4 Logistic growth – the transcritical bifurcation

In the previous section, the stability of homogeneous steady states was examined. Now we extend our interest to situations, where the stability properties of a steady state change in dependence of a parameter, i.e. *bifurcations.* We consider the simplest nonlinear model of population dynamics.

The dynamics of a spatially homogeneous population with density $p(t)$, living in an environment with limit resources, can be described by the ordinary differential equation

$$\frac{dp}{dt} = \alpha \left(1 - \frac{p}{\bar{p}}\right) p = \alpha p - \beta p^2 \,,$$

where the right hand side is called *logistic growth* with the growth rate α of small populations and the critical density p_0 above which the growth rate is negative. By an appropriate nondimensionalization, the second parameter can be eliminated:

$$\frac{dp}{dt} = ap - p^2 \,,$$

where a is a dimensionless parameter. Although biologically irrelevant, it is instructive to observe what happens when a takes both positive and negative values. A bifurcation then occurs at $a = 0$ where the trivial steady state $p = 0$ changes its stability properties from stable for $a < 0$ to unstable for $a > 0$. The stability properties of the second steady state $p = a$ are just opposite. We say that the nontrivial steady state *bifurcates* from the trivial steady state at $a = 0$. There an *exchange of stability* between the two steady states occurs. This situation is called a *transcritical bifurcation.*

Actually, exactly the same happens, if we allow a spatially nonhomogeneous diffusing population $p(x,t)$ on a bounded domain with zero flux boundary conditions:

$$\begin{aligned} &\partial_t p = \Delta p + ap - p^2 \,, \quad \text{for } x \in \Omega \subset \mathbb{R}^d \,, \\ &\nu \cdot \nabla p = 0 \,, \quad \text{on } \partial\Omega \,. \end{aligned}$$

A transcritical bifurcation with the homogeneous steady states $p = 0$ and $p = a$ occurs.

A more interesting situation occurs under different boundary conditions. Consider a situation where Ω does not represent a container. The individuals can leave the domain; however, the environment is hostile to them outside of Ω, such that they cannot survive there. This can be modelled by the

(*Dirichlet*) boundary conditions $p = 0$ on $\partial\Omega$. For computational simplicity we consider a one-dimensional situation:

$$\begin{aligned} &\partial_t p = \partial_x^2 p + ap - p^2\,, \quad \text{for } 0 < x < \pi\,, \\ &p(0,t) = p(\pi,t) = 0\,. \end{aligned} \tag{1}$$

The length π of the domain has been achieved by the nondimensionalization. The Dirichlet boundary conditions still permit the trivial steady state $p = 0$, but eliminate the other homogeneous steady state. Linearization at $p = 0$ leads to the eigenvalues $\lambda_j = a - j^2$, $j = 1, 2, \ldots$, and the according eigenfunctions $\varphi_j(x) = \sin(jx)$. So the trivial steady state is stable even for positive $a < 1$. Diffusion subject to Dirichlet boundary conditions has a stabilizing effect. At $a = 1$, the largest eigenvalue λ_1 changes sign.

We shall demonstrate by an asymptotic analysis for values of a close to 1 that again a transcritical bifurcation occurs. We choose a parameter ε taking small positive and negative values and set $a = 1 + \varepsilon$. For the solution we make the ansatz

$$p(x,t) = \varepsilon p_0(x, |\varepsilon| t) + \varepsilon^2 p_1(x, |\varepsilon| t) + O(\varepsilon^3)\,,$$

which takes into account that we expect solutions close to the trivial steady state varying slowly. Substitution in (1) and comparing coefficients of equal powers of ε leads to

$$\begin{aligned} &0 = \partial_x^2 p_0 + p_0\,, \quad p_0(0,\tau) = p_0(\pi,\tau) = 0\,, \\ &\sigma \partial_\tau p_0 = \partial_x^2 p_1 + p_1 + p_0 - p_0^2\,, \quad p_1(0,\tau) = p_1(\pi,\tau) = 0\,, \end{aligned} \tag{2}$$

where we have set $\tau = \varepsilon t$ and $\sigma = \operatorname{sign} \varepsilon$. The first line is a linear problem for p_0 with the solution $p_0(x,\tau) = A(\tau) \sin x$, where $A(\tau)$ can be chosen arbitrarily. The second line can be seen as an inhomogeneous version of the same linear problem, now with the unknown p_1, if p_0 is considered as known.

Now the idea is the following: Since the kernel of the linear problem is nontrivial, we expect the inhomogeneous problem to require a solvability condition on the inhomogeneity. This solvability condition will provide the missing information for determining the leading term p_0 completely.

Actually, the solvability condition is obtained by multiplication with $\sin x$ and integration with respect to x from 0 to π. Doing this with (2) after substitution of the general solution for p_0 gives an ordinary differential equation for the as yet unknown coefficient $A(\tau)$:

$$\frac{dA}{d\tau} = \sigma \left(A - \frac{8}{3\pi} A^2 \right)\,.$$

We deduce the existence of a second steady state which is close to $(a - 1)\frac{3\pi}{8}\sin x$ for a close to 1, and which is unstable for $a < 1$ and stable for $a > 1$. Thus, a transcritical bifurcation occurs with a bifurcating nonhomogeneous steady state. The bifurcating solution is biologically relevant only for $a > 1$, since it is negative for $a < 1$.