The philosophical (ir)relevance of Gödel's proof

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'Gödel's proof'[1] is one of the classic results which have shown the limitations of logic in the first half of the twentieth century. According to Gödel a formal system in which it is possible to formulate the arithmetics of whole numbers necessarily produces some undecidable statements. Thus a straightforward version of the Hilbert program must fail.

That's for that. But what does this imply for philosophy *beyond* the field of foundations of mathematics? Is *the whole world* to be seen as a 'victim of Gödelisation'?—The importance of Gödel's proof for the foundational debate in mathematics and for any kind of philosophy of mathematics is out of question. What I will ask here, however, is whether there is a philosophical relevance of Gödel's proof *outside* of the field of philosophy of mathematics. What can we learn from Gödel's proof if we are not concerned with mathematics but, roughly speaking, with 'the world out there'?

A Gödelian question in that field of philosophy could be the following: 'Is it possible to provide a formal language in which we are able to formulate any meaningful statement about the real (spatio-temporal) world so that in a particular *true* semantic interpretation of that language it must be *decidable* for any such statement whether or not it is true in that interpretation?' In a way such a question defines a counterpart to the Hilbert program for the external world. What I will show here is that, fortunately, we do not have reason to believe that such a real-world-version of the Hilbert program will fail. There is a straightforward way to show that languages about the real world *can* be defined in such a way that Gödel's proof (or any other limitation result from the mathematical foundation debate) is no challenge to them. Therefore, what I will hold here is that Gödel's proof has no real philosophical relevance outside of the field of foundations of mathematics.

We obtain this result on the basis of two philosophical assumptions. First, we assume that there is no need for an ontology of the real (spatio-temporal) world to include such things as mathematical objects. A real-world-ontology can be constructed in a much simpler way, because we can *assume* the existence of mathematics and can use mathematics in a *naive* way. Second, we assume

[1] Kurt Gödel (1931): Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik* 38, 173-198.

that we should be able to give a description of the spatio-temporal world with a repertoire of *only finitely many objects* which is at least approximately complete and at least approximately true.

Given those restricting assumptions we ask on the other hand for a language with a maximum of expressive possibilities. The language should be a relational language, constructed in a many-sorted way, so that we can modulate higher order and other complex kinds of relations (in the sense of Henkin-semantics). Further we claim that our language has a device for quantification over structures so that we can formulate statements like 'ϕ is necessarily true', 'ϕ is valid' or even 'ϕ is true in the actual world' in that language.

The language $\mathsf{FIN}(\mathcal{S}, \mathcal{P}, \mathcal{P}^*, \mathfrak{M})$ is divided into a non-modal and a modal part. The non-modal part is defined by a finite set $\mathcal{S}$ of finite and two by two disjoint sets—the *sorts* of the language—and a finite set $\mathcal{P}$ of 'predicates' or 'meaningful combinations of sorts'. Let N be the set of all finite and non-empty sequences of elements of $\mathcal{S}$; then $\mathcal{P}$ is defined as a finite subset of N. We define:

$$\mathcal{P}_\times := \bigcup_{(s_j)_{j=1}^i \in \mathcal{P}} s_1 \times \ldots \times s_i.$$

In order to define FIN as a free logic we require that every sort of $\mathcal{S}$ is an element of $\mathcal{P}$. A structure $\mathfrak{A}$ is then defined as a subset of $\mathcal{P}_\times$ so that for every atomic formula $c_1, \ldots, c_i$ it holds:

$$c_1, \ldots, c_i \in \mathfrak{A} \to (c_j)_{j=1}^i : c_j \in \mathfrak{A}.$$

Therefore we can define for every term the existence predicate $E(c)$ as fulfilled iff $c \in \mathfrak{A}$. We call $\mathbb{A}$ the (finite) set of all FIN-structures.

Now we define the modal part of the language. Let N' be the set of all finite sequences of elements of $\mathcal{S} \cup \{\mathbb{A}\}$ and $N^* := N' \setminus \mathcal{P}$. Then $\mathcal{P}^*$ is defined as a finite subset of N^* and we define:

$$\mathcal{P}^*_\times := \bigcup_{(s_j)_{j=1}^i \in \mathcal{P}^*} s_1 \times \ldots \times s_i.$$

The *modal model* $\mathfrak{M}$ is simply a subset of $\mathcal{P}^*_\times$. Every element of a sort out of $\mathcal{S} \cup \{\mathbb{A}\}$ we define as a *constant*. Additionally, we have an $\mathbb{A}$-constant $\mathtt{SELF}$ with $\mathtt{SELF} \notin \mathbb{A}$ and define, for structures $\mathfrak{A}$ and every constant $c \neq \mathtt{SELF}$:

$$\mathfrak{A}(c) := c,$$
$$\mathfrak{A}(\mathtt{SELF}) := \mathfrak{A}.$$

For structures $\mathfrak{A}$ we define $E(\mathfrak{A}) := \mathfrak{A}$ (because the modal part of the language is not defined in the sense of a free logic). Every sequence of constants is an *atomic formula*. The syntax of FIN is defined as:

$$\phi ::= \ p \mid a \Vdash \phi \mid \neg\phi \mid \phi \wedge \phi,$$

where p ranges over atomic formulae and a over $\mathbb{A}$-constants. In addition to the usual rules for $\neg$ and $\wedge$ we define, for every atomic formula $(c_1, \ldots, c_n)$, every $\mathbb{A}$-constant a, every structure $\mathfrak{A}$ and every formula ϕ:

$$\begin{array}{lll} \mathfrak{A} \models (c_1, \ldots, c_n) & \text{iff} & (c_1, \ldots, c_n) \in \mathfrak{A} \quad \text{or} \quad (\mathfrak{A}(c_1), \ldots, \mathfrak{A}(c_n)) \in \mathfrak{M}, \\ \mathfrak{A} \models a \Vdash \phi & \text{iff} & \mathfrak{A}(a) \models \phi. \end{array}$$

FIN is decidable, because there are only finitely many atomic formulae and thus the language can be shown to be equivalent to a propositional logic over a finite set of propositional constants. We can realize quantification in this setting via some finite conjunctions and disjunctions. Let $c \in s$ be a constant, $c_1, \ldots, c_n$ a sequence which contains every element of s, and x an arbitrary symbol which we call an *s-variable*. Then we define, for every formula ϕ:

$$\forall x \phi \left[\frac{x}{c}\right] := \left(E(c_1) \rightarrow \phi\left[\frac{c_1}{x}\right]\right) \wedge \ldots \wedge \left(E(c_n) \rightarrow \phi\left[\frac{c_n}{x}\right]\right),$$

We introduce modal operators $\Box$, in the usual sense:

$$\Box\phi \quad \text{iff} \quad \forall a : (R, \texttt{SELF}, a) \rightarrow a \Vdash \phi,$$

R is interpreted as a relation over possible worlds, a is an $\mathbb{A}$-variable. If R is defined as a total relation, so that $(R, \mathfrak{A}, \mathfrak{A}')$ is valid for every pair $(\mathfrak{A}, \mathfrak{A}')$ of structures we obtain the special case that $\Box\phi$ expresses 'ϕ is valid'.

If $\mathcal{S}$ contains all the objects of the real world, $\mathcal{P}$ defines all the relevant relations between those objects, and $\mathcal{P}^*$ all the relevant relations over possible worlds (or other 'stable' relations), then we can formulate *any* possible (modal or non-modal) statement about the real world as a FIN-formula. Finally, if `actual` is a FIN-structure that depicts exactly those relations or propositions which are true in the *actual* world, then a particular statement ϕ about the real world is *true* iff it is satisfied in `actual`. Therefore, we can define (as an element of the language FIN) the *truth-predicate* $T(\phi)$, for every formula ϕ:

$$T(\phi) := \texttt{actual} \Vdash \phi.$$