PHILOSOPHICAL LOGIC SHOULD USE MATHEMATICS, NOT META-MATHEMATICS.*

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Suppes' proposal

In his *Introduction to Logic* from 1957 Patrick Suppes presented the layout of a new formal account of scientific theories which is now best-known as the 'semantic view' of theories.

Suppes' proposal was to define a theory not via axioms of elementary (first-order) logic but by defining a so-called set-theoretical predicate, i. e. a theory is constructed by a particular number of sets, relations and functions that fulfill some particular axioms.

☞ *The language* which is used for the specification of a theory here is not first-order logic but set theory.

Can we 'Suppesisize' logics?

Actually, trying to define logics outside of a logical framework, at the first glance sounds like a bad joke.

☞ The point is the difference between *mathematical logic*, i. e. the meta-mathematical perspective of logic and *philosophical logic*, i. e. the non-mathematical perspective of logic.

In other words: we can take a logic[a] alternatively *either* as a mathematical entity *or* as a philosophical (and non-mathematical) entity.

[a]In what follows I assume tacitly that there is *a plurality* of entities which we like to call 'logic'.

Mathematical logic, informally

A logic as a mathematical entity is a language that enables us to define some things (classes, categories, sets) whose properties are such rather simple things like fulfilling the law of induction or the continuums hypothesis, e. g. such a language should enable us to define such things like the set of all natural or all real numbers.

☞ In other words: logic as a mathematical entity enables us to specify something like *an ontology for mathematics.*

The discussion, then, is concerned with questions like: Is it possible to specify a class of mathematical objects x in a language y up to isomorphism? Is there a language y that enables us to specify x so that at the same time a statement is provable in y (with x-axioms) if and only if this statement is true?

Philosophical logic, informally

A philosophical logic is a language that has some elementary objects like names for *individuals, properties* and *functions*, like *variables, quantifiers, modal operators* and other *intensional devices*.

☞ Given a particular language L in the sense just described, the principal layout of philosophical investigations about L is this:

> Given a universe that consists of a suitable family of sets O of (possible) individuals, (possible) properties, functions, etc.: how can we use L here, in order to express everything what can be expressed in principle about such a given 'ontology' O?

Mathematical logic, technically

Given a language L in the sense described above there are at least two main strategies for *mathematical reasoning about* L: the syntactic and the semantic strategy

(1) We can specify a deductive system D for L, i. e. a relation over the power set of L that assigns to every set of formulas a set of logical consequences.

(2) We can specify a model theoretic framework M for L, i. e. a set of rules that shows how to connect L with a particular ontology O. [a]

☞ *Technically, a mathematical logic is a language L plus a deductive system D and/or a model theoretic framework M.*

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[a] Most of the questions discussed in mathematical logic are questions about the interconnection between (1) and (2), of course.

Philosophical logic, technically

In principle a philosophical logic is nothing but a language L plus a particular ontology O and a set of rules that show how to connect L to O.

☞ Given that significant *directness* of the connection between language and ontology a modification suggests itself which appears to be technically trivial but philosophically highly illuminating:

Just introduce your language in such a way that every particular *name* of L is thought to denote one particular object so that the distinction between L and O collapses.

☞ A philosophical logic, then, is nothing else than an *interpreted language* L'.

But what exactly is an 'interpreted language'?

Let L_a be a language of propositional logic and L_p a language of first-order logic. Then we could state that L_a in fact is interpreted because every propositional constant represents a particular 'proposition'; to take L_p as an interpreted language we simply would have to stipulate that every individual constant and every predicate constant of the language shall designate some particular object (cf. Kripkes notion of *direct reference*).

But now, the entities (formulas) of those languages clearly do not have a reference in itself. Neither the formulas of L_a nor the formulas of L_p would have a plausible definition for truth values, as long as we *only* stipulate direct reference.

An interpreted language must be constructed in such a way that not only the names of the language have fixed designations but that every formula has a fixed truth-value!

Formulas like p (atomic proposition) or $p \to q$, etc. does not have a truth value, because they are neither tautologies nor contradictions. Thus, to get an interpreted version of a language like L_a or L_p we need formulas of the form

$$w \Vdash p,\ w \Vdash p \to q,\ \text{etc.},$$

where w designates a 'structure' and $\Vdash$ that the formula on the right is satisfied in the structure on the left.

An interpreted propositional language

The language L'_a is constructed like L_a but with two additional linguistic elements: (1) if A is the set of all propositional constants of L_a then we have the power-set W of A as an additional category of names in L'_a, (2) there is an additional language device $\Vdash$ for which we have the syntactic rule:

> if ϕ is a formula and w is an element of W then $w \Vdash \phi$ is also a formula.

Now we restrict L'_a to all those formulas which have the form $w \Vdash \phi$ and we define a truth value for every formula in the obvious sense that ϕ is true, iff it is true in a world where w represents the set of all true atomic propositions.

An interpreted first-order language

In L'_p, again, we have a set of 'worlds' W and a device $\Vdash$ that express truth of a formula in a world.

A world $w = (D_\exists, \alpha)$ consists of a subset $D_\exists$ of the set D of all individual constant (i. e. all rigid designators) of L'_p and a function α that assigns to the predicates and functions of the language relations and functions over $D_\exists$.

☞ L'_p is a free logic.

☞ In L'_p we can express satisfaction over structures whose domains are subsets of D.

Modality in interpreted languages

To enrich an interpreted language like L'_a and L'_p with modal aspects we have to add

1. an additional set of predicates that range over the set W of worlds,
2. an additional set of variables that range over W,
3. a constant $\aleph$ that assigns on every place of a formula the world which is actual on this place,
4. a modal interpretation $\mathfrak{W}$ that assigns to every modal predicate a relation over W.

If r is a binary modal predicate we can define

$$\Box\phi := \forall y : r(\aleph, y) \rightarrow y \Vdash \phi$$

and have a perfect expression for modality in the Kripkean sense.

Meta-logical aspects

If we define logical consequence via conjunction and implication (and with possibly infinitely long formulas) there remains one central meta-logical question: decidability of a formula.

Roughly speaking, the formulas of an interpreted language are decidable, iff the set W of all worlds of the language is finite.

In this case there is also no need for a calculus, because we can decide every formula via truth table method.

Conclusion

☞ Interpreted languages are a presentation of logic in a set theoretic framework. Thus we indeed have an aspect of logic here that uses mathematics not meta-mathematics ('Suppesization' of logics).

☞ For the philosophical aspects of logics the mathematical point of view has the same status like the 'received view' of scientific theories had in philosophy of science in the 50th.

Ultimately, logicians should use interpreted languages if they like to express philosophical features of logics.