cosmological spacetimes (Rodnianski-Speck, 2014)

We give a brief comment on the relevance of this feature of a spacetime.

Remark 1.21. *A certain spacetime as a model for an isolated region in the universe (asymptotically flat spacetimes) or for the universe itself (cosmological spacetimes) is in physics often chosen to be one of the explicit solutions such as Minkowski space, the Schwarzschild spacetime or a Robertson-Walker universe. However, even within the framework of General Relativity due to the presence of small inhomogeneities or small amounts of matter, the correct model would always be the MGHD of a perturbation in the sense defined above.*
If now the MGHD of such a perturbation would substantially deviate asympotitcally from the background the model would be unfitting and useless.
Therefore, understanding non-explicit *spacetimes in a neighborhood of explicit ones in the sense of the nonlinear stability problem is fundamental for applications.*

One of the most challenging open problems currently under investigation concerns the *black hole stability problem.* Recent progress on this problem has been made by

- Linear stabiliy of black holes (Dafermos-Holzegel-Rodnianski, 2016)
- Nonlinear stability of Kerr-deSitter (Hintz-Vasy, 2016)

2. The CMCSH gauge condition

We consider the constant-mean-curvature-spatial-harmonic gauge condition (CMCSH) originally introduced by Lars Andersson and Vincent Moncrief in [1]. In this gauge, the Einstein equations take the form an elliptic-hyperbolic system.
We remark that all details given in the following can be found in [1].

2.1. **Setup.**

We consider in the following an initial data set (M, g_0, k_0), with the restriction

$$\tau_0 \equiv \mathsf{tr}_{g_0} k_0 = g_0^{ab} k_{0\,ab} = \text{const}, \tag{2.1}$$

i.e. constant-mean curvature initial data. In general we use the notation

$$\tau = g^{ab} k_{ab}. \tag{2.2}$$

Also we assume the manifold M to be compact, in particular we consider cosmological spacetimes. We will point out below inhowfar this approach is nevertheless general for the class of spacetimes we treat.

Let us now consider the MGHD $(\overline{M}, \overline{g})$ of this initial data set and use the ansatz (1.2) for the spacetime metric. Imposing the Einstein equations on the metric (1.2) results in the constraint equations (1.5) and the following evolution equation for the second fundamental form, which is equivalent to the "spatial part" of the Einstein equations

$$\overline{R}_{ab} = 0 \tag{2.3}$$

and reads

$$\partial_t k_{ab} = -\nabla_a \nabla_b N + N \left[R_{ab} + \tau k_{ab} - 2 k_{ac} k^{cb} \right] + X^c \nabla_c k_{ab} + k_{ac} \nabla_b X^c + k_{cb} \nabla_a X^c, \tag{2.4}$$

where R_{ab} denotes the Ricci tensor of the metric g and τ the mean curvature function. We interpret this as an evolution equation for the second fundamental form. Together with the definition of the second fundamental form, which we rewrite as an evolution equation for the metric,

$$\partial_t g_{ab} = -2N k_{ab} + \nabla_a X_b + \nabla_b X_a \tag{2.5}$$

these two equations constitute the evolution equations - so far no gauge is fixed yet.
It remains to obtain equations for the lapse function and for the shift vector field. Regarding an equation for the lapse one may consider the analogous evolution equation for k^a_b and take the trace, which results in

$$\partial_t \tau = -\Delta N + N k_{ab} k^{ab} + X^a \nabla_a \tau. \tag{2.6}$$

With the suitable choice of gauge this will be turned into an elliptic equation for the lapse function.

2.2. **CMCSH gauge.** We will now impose two gauge conditions.
First, we impose that the spacetime can be foliated by hypersurfaces on each of which the mean curvature is constant and such that value of the mean curvature is a specific time function. In particular, we choose

$$t = \tau. \tag{2.7}$$

This is the so-called CMC condition. For the second condition we need to introduce some structure. Let therefore γ be some smooth fixed Riemannian metric on M and let Γ and $\widehat{\Gamma}$ denote the Christoffel symbols of g and γ, respectively.
Then, the object

$$g^{ij}(\Gamma^a_{ij} - \widehat{\Gamma}^a_{ij}) \equiv V^a \tag{2.8}$$

is a well-defined vector field on M. In particular, a well-defined object on M independent of local coordinates. Therefore the following choice of gauge, which is the so-called *spatial harmonic gauge*, is justified. We impose the second gauge condition,

$$V^a = 0. \tag{2.9}$$

2.3. **The reduced system.** Combining the gauge conditions with the Einstein equations yields the *reduced system of Einstein's equations*, which we present in the following.

2.3.1. *The lapse equation.* As a consequence of the CMC condition, equation (2.6) simplifies to

$$\Delta N = N|k|^2_g - 1, \tag{2.10}$$

which is an elliptic equation for N.

2.3.2. *The shift equation.* Taking the time derivative of the condition (2.9) and using the evolution equation for g yields an elliptic equation for the shift vector field, reading[9]

$$\Delta X^i + R^i_j X^j - \mathcal{L}_X V^i = (-2Nk^{mn} + 2\nabla^m X^n)(\Gamma^i_{mn} - \widehat{\Gamma}^i_{mn}) + 2\nabla^m N k^i_m - \nabla^i N\tau. \tag{2.11}$$

2.3.3. *Expanding the Ricci tensor in spatial harmonic gauge.* A third fundamental implication of the spatial harmonic gauge condition is a decomposition of the Ricci tensor of g. First we define for any symmetric 2-tensor h,

$$\hat{\Delta}_g h_{ij} \equiv \frac{1}{\sqrt{g}}\hat{\nabla}_m(g^{mn}\sqrt{g}\hat{\nabla}_n h_{ij}), \tag{2.12}$$

where $\hat{\nabla}$ is the covariant derivative of γ. Then, the (SHG) implies

$$\hat{\Delta}_g h_{ij} = g^{mn}\hat{\nabla}_m\hat{\nabla}_n h_{ij}. \tag{2.13}$$

Moreover, the following decomposition of the Ricci-tensor holds,

$$R_{ij} = -\frac{1}{2}\hat{\Delta}_g g_{ij} + S_{ij}[g, \partial g] + \delta_{ij}, \tag{2.14}$$

[9] The Lie derivative of a tensor field T^β_α w.r.t. a vector field X is given by $(\mathcal{L}_X T)^b_a = X^i\partial_i T^b_a - \partial_i X^b T^i_a + \partial_a X^i T^b_i$.

where $\delta_{ij} = \frac{1}{2}(\nabla_i V_j + \nabla_j V_i)$ and S_{ij} is at most quadratic in first derivatives of g.

Eventhough the term δ_{ij} vanishes due to the gauge condition, we keep it in the equations as this term will be used below to prove the propagation of the gauge conditions by the reduced system. Details on this issue will follow below.

2.3.4. *The reduced system.* The reduced system reads

$$
\begin{aligned}
\partial_t g_{ij} &= -2Nk_{ij} + \mathcal{L}_X g_{ij} \\
\partial_t k_{ij} &= -\nabla_i \nabla_j N + N(-\frac{1}{2}\hat{\Delta}_g g_{ij} + S_{ij}[g, \partial g] + \tau k_{ij} - 2k_{im}k^m_j) \\
&\quad + \mathcal{L}_X k_{ij} \\
-\Delta N + |k|^2_g N &= 1 \\
\Delta X^i + R^i_j Y^j - \mathcal{L}_X V^i &= (-2Nk^{mn} + 2\nabla^m X^n)(\Gamma^i_{mn} - \widehat{\Gamma}^i_{mn}) \\
&\quad + 2\nabla^m N k^i_m - \tau \nabla^i N
\end{aligned}
\tag{2.15}
$$

Remark 2.16. *We will show below that this system has a local-in-time solution, which is a priori not clear. Moreover, it is not clear whether this solution solves the Einstein equations, which will later be shown by showing that the gauge conditions and the constraints are propagated.*

We make two further remarks before we turn to the local-existence problem for the system (2.15).

Remark 2.17. *The system* (2.15) *is elliptic-hyperbolic in the sense that it consists of two hyperbolic evolution equations for g and k, respectively, and two elliptic equations which hold on each slice of the foliation M_t.*

Remark 2.18. *In the harmonic gauge (considered for instance in* [6]*) the Einstein equations take the form of a nonlinear (quasilinear) wave equation for the spacetime metric $\overline{g}$. We can observe that in CMCSH gauge there is also a quasilinear wave equation hidden. This is apparent when we set the shift vector to zero and the lapse function to be constant, $N = 1$ and then replace k in its evolution equation by $\partial_t g$. Then in combination with the generalized Laplacian on the right-hand side we obtain the modified wave operator*

$$\widehat{\Box}_g \equiv -\partial_t^2 + \hat{\Delta}_g \tag{2.19}$$

acting on g.

3. Local existence

We prove in this section that the system (2.15) has a local-in-time solution. Moreover we show that this solution solves the Einstein equations.

3.1. **Preliminaries.**

3.1.1. *Notations.* Let us introduce some notations relevant for the local-existence statement. Recall that γ is a once and for all fixed smooth manifold on M of negative sectional curvature with covariant derivative $\hat{\nabla}$. We define the standard Sobolev spaces for tensor fields with respect to γ and denote the corresponding norms by

$$\|.\|_{H^s}. \tag{3.1}$$

We also use the combined norm

$$\|(g,k)\|_{\mathcal{H}^s} = \|g\|_{H^s} + \|k\|_{H^{s-1}} \tag{3.2}$$

and

$$\mathcal{H}^s = H^s \times H^{s-1}, \quad \mathcal{W}^{1,\infty} = W^{1,\infty} \times L^\infty. \tag{3.3}$$

To compare the metric g with γ we introduce the constant $\Lambda[g] \geq 1$ such that

$$\Lambda^{-1} g(Y,Y) \leq \gamma(Y,Y) \leq \Lambda g(Y,Y), \quad Y \in TM \tag{3.4}$$

and

$$\Lambda[\overline{g}] = \Lambda[g] + \|N\|_\infty + \|N^{-1}\|_\infty + \|X\|_\infty \tag{3.5}$$

We denote by

$$\Lambda(t) = \sup_{t\in[0,t]} \Lambda[\overline{g}(t)]. \tag{3.6}$$

Also we denote

$$D\overline{g} = (DN, DX, Dg) \tag{3.7}$$

for the first order spatial derivatives D.

3.1.2. *Local-well-posedness.* Let us elaborate briefly on local-well-posedness of an evolutionary system of PDEs.
A system of the type (2.15) is locally well posed if, in a suitable class of regularity, the following holds.

Definition 3.8. (1) *For given initial data $(g_0, k_0) \in H^s(M) \times H^{s-1}$ for s sufficiently large at $t = 0$ there exists a time $t_* > 0$ and a unique solution*

$$(g,k,N,X) \in \bigcap_{0\leq j\leq k-1} C^0([0,t_*], H^s \times H^{s-1} \times H^{s+1} \times H^{s+1}) \cap C^1([0,t_*], H^{s-1} \times H^{s-2} \times H^s \times H^s) = \tag{3.9}$$

to (2.15) *that coincides with the initial data at $t = 0$.*

(2) *The initial value problem is* Cauchy stable *iff the map*

$$H^s \times H^{s-1} \ni (g_0, k_0) \mapsto (g, k, N, X) \in \mathcal{C}^{0,1}([0, t_*], \mathcal{H}^s \times H^{s+1} \times H^{s+1}) \tag{3.10}$$

is continuous.

(3) *A continuation criterion holds: The maximal time of existence T_* is characterized by the blow-up of a suitable norm of the solution such as: Either $T_* = \infty$ or*

$$\limsup_{t \nearrow T_+} (\Lambda[\overline{g}] + \|D\overline{g}\|_\infty + \|k\|_\infty) = +\infty. \tag{3.11}$$

Remark 3.12. *If we intend to analyze the nonlinear stability of a certain background solution, i.e. its global behavior for small perturbations, then the local-well posedness is used in the following sense. First, we know that a short-time solution exists, which means that our choice of gauge is valid; in the present case that a CMC foliation exists. In particular, we can work in the CMCSH system to analyze the solution. Secondly, we now by the Cauchy stability of the system that the smaller we choose the perturbation, the closer the solution will be to the background. In particular, smallness of the perturbation holds for any finite amount of time by making the initial perturbation small. Finally, the continuation criterion allows us to show global existence of the solution in time by showing that the above norms are bounded throughout the evolution.*

3.2. **The local-existence theorem in CMCSH gauge.** The following theorem solves the local-existence problem for the vacuum Einstein equations in CMCSH gauge.

Theorem 3.13 (Andersson, Moncrief 2003)**.** *Let $s > n/2 + 1 = 5/2$ (in $\dim M = 3$) and let M be such that there exists a metric γ on M with negative sectional curvature. Consider the class of CMC initial data on M, $(g_0, k_0) \in H^s \times H^{s-1}$. Then the initial value problem for the reduced Einstein equations* (2.15) *is locally well-posed and the Lorentz-metric $\overline{g}$ constructed from the solution (g, N, X) solves the vacuum Einstein equations.*

Furthermore, let $t_0 = \mathrm{tr}_{g_0} k_0 < 0$ and let (T_-, T_+) be the maximal existence interval in CMC time $t = \tau$. Then either

$$(T_-, T_+) = (-\infty, 0) \tag{3.14}$$

or

$$\limsup (\Lambda[\overline{g}] + \|D\overline{g}\|_\infty + \|k\|_\infty) = +\infty. \tag{3.15}$$

as $t \nearrow T_+$ or as $t \searrow T_-$

In the remainder of this section we give a sketch of a proof of this theorem.

3.3. **A sketch of proof of local-well posedness.** The following method to prove LWP for the reduced system is taken from [1].

This proof is very technical and uses several different tools from functional analysis. We shall give here an outline with the main steps and ideas to construct the desired solutions and establish its properties.

The scheme of proof is the following.

(1) Set up a sequence of approximative smooth solutions to the system by iteratively solving the system of equations, where the respective right-hand sides of the equations are determined by the previous iteration step, which requires to solve only linear equations in each step.
(2) Show by invoking smallness of the time interval boundedness of this sequence in a norm of high regularity by an energy estimate.
(3) Show convergence of the sequence in a norm of low regularity.
(4) Upgrade the convergence to the high regularity norm by interpolation. (this establishes the existence and uniqueness by a fixed point argument)
(5) Continuity of the solution follows by some standard functional analytic arguments in combination with re-evaluating energy estimates in the proof of local existence.
(6) Cauchy stability follows by considering the system for differences of solution corresponding to different initial data.
(7) The continuation criterion follows from a re-interpretation of an energy estimate valid for the solution.

3.3.1. *Defining the iterative sequence.* Let us abbreviate the system of evolution equations by

$$
\begin{aligned}
L[U]U &= \mathcal{F}[U], \\
U\big|_{t=0} &= U^0
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
U &= (u,v) = (g,-2k) \\
L[U] &= L[(N,g,X)[U]] = \begin{pmatrix} \partial_t - \hat{\nabla}_X & -N \\ -N\hat{\Delta}_g & \partial_t - \hat{\nabla}_X \end{pmatrix}
\end{aligned}
\tag{3.17}
$$

and $\mathcal{F} = \mathcal{F}[U]$ denotes the corresponding right-hand side to make the system above equivalent to the evolution equations for (g,k). Here, the expression $(N,g,X)[U]$ is meant to give (N,X) as the solution to the elliptic system given in terms of g and k. We introduce the solution mapping

$$
\sigma : (g,k) \mapsto (N,X)
\tag{3.18}
$$

by solving the elliptic system

$$
\begin{aligned}
-\Delta N + |k|_g^2 N &= 1 \\
\Delta X^i + R^i_j Y^j - \mathcal{L}_X V^i &= (-2Nk^{mn} + 2\nabla^m X^n)(\Gamma^i_{mn} - \widehat{\Gamma}^i_{mn}) \\
&\quad + 2\nabla^m N k^i_m - \tau \nabla^i N
\end{aligned}
\tag{3.19}
$$

This poses no problem as $-\Delta + |k|^2$ is an isomorphism and so is $\Delta + R^i_j + ...$ under our assumption that γ is of negative sectional curvature.

Lemma 3.20. *Let γ be of negative sectional curvature, $\tau \neq 0$ and $V = 0$. Then*

$$P : H^{s+1} \to H^{s-1} \tag{3.21}$$

is an isomorphism, where

$$PY^i = \Delta Y^i + R^i_j Y^j - \mathcal{L}_Y V^i - 2\nabla^m Y^n (\Gamma^i_{mn} - \widehat{\Gamma}^i_{mn}). \tag{3.22}$$

Moreover,

$$B : H^{s+1} \to H^{s-1} \tag{3.23}$$

is an isomorphism, where

$$B = \Delta - |k|^2_g. \tag{3.24}$$

For what follows it is important to make the following observation.

$$\|\sigma(g_1, k_1) - \sigma(g_2, k_2)\|_{\mathcal{H}^{s+1}} \leq C(R)\|(g_1, k_1) - (g_2, k_2)\|_{\mathcal{H}^s} \tag{3.25}$$

for $(g_i, k_i) \in B^s_R(g_0, k_0)$. A similar estimate holds for $\mathcal{F} = \mathcal{F}[g, k]$.

Iteration. To define the sequence of iterative solution we first approximate the initial data U^0 by smooth functions. Let

$$\{U^0_m\}^\infty_{m=0} \subset C^\infty \cap B^s_R(U^0) \tag{3.26}$$

be a sequence that converges to U^0 in $\mathcal{H}^s$ as $m \to \infty$. Based on this approximation of the initial data we define now the sequence of approximative solutions

$$\{U_m\}^\infty_{m=0} \tag{3.27}$$

on $M \times [0, t]$ inductively.
The first element is given by

$$U_0(t) \equiv U^0_0. \tag{3.28}$$

We the define $\mathcal{F}_0 = 0$ and $L_0 = L[U_0]$.
Then for each $m > 0$ we define

$$\begin{aligned} (N_m, X_m) &= \sigma(U_m) \\ L_m &= L[U_m] \\ \mathcal{F}_m &= \mathcal{F}[U_m] \end{aligned} \tag{3.29}$$

and define U_{m+1} as the solution to the **linear hyperbolic system**

$$\begin{aligned} L_m U_{m+1} &= \mathcal{F}_m \\ U_{m+1}\big|_{t=0} &= U^0_{m+1}. \end{aligned} \tag{3.30}$$

(cf. Sogge, *Lectures on Nonlinear wave equations*, Theorem 3.3).

In the following we will analyze this sequence and show that it converges to a limit U that solves the nonlinear system.

3.3.2. *Boundedness in high regularity.* To prove boundedness of the sequence the essential idea is to derive an **energy estimate** from the evolution equation.
We define the low-order energy by (recall $U_m = (u_m, v_m)$)

$$\mathbf{E}(t, U_{m+1}) \equiv \frac{1}{2}\int_M \left(|u_{m+1}|^2 + |\hat{\nabla} u_{m+1}|^2_{g_m} + |v_{m+1}|^2\right)\mu_{g_m} \tag{3.31}$$

and the higher order energies by

$$\mathbf{E}_s(t, U_{m+1}) \equiv \|U_{m+1}\|_{L^\infty([0,t],\mathcal{H}^s)}. \tag{3.32}$$

Then

$$\mathbf{E}_s(t, U_{m+1}) \leq C \sum_{\ell \leq s} \mathbf{E}(t, \hat{\nabla}^\ell U_{m+1}). \tag{3.33}$$

We define the notation

$$\rho_m = -\frac{1}{2}(\partial_t g_m - \mathcal{L}_{X_m} g_m). \tag{3.34}$$

The following lemma states the energy estimate.

Lemma 3.35.

$$\begin{aligned} \mathbf{E}_s(t, U_{m+1}) &\leq C e^{\left(C \int_0^t (\|\rho_m\|_\infty + \|D\overline{g}_m\|_\infty) dt\right)} \\ &\quad \times \left(\mathbf{E}_s(0, U_{m+1}) + \|\mathcal{F}_m\|_{L^1([0,t],\mathcal{H}^s)} + \int_0^t \|\overline{g}\|_{H^s} \|U_{m+1}\|_{\mathcal{W}^{1,\infty}} dt\right) \end{aligned} \tag{3.36}$$

for $C = C(t, \Lambda[g_m, N_m, X_m])$.
If $s > 3/2 + 1$, then

$$\begin{aligned} \mathbf{E}_s(t, U_{m+1}) &\leq C e^{\left(C \int_0^t (\|\rho_m\|_{H^{s-1}} + \|\overline{g}_m\|_{H^s}) dt\right)} \\ &\quad \times \left(\mathbf{E}_s(0, U_{m+1}) + \|\mathcal{F}_m\|_{L^1([0,t],\mathcal{H}^s)}\right). \end{aligned} \tag{3.37}$$

Proof. We do the computation for the first order energy. Taking the time derivative of $\mathbf{E}$ and using the linear hyperbolic system yields

$$\begin{aligned} \partial_t \mathbf{E}(t, U_{m+1}) = &\int_M \Big(\langle u_{m+1}, N_m v_{m+1} + F_{m,u}\rangle + \langle \hat{\nabla}_i u_{m+1}, \hat{\nabla}_j F_{m,u}\rangle g_m^{ij} \\ &+ \langle \hat{\nabla}_i u_{m+1}, \mathsf{Riem}_{rn} u_{m+1}\rangle X_m^r g_m^{ij} + \langle v_{m+1}, F_{m,v}\rangle\Big)\mu_{g_m} \\ &+ \int_M \Big(\langle \hat{\nabla}_i u_{m+1}, \hat{\nabla}_j u_{m+1}\rangle \rho_m^{ij} \\ &\qquad - \frac{1}{2}(|u_{m+1}|^2 + |\hat{\nabla} u_{m+1}|^2_{g_m} + |v_{m+1}|) g_m^{ij} \rho_{m,ij}\Big)\mu_{g_m}, \end{aligned} \tag{3.38}$$

where $\mathsf{Riem}_{rn} u_{ij} = \hat{R}^l_{irn} u_{lj} + \hat{R}^l_{jrn} u_{il}$.

An application of the Cauchy-Schwarz inequality yields

$$|\partial_t \mathbf{E}| \leq C(\Lambda[\overline{g}_m]) \left(\sqrt{\mathbf{E}} \|\mathcal{F}_m\|_{\mathcal{H}^1} + (1 + \|\rho_m\|_\infty)\mathbf{E}\right). \tag{3.39}$$

This works similar for the higher orders.

Turning this into an equation for $\sqrt{\mathbf{E}}$ and integrating yields

$$\sqrt{\mathbf{E}(t)} \leq \sqrt{\mathbf{E}(0)} + C\left(\|\mathcal{F}\|_{L^1((0,t),\mathcal{H}^1)} + \int_0^t (1+\|\rho(s)\|_\infty)\sqrt{\mathbf{E}(s)}ds \right) \tag{3.40}$$

An application of Gronwall's inequality[10] yields the result. □

Using this energy estimate we are now going to prove boundedness of the sequence.

Lemma 3.41. *There exists a time $t_* > 0$ such that*

$$\{U_m\} \subset L^\infty\left([0,t_*], B_R^s(U^0)\right). \tag{3.42}$$

Proof. We do an induction and assume

$$U_{\ell-1} \in L^\infty\left([0,t_*], B_R^s(U^0)\right). \tag{3.43}$$

We consider the system

$$L_{\ell-1}(U_\ell - U_0) = \mathcal{F}_{\ell-1} - L_{\ell-1}U_0 \tag{3.44}$$

and apply the energy estimate. Observe before that by the assumption there exists a constant C_R depending only on R that bounds the coefficient in the energy estimate, since this only depends on $\ell-1$. In particular the energy estimate reduces to

$$\mathbf{E}_s(t, U_\ell - U_0) \leq C_R\left(\|U_\ell^0 - U_0^0\|_{\mathcal{H}^s} + \|\mathcal{F}_{\ell-1}\|_{L^1([0,t],\mathcal{H}^s)} + \|L_{\ell-1}U_0\|_{L^1([0,t],\mathcal{H}^s)}\right) \tag{3.45}$$

The left-hand side bounds the norm we wish to estimate, $\|U_\ell - U_0\|_{L^\infty([0,t],\mathcal{H}^s)}$ so we need to show that the right-hand side is suffciently small.

Note, that we still have the freedom to choose the time t_* small and that we may manipulate the data of the approximation.
Therefore, we may retrospectively impose the following condition on the initial data approximation,

$$C_R\|U_m^0 - U_{m'}^0\|_{\mathcal{H}^s} < \frac{R}{4}. \tag{3.46}$$

Also by choosing t_* sufficiently small in view of the elementary estimate

$$\|.\|_{L^1([0,t_*])} \leq t_*\|.\|_\infty \tag{3.47}$$

we can ensure

$$\|U_\ell - U_0\|_{L^\infty([0,t_*],\mathcal{H}^s)} < \frac{R}{2}. \tag{3.48}$$

Now, we only need to estimate

$$\begin{aligned} \|U_\ell - U^0\|_{L^\infty([0,t_*],\mathcal{H}^s)} &\leq \|U_\ell - U_0\|_{L^\infty([0,t_*],\mathcal{H}^s)} + \|U_0 - U^0\|_{L^\infty([0,t_*],\mathcal{H}^s)} \\ &< \frac{R}{2} + \|U_0^0 - U^0\|_{\mathcal{H}^s} \end{aligned} \tag{3.49}$$

[10] see appendix

Again by retrospectively imposing smallness by construction of the second term on the left-hand side, which only concerns the approximation of the initial data, the claim follows. ☐

3.3.3. *Convergence in low-regularity.*

Lemma 3.50. *There exists a $t_* > 0$ such that $\{U_\ell\}$ is a Cauchy sequence in $L^\infty([0,t_*],\mathcal{H}^1)\cap C^{0,1}([0,t_*],\mathcal{H}^0)$.*

Proof. We know that a time $t_* > 0$ exists for which the sequence is bounded. Here, we may decrease this t_* to obtain convergence.

We consider the equation

$$L_{\ell-1}(U_\ell - U_{\ell'}) = \mathcal{F}_{\ell-1} - \mathcal{F}_{\ell'-1} - (L_{\ell-1} - L_{\ell'-1})U_{\ell'}. \tag{3.51}$$

It is of the same form as the evolution equations above, so we can use the energy estimate and obtain immediately for $t \le t_*$

$$\begin{aligned}\|U_\ell - U_{\ell'}\|_{L^\infty([0,t],\mathcal{H}^1)} \le C_R\Big(&\|U^0_\ell - U^0_{\ell'}\|_{\mathcal{H}^1} + \|\mathcal{F}_{\ell-1} - \mathcal{F}_{\ell'-1}\|_{L^1([0,t],\mathcal{H}^1)}\\ &+ \|(L_{\ell-1} - L_{\ell'-1})U_{\ell'}\|_{L^1([0,t],\mathcal{H}^1)}\Big)\end{aligned} \tag{3.52}$$

Using the fact that the difference of the $\mathcal{F}$-terms and the L-terms can be estimated by the difference of the U terms and that $DU_{\ell'}$ is uniformly bounded in $\mathcal{H}^1$ by the previous lemma, we obtain, choosing t_* sufficiently small,

$$\|U_\ell - U_{\ell'}\|_{L^\infty([0,t_*],\mathcal{H}^1)} \le C_R\|U^0_\ell - U^0_{\ell'}\|_{\mathcal{H}^1} + \frac{1}{2}\|U_{\ell-1} - U_{\ell'-1}\|_{L^\infty([0,t_*],\mathcal{H}^1)}. \tag{3.53}$$

From this inequality we will deduce the convergence as follows. As we are only interested in the limit of the sequence, we can consider a thinned out subsequence approximating the initial data by abuse of notation also denoted by $\{U^0_m\}$ such that

$$C_R \cdot \sum_{\ell\ge 1} \|U^0_{\ell+1} - U^0_\ell\|_{\mathcal{H}^1} < \frac{R}{2}. \tag{3.54}$$

Using the notations

$$\begin{aligned} a_\ell &:= \|U_\ell - U_{\ell-1}\|_{L^\infty([0,t_*],\mathcal{H}^1)}\\ b_\ell &:= C_R\|U^0_{\ell+1} - U^0_\ell\|_{\mathcal{H}^1}\end{aligned} \tag{3.55}$$

the above inequality then reads

$$a_{\ell+1} \le b_m + \frac{1}{2}a_\ell \tag{3.56}$$

for $m \ge 1$. Taking the sum $\sum_{\ell\ge 1}$ then yields

$$\sum_{\ell\ge 1} a_\ell \le 2\sum_{\ell\ge 1} b_\ell + 2a_1 < \infty \tag{3.57}$$

and in particular $\|U_\ell - U_{\ell-1}\|_{L^\infty([0,t_*],\mathcal{H}^1)} = a_\ell \to 0$, i.e. $\{U_\ell\}$ is Cauchy. ☐

3.3.4. *Regularity of the solution.* By some technical steps one can then show that the sequence converges weakly in higher regularity, the limit U in fact solves the equation and has the regularity properties as required. For details we refer to [1].

3.3.5. *Cauchy stability.* The approach to proving Cauchy stability is to consider two solution with different intial data and derive an evolution equation for their difference. This equation can then be analyzed using again the energy estimate from above.

3.3.6. *Continuation criterion.* For the continuation criterion we observe that the solution U fulfills the integral estimate

$$\mathbf{E}_s(t,U) \leq C\mathbf{E}_s(0,U)e^{\left(C\int_0^t(\|\rho\|_\infty+\|D\overline{g}\|_\infty)dt\right)} \tag{3.58}$$

and in turn the energy cannot blow-up towards the end of the interval unless the characteristic quantities do. This provides a uniformly bounded sequence of initial data arbitrarily close to the potential blow-up time, which in turn allows to extend the solution and contradicts the assumption of maximality.

We conclude that all essential steps in the analysis of the local-existence problem are based on the observation that the approximative solutions and the solution itself fulfil a type of energy estimate.

3.3.7. *Conservation of gauges.* It remains to prove that the solution we constructed above in fact solves the Einstein equations and not just the reduced system. This follows from the *propagation of the gauge conditions and constraint equations.*
We know that they are fulfilled initially at $t = 0$. In fact they hold throughout the evolution, which yet again follows from an energy argument.
Let us define

$$\begin{aligned} A &= \mathsf{tr}_g k - t \\ V^k &= g^{ij}(\Gamma^k_{ij} - \widehat{\Gamma}^k_{ij}) \\ F &= R(g) + (\mathsf{tr}_g k)^2 - |k|^2_g - \nabla_i V^i \\ D_i &= \nabla_i(\mathsf{tr}_g k) - 2\nabla^m k_{mi}. \end{aligned} \tag{3.59}$$

Moreover, we consider the H^1-energy

$$\mathcal{E} \equiv \int (|A|^2 + |\nabla A|^2 + |F|^2 + |V|^2 + |\nabla V|^2 + |D|^2)\mu_g. \tag{3.60}$$

Vanishing of this energy is equivalent to validity of gauge conditions and constraint equations. Straightforward computations yield

Lemma 3.61. *Let (g,k,N,X) be a solution to reduced system, sufficiently regular, then there exists a constant $C = C(g,k,N)$ such that*

$$|\partial_t \mathcal{E}| \leq C\mathcal{E}. \tag{3.62}$$

An application of Gronwall's identity immediately implies that $\mathcal{E}$ is identically zero along the time interval.

4. Global existence and stability

We consider now the so-called *Milne model*,

$$\left((0,\infty)\times M, -dt^2+\frac{t^2}{9}\gamma\right), \tag{4.1}$$

where (M,γ) is a compact Riemannian Einstein manifold with Einstein constant $-2/9$, i.e.

$$\mathsf{Ric}[\gamma]_{ij} = -\frac{2}{9}\gamma. \tag{4.2}$$

The Milne model is a future complete solution to the Einstein vacuum equations.

The stability of this model discussed in the following was proven in [2]. We refer to this paper for details.

4.1. **CMC foliation of the Milne model.** The Milne model can be CMC foliated, which can be analyzed by a simple computation and comparison with the CMCSH reduced Einstein equations. In this formalism the Milne model is given by

$$\begin{aligned} g_b &= \frac{t^2}{9}\gamma \\ k_b &= -\frac{1}{t}g_b \\ \tau &= -\frac{3}{t}\in(-\infty,0) \\ N_b &= 1 \\ X_b &= 0 \end{aligned} \tag{4.3}$$

So, if we consider for instance the initial data set induced by the Milne model at $t=1$ we obtain

$$(M, g_{b,0}, k_{b,0}) = (\frac{1}{9}\gamma, -\frac{1}{9}\gamma), \tau_0 = -3. \tag{4.4}$$

Remark 4.5. *We observe here that the second fundamental form, k is a multiple of the metric γ (an in particular of g). If we consider a perturbation of this tensor field, it will have a part which is not proportional to the metric, which is the the so-called* trace-free *part. So for what follows we decompose the general second fundamental form into a tracefree part and the trace-part,*

$$k = \Sigma + \frac{\tau}{3}g. \tag{4.6}$$

4.2. **Nonlinear stability of the Milne-model.** We wish to prove the following theorem.

Theorem 4.7 (Andersson-Moncrief, [2])**.** *Let $s>\frac{n}{2}+1$. Then there exists a $\varepsilon>0$ such that for all initial data sets (M,g_0,k_0) with*

$$\|g_0-g_{b,0}\|_{H^s}+\|k_0-k_{b,0}\|_{H^{s-1}}<\varepsilon \tag{4.8}$$

their respective MGHD has a foliation by CMC-surfaces taking values in $[\tau_0, 0)$, *it is future complete and the spatial geometry converges in a suitable sense (to be determined later) to the corresponding geometry of the Milne spacetime.*

We prove this theorem in the remainder of these notes.

4.3. **Rescaling the equations.** The spatial geometry of the spatial slices in the background geometry grows in time. If this geometry is appropriately rescaled in terms of the mean curvature we obtain a geometry which is "constant" in time. This turns out to be helpful for the global analysis of the MGHD of perturbations. Abusing the notation we now rename the original quantities with a $\widetilde{\cdot}$-symbol to denote the rescaled quantities by the original symbols. Obviously, the rescaling to obtain a "constant" metric is given by

$$g := \tau^2 \widetilde{g} \tag{4.9}$$

By a careful analysis of the physical dimension of the respective quantities (cf. [2]) the natural rescaling for lapse, shift and tracefree part of the second fundamental form is

$$\begin{aligned} \Sigma &:= \tau(\widetilde{k} - \frac{\tau}{3}\widetilde{g}) \\ N &:= \tau^2 \widetilde{N} \\ X^i &:= \tau \widetilde{X}^i. \end{aligned} \tag{4.10}$$

This rescaling yields a new reduced system for the variables (g, Σ, N, X), which is *almost autonomous*. To erase any explicit dependence on the mean-curvature time τ we introduce the new time variable

$$T := -\ln\left(\frac{\tau}{\tau_0}\right) \tag{4.11}$$

yielding

$$-\frac{\partial}{\partial T} = \tau \partial_\tau. \tag{4.12}$$

The range of T is $\mathbb{R}$, where $T \to \infty$ corresponds to $\tau \nearrow 0$. In these new variables the background spacetime (the Milne model) takes the form

$$\begin{aligned} g_b &= \gamma \\ \Sigma_b &= 0 \\ N_b &= 3 \\ X_b &= 0. \end{aligned} \tag{4.13}$$

When we introduce for s sufficiently large the Sobolev norm

$$\|.\|_{H^s} \tag{4.14}$$

with respect to some once for all fixed metric on M we wish to show for the development (g, Σ, N, X) of a sufficiently small initial perturbation (g_0, k_0)

$$\|g - \gamma\|_{H^s} + \|\Sigma\|_{H^{s-1}} + \|N - 3\|_{H^{s+1}} + \|X\|_{H^{s+1}} \overset{T\to\infty}{\longrightarrow} 0. \tag{4.15}$$

4.4. **The rescaled system.** We give the rescaled elliptic-hyperbolic system.

(4.16)
$$\begin{aligned}
\partial_T g_{ij} &= 2N\Sigma_{ij} + 2(\frac{N}{3}-1)g_{ij} - \mathcal{L}_X g_{ij}\\
\partial_T \Sigma_{ij} &= -2\Sigma_{ij} - N(R_{ij} - \delta_{ij} + \frac{2}{9}g_{ij})\\
&\quad + \nabla_i\nabla_j N + 2N\Sigma_{im}\Sigma^m_j - \frac{1}{3}(\frac{N}{3}-1)g_{ij}\\
&\quad - (\frac{N}{3}-1)\Sigma_{ij} - \mathcal{L}_X\Sigma_{ij}\\
-\Delta N + (|\Sigma|^2_g + \frac{1}{3})N &= 1\\
\Delta X^i + R^i_j X^j - \mathcal{L}_X V^i - 2\nabla^m X^n(\Gamma^i_{mn} - \widehat{\Gamma}^i_{mn}) &= -2N\Sigma^{mn}(\Gamma^i_{mn} - \widehat{\Gamma}^i_{mn}) - 2(\frac{N}{3}-1)\\
&\quad + 2(\nabla^m N)\Sigma^i_m - \nabla^i(\frac{N}{3}-1)
\end{aligned}$$

We use the system in the following to derive a priori estimates for its solutions. We start with the elliptic system. Before let us make a global bootstrap assumption of the form

$$(g,k) \in B^s_\delta(\gamma, 0) \tag{4.17}$$

for δ sufficiently small but large with respect to the initial perturbation of order ε. By a bootstrap argument we will be able to justify this assumption.

4.5. **Elliptic estimates.** Under the smallness assumption the following elliptic estimates hold.

Lemma 4.18.

(4.19)
$$\begin{aligned}
\|\frac{N}{3} - 1\|_{H^{s+1}} &\leq C(\delta)\|\Sigma\|^2_{H^{s-1}}\\
\|X\|_{H^{s+1}} &\leq C(\delta)\left(\|\Sigma\|^2_{H^{s-1}} + \|g-\gamma\|^2_{H^s}\right)
\end{aligned}$$

4.6. **Energy estimates.** Before writing down the appropriate energies for Σ and $g-\gamma$ we refine the evolution equation for Σ a bit more using the following Lemma.

Lemma 4.20.

$$R_{ij} - \partial_{ij} + \frac{2}{9}g_{ij} = \frac{1}{2}\mathcal{L}_{g,\gamma}(g-\gamma) + J_{ij}, \tag{4.21}$$

where

$$\|J\|_{H^{s-1}} \leq C(\delta)\|g-\gamma\|^2_{H^s} \tag{4.22}$$

and[11]

$$\mathcal{L}_{g,\gamma}h_{ij} = -\Delta_{g,\gamma}h_{ij} - 2R[\gamma]_{icjd}h^{cd}. \tag{4.23}$$

[11] Here, $\Delta_{g,\gamma}$ is the same operator as $\hat{\Delta}_g$ before.

To define the energies we need to define two numbers depending on the lowest positive eigenvalue of the operator $\mathcal{L}_{\gamma,\gamma}$, denoted by λ_0 (we know $\lambda_0 \geq 1/9$). Let

$$\alpha = \begin{cases} 1 & \lambda_0 > 1/9 \\ 1-\delta_\alpha & \lambda_0 = 1/9, \end{cases} \tag{4.24}$$

where $\delta_\alpha = \sqrt{1-9(\lambda_0-\varepsilon')}$ for some small ε' and

$$c_E = \begin{cases} 1 & \lambda_0 > 1/9 \\ 9(\lambda_0-\varepsilon') & \lambda_0 = 1/9. \end{cases} \tag{4.25}$$

For convenience we define now

$$u = g - \gamma, \quad v = 6\Sigma, \quad \omega = \frac{N}{3} \tag{4.26}$$

Then the evolution equations reduce to

$$\begin{aligned} \partial_T u &= \omega v - X^i \hat{\nabla}_i u + \mathcal{F}_u \\ \partial_T v &= -2v - 9\omega \mathcal{L}_{g,\gamma} u - X^i \hat{\nabla} v + \mathcal{F}_v \end{aligned} \tag{4.27}$$

where

$$\begin{aligned} \|\mathcal{F}_u\|_{H^s} &\leq C(\|u\|^2_{H^s} + \|u\|^2_{H^{s-1}}) \\ \|\mathcal{F}_v\|_{H^{s-1}} &\leq C(\|u\|^2_{H^s} + \|u\|^2_{H^{s-1}}) \end{aligned} \tag{4.28}$$

We define the quadratic part of the energy by

$$\mathcal{E}_s = \frac{1}{2}\int_M \langle v, \mathcal{L}^{s-1}_{g,\gamma} v\rangle \mu_g + \frac{9}{2}\int_M \langle u, \mathcal{L}^s_{g,\gamma} u\rangle \mu_g \tag{4.29}$$

and the diagonal part by

$$D_s = \int_M \langle v, \mathcal{L}^{s-1}_{g,\gamma} u\rangle \mu_g \tag{4.30}$$

The total energy of order ℓ is given by

$$\mathbf{E}_\ell(u,v) = \sum_{i=1}^{\ell} \mathcal{E}_i + c_E D_i. \tag{4.31}$$

Lemma 4.32 (Coercivity)**.** *For $\delta > 0$ sufficiently small*

$$\|g-\gamma\|^2_{H^s} + \|\Sigma\|^2_{H^{s-1}} \leq C(\delta)\mathbf{E}_s. \tag{4.33}$$

Note that the choice of c_E is crucial for the previous lemma.

Lemma 4.34 (First order energy identity)**.** *(case $c_E = 1$ for simplicity)*

$$\begin{aligned} \partial_T \mathcal{E}_1 &= -2\int_M |v|^2 \mu_g + U \\ \partial_T D_1 &= -2\int_M \langle v, u\rangle \mu_g + \int_M |v|^2\mu_g - 9\int_M \langle \mathcal{L}_{g,\gamma} u, u\rangle \mu_g + V, \end{aligned} \tag{4.35}$$

where U and V are higher order, i.e. they are bounded by $\|u\|^3_{H^s} + \|v\|^3_{H^{s-1}}$. In combination this yields

$$\partial_T \mathbf{E}_1 = -2E_1 - 2D_1 + W, \tag{4.36}$$

where $W = U + V$.

A similar equality also holds for the higher order norms, so in total we obtain

$$\partial_T \mathbf{E}_s \le -2\mathbf{E}_s + C\mathbf{E}_s^{3/2}. \tag{4.37}$$

So for the squareroot $Y = \sqrt{\mathbf{E}_s}$

$$\dot{Y} \le -Y + CY^2, \tag{4.38}$$

where the equality equation is solved by

$$y = \frac{1}{C + e^{(T-T_0)}(y(0)^{-1} - C)} \lesssim e^{-T}. \tag{4.39}$$

In particular,

$$\|g - \gamma\|_{H^s} + \|\Sigma\|_{H^{s-1}} \lesssim e^{-T}. \tag{4.40}$$

The continuation criterion implies global existence and the rate of decay timelike- and null-geodesic completeness towards the future (cf. a theorem by Choquet-Bruhat/Cotsakis).

Appendix

Lemma 4.41. *Assume*

$$\mathbf{E}(t) \leq G(t) + \int_{t_0}^{t} k(s)\mathbf{E}(s)ds \tag{4.42}$$

for all $t \in [t_0, t_*]$.
Then, if $\mathbf{E}$, k *and* G *non-negative,* $\mathbf{E} \in L^\infty([t_0, t_*])$, $k \in L^1([t_0, t_*])$ *and* G *non-decreasing, we have*

$$\mathbf{E}(t) \leq G(t) \exp\left(\int_{t_0}^{t} k(s)ds\right). \tag{4.43}$$

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