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Quasi-isometries between graphs and trees ☆

Bernhard Krön [a], Rögnvaldur G. Möller [b]

[a] University of Vienna, Faculty of Mathematics, Nordbergstraße 15, 1090 Wien, Austria
[b] Science Institute, University of Iceland, Dunhaga 3, 107 Reykjavik, Iceland

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Abstract

Criteria for quasi-isometry between trees and general graphs as well as for quasi-isometries between metrically almost transitive graphs and trees are found. Thereby we use different concepts of thickness for graphs, ends and end spaces. A metrically almost transitive graph is quasi-isometric to a tree if and only if it has only thin metric ends (in the sense of Definition 3.6). If a graph is quasi-isometric to a tree then there is a one-to-one correspondence between the metric ends and those d-fibers which contain a quasi-geodesic. The graphs considered in this paper are not necessarily locally finite.

Keywords: Quasi-isometry; Tree; Ends of graphs; d-fiber; Quasi-geodesic

0. Introduction

The concept of quasi-isometry plays a central role in recent developments in group theory and geometry. It is a weakened form of isometry which preserves many geometric properties.

Ends of graphs are usually defined as equivalence classes of rays. In locally finite graphs (i.e., each vertex is adjacent to only finitely many other vertices) there is one standard notion of ends. In non-locally finite graphs, there are mainly three different types of ends: edge ends, metric ends and vertex ends. For surveys which treat all three types of ends we refer to Hien [7] and Krön [10]. The notion of ends was first introduced by Freudenthal in his thesis [2] in 1931, see also [3]. He considered Hausdorff spaces with a countable basis. His ends are defined as equivalence classes of descending sequences $(G_n)_{n\in\mathbb{N}}$ of compactly bounded connected open

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E-mail addresses: bernhard.kroen@univie.ac.at (B. Krön), roggi@raunvis.hi.is (R.G. Möller).

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sets such that $\bigcap_{n\in\mathbb{N}} \overline{G}_n$ is empty. In 1945 he introduced ends of locally finite graphs in [4] in a similar way. Independently, Halin introduced ends of graphs in [5] in a more graph theoretical way as equivalence classes of rays (i.e., one-way infinite paths of distinct vertices): two rays R_1 and R_2 are equivalent if there is a third ray which has infinitely many vertices in common with R_1 and R_2. It is not hard to see that this is equivalent to saying that the rays R_1 and R_2 are not equivalent if and only if there is a finite set of vertices F such that every path from R_1 to R_2 contains an element of F. In other words, R_1 and R_2 are equivalent if they cannot be separated by a "small" set of vertices, where small means finite. We will refer to the latter type of ends as vertex ends. Vertex ends are the most common type of ends in recent graph theoretical publications. The relation between Freudenthal and Halin ends was discussed by Diestel and Kühn in [1].

Metric ends were first mentioned in [7]. The basic idea is "measuring instead of counting," which means that sets of vertices are considered as "small" if they are bounded with respect to the natural graph metric instead of calling them "small" if they are finite. This can be useful whenever the metric of the graph plays an important role. In [10, Theorem 6] it is shown that a quasi-isometry between two graphs (see Definition 2.1) extends to a homeomorphism of the relevant metric end spaces (see Definition 4.7 and [10, Section 5]). A graph is said to be *almost transitive* if the automorphism group has only finitely many orbits on the set of vertices. When considering metric ends the natural symmetry condition is that of *metrical almost transitivity* (see [11]). A graph is said to be *metrically almost transitive* if there is a vertex v and a constant c such that the distance of any vertex to the orbit of v under the automorphism group is at most c. Hence metrical almost transitivity is a more general concept than almost transitivity.

The strategy in this paper is to start with characterizing general graphs which are quasi-isometric to trees and then to use the results in order to characterize metrically almost transitive graphs which are quasi-isometric to trees by looking only at the metric end space and without considering the local structure of the graph itself. The latter criterion is given in terms of the "thickness" of the ends and of the end space. A further aim is to analyze the relation between metric ends and the d-fibers defined in [8] and [9] in this context. Metric ends and fibers are discussed with regard to group actions in [12].

In Section 1 we give elementary definitions and fix some basic notation.

Section 2 is devoted to the study of quasi-isometries between general graphs and trees. If a graph X is quasi-isometric to a tree then we give a canonical construction of a tree which is quasi-isometric to X. We also find a general necessary and sufficient condition for a graph to be quasi-isometric to a tree, see Definition 2.4 and Theorem 2.8.

Jung and Niemeyer introduced two different types of *fibers* in [8] and [9]. Fibers can be viewed as refinements of ends. Whereas rays are in the same end if they cannot be separated be a "small" set, two rays are in the same fiber if they are "close" to each other. There are some connections between ends and fibers but there are also essential differences. In the present paper we will be interested in the so-called d-fibers, see Definition 3.2. Jung and Niemeyer called a subgraph H of X *metric* if the metrics $d_H(\cdot \mid \cdot)$ and $d_X(\cdot \mid \cdot)$ are equivalent on VH. Thus there is a constant c such that $d_H(x, y) \leqslant c \cdot d_X(x, y)$, for all x and y in VH. Double rays (i.e., two-way infinite paths of distinct vertices) which are metric in the sense of Jung and Niemeyer were called *quasi-axes* by Polat and Watkins in [14]. We will call rays which are metric in the sense of Jung and Niemeyer *quasi-geodesics* (see Definition 3.8). This term is used similarly in the theory of hyperbolic groups, where it denotes a quasi-isometric embedding of the real numbers.

In Section 3 it is shown that if a graph is quasi-isometric to a tree then there is a one-to-one correspondence between the metric ends and those d-fibers which contain quasi-geodesics (see Theorem 3.9). There are graphs which are not metrically almost transitive and which have this one-to-one correspondence but which are not quasi-isometric to any tree (see Examples 3.7(i) and 3.11).

In Section 4 we state some basic results for metrically almost transitive graphs. *Star balls* were defined in [11]. These are balls B (a *ball* in a graph X is a set of vertices of the type $\{v \in VX \mid d_X(u,v) \leqslant n\}$ where u is some vertex in X and n is a number) such that there is no upper bound on the diameters of those components of the complement of B which have a bounded diameter. We will repeatedly use the fact that there are no star balls in a metrically almost transitive graph. (Note that star-balls may only occur in non-locally finite graphs.) We also prove that if a metrically almost transitive graph has only one metric end, then this end is thick. For the locally finite case see Halin [6, Theorem 9] and Thomassen [16, Proposition 5.6].

In Section 5 we study quasi-isometries between metrically almost transitive graphs and trees. Every metrically almost transitive graph is quasi-isometric to a transitive graph (see Theorem 5.2). The main result of this section (Theorem 5.5) is that a metrically almost transitive graph is quasi-isometric to a tree if and only if it has only thin metric ends (see Definition 3.6). If a metrically almost transitive graph is quasi-isometric to some tree then it is also quasi-isometric to a tree without vertices of degree one.

1. Preliminaries

A graph is a pair $X = (VX, EX)$ with vertex set VX and edge set EX. Edges are two element subsets of VX. Hence our graphs are undirected and have neither loops nor multiple edges. Two vertices x and y are said to be *adjacent*, or *neighbours*, if $\{x, y\}$ is an edge. A graph is *locally finite* if each vertex has only finitely many neighbours. Let C be a set of vertices. We define the *boundary* NC of C as the set of those vertices in X that are not in C but are adjacent to some vertex in C. And we define the *inner boundary* IC of C as the set of those vertices in C that are adjacent to some vertex which is not in C, so $IC = N(VX \setminus C)$. A *walk* of length n from x to y is an $n+1$-tuple $(x = x_0, x_1, \dots, x_n = y)$ such that x_i and x_{i+1} are adjacent for $i = 0, 1, \dots, n-1$. A walk consisting of distinct vertices is called a *path*. Let $\pi_1 = (v_0, v_1, \dots, v_m)$ and $\pi_2 = (w_0, w_1, \dots, w_n)$ be walks with $v_m = w_0$. Then the *concatenation* $\pi_1 \circ \pi_2$ of π_1 and π_2 is the walk $(v_0, v_1, \dots, v_m = w_0, w_1, \dots, w_n)$. A *ray* is a sequence $(x_0, x_1, \dots)$ of distinct vertices such that x_i and x_{i+1} are adjacent for $i \geqslant 0$. A *geodetic* path from x to y is a path from x to y of minimal length. The *distance* $d_X(x, y)$ between two vertices is the length of a geodetic path from x to y. Let A be a set of vertices. We set $d_X(x, A) = \min\{d_X(x, y) \mid y \in A\}$. The set A is *connected* if any two vertices in A can be connected by a path which is contained A. The *components* of A are the maximal connected subsets of A. If the graph is connected (i.e., VX is connected) then d_X is a metric on VX. We write $\operatorname{diam}_X A$ for the diameter of A with respect to this metric. A *ball* with *center* $o \in VX$ and *radius* r is the set $B_X(o, r) = \{y \in VX \mid d_X(x, y) \leqslant r\}$, where r is an integer and $r \geqslant -1$. Note that $B_X(o, -1) = \emptyset$ and $B_X(o, 0) = \{o\}$. A set of vertices F *separates* vertices x and y if there is no path from x to y which is disjoint from F. In particular, any vertex in F is separated by F from any other vertex. The set F *separates* a set of vertices A from a vertex x or from a set of vertices B if F separates any vertex in A from x or any vertex in A from any vertex in B, respectively.

2. Quasi-isometries between graphs and trees

Definition 2.1. Two connected graphs X and Y are *quasi-isometric* if there are functions $\phi : VX \to VY$ *and* $\psi : VY \to VX$ and constants a, b, c and d such that for all x, x_1 and x_2

in VX and y, y_1 and y_2 in VY, the following conditions hold:

(Q1) $d_Y(\phi(x_1), \phi(x_2)) \leqslant a \cdot d_X(x_1, x_2)$ (boundedness of ϕ),
(Q2) $d_X(\psi(y_1), \psi(y_2)) \leqslant b \cdot d_Y(y_1, y_2)$ (boundedness of ψ),
(Q3) $d_X(\psi\phi(x), x) \leqslant c$ (quasi-injectivity of ϕ),
(Q4) $d_Y(\phi\psi(y), y) \leqslant d$ (quasi-surjectivity of ϕ).

A function $\phi: VX \to VY$ is said to be a *quasi-isometry* if there exists a function $\psi: VY \to VX$ and constants a, b, c, d such that conditions (Q1)–(Q4) above hold.

In general metric space, the axioms (Q1) and (Q2) require an additional additive constant. In graphs, the positive values of the metric on the vertices cannot be arbitrarily close to zero. This is the reason why we omit these constants for graphs. For more details we refer to [11, Lemma 11]. Note that in general metric space, quasi-isometries are neither surjective nor injective. The proof of the following lemma is easy and is left to the reader.

Lemma 2.2. *Let ψ be a quasi-isometry from Y to X and let $A_1, A_2, \ldots$ be sets of vertices in Y. Then*

$$\lim_{n\to\infty} \operatorname{diam}_X \psi(A_n) = \infty \quad \Leftrightarrow \quad \lim_{n\to\infty} \operatorname{diam}_Y A_n = \infty.$$

Definition 2.3. Let o be a vertex of a connected graph X and let $n \geqslant 0$ be an integer. The components of the complement of a ball $B_X(o, n)$ in VX are called *radial cuts of X with center o and coradius n*. Let $\mathcal{C}_o$ denote the set of all radial cuts with center o.

Note that the coradius and the center are not necessarily determined by a given radial cut.

Definition 2.4. A graph X is *thin* if there is a constant M such that $\operatorname{diam}_X NC < M$, for any radial cut C.

Note that $\operatorname{diam}_X NC \leqslant \operatorname{diam}_X IC + 2$ and $\operatorname{diam}_X IC \leqslant \operatorname{diam}_X NC + 2$ for any set of vertices C. This means that in Definition 2.4, we could as well use the inner boundary IC instead of the usual boundary NC.

Lemma 2.5. *Let C be a radial cut with center o and let F be a set of vertices such that $F \cap IC \neq \emptyset$ and such that F separates C from o. Then*

$$\operatorname{diam}_X IC \leqslant 3 \operatorname{diam}_X F.$$

Proof. There is nothing to prove if F is unbounded. Otherwise let r be the smallest number such that $F \subseteq B_X(o, r)$ and let r' be the smallest number such that $B_X(o, r') \cap F \neq \emptyset$. Then $r - r' \leqslant \operatorname{diam}_X F$. Any path from a vertex x_1 in IC to o must hit F and the sets IC and F are both contained in $\{x \in VX \mid r' \leqslant d_X(o, x) \leqslant r\}$. Hence $d_X(x_1, F) \leqslant r - r'$. For any pair of vertices x_1 and x_2 in IC we obtain

$$\begin{aligned} d_X(x_1, x_2) &\leqslant d_X(x_1, F) + \operatorname{diam}_X F + d_X(F, x_2) \\ &\leqslant 2(r - r') + \operatorname{diam}_X F \leqslant 3 \operatorname{diam}_X F. \qquad \square \end{aligned}$$

Lemma 2.6. *Let v and w be two vertices in VX. Then*

$$\sup\{\operatorname{diam}_X IC \mid C \in \mathcal{C}_v\} < \infty \quad \Leftrightarrow \quad \sup\{\operatorname{diam}_X IC \mid C \in \mathcal{C}_w\} < \infty.$$

Proof. Let D_v be a radial cut with center v and coradius r_v such that $r_v > 2d_X(v,w)$. Let D_w be the radial cut with center w and maximal coradius r_w such that $D_v \subseteq D_w$. There is a vertex x in $ID_v \cap ID_w$. The inequality $d_X(x,v) \leqslant d_X(x,w) + d_X(w,v)$ implies $r_v \leqslant r_w + d_X(w,v)$. By $r_v > 2d_X(v,w)$ we get $r_w > d_X(v,w)$, and therefore v is not an element of D_w. Thus ID_w separates v from D_v and, by Lemma 2.5, we obtain $\operatorname{diam}_X ID_v \leqslant 3 \operatorname{diam}_X ID_w$. This implies

$$\sup\{\operatorname{diam}_X IC \mid C \in \mathcal{C}_w\} < \infty \quad \Rightarrow \quad \sup\{\operatorname{diam}_X IC \mid C \in \mathcal{C}_v\} < \infty,$$

because all but finitely many cuts in $\mathcal{C}_v$ and $\mathcal{C}_w$ satisfy such an inequality. We obtain the other implication with the same arguments after exchanging v and w. $\square$

Corollary 2.7. *A graph is thin whenever the condition*

$$\sup\{\operatorname{diam}_X IC \mid C \in \mathcal{C}_v\} < \infty$$

is satisfied for some vertex v.

In Definition 2.3, we defined radial cuts $B_X(o,n)$ for coradii $n \geqslant 0$. For $n = -1$, this definition still makes sense. The ball $B_X(o,-1)$ is empty. For any center o, the radial cut with coradius $n = -1$ is the whole set of vertices VX. Let $\mathcal{C}_o$ denote the set of all radial cuts with center o and coradius $n \geqslant -1$.

We define a graph T_o by setting $VT_o = \mathcal{C}_o$ and by defining two elements C and D in VT_o with coradii r_C and r_D to be adjacent if either $C \subsetneq D$ or $D \subsetneq C$ and if $|r_C - r_D| = 1$. Then T_o is a connected tree which we call *radial cut tree of X with center o*. The construction of T_o is similar to the construction of structure trees, see [11,13,17] and the references therein. The difference is that in structure tree theory the set of cuts has to be invariant under the action of automorphisms, with the result that the automorphisms of X induce a group action on the structure tree. That is, there is a homomorphism from the automorphisms of X to the automorphisms of the structure tree. The set $\mathcal{C}_o$ is in general not invariant under automorphisms of X.

For a vertex o we define $\phi_o : VX \to VT_o$ where $\phi_o(x)$ is the component C in $\mathcal{C}_o$ such that $x \in IC$ and $\psi_o : VT_o \to VX$ where $\psi_o(C)$ is any vertex in IC. Note that ϕ_o is surjective and ψ_o is injective.

Next we formulate a criterion for a graph to be quasi-isometric to a tree using the construction of the radial cut tree T_o and the functions ϕ_o and ψ_o. The proof of this theorem will be split up into a series of lemmas.

Theorem 2.8. *Let X be a connected graph and let o be a vertex. Then the following statements are equivalent*:

(1) *X is thin.*
(2) *X is quasi-isometric to T_o with quasi-isometries ϕ_o and ψ_o.*
(3) *X is quasi-isometric to a tree.*

For the following three lemmas let $\phi : VX \to VY$ and $\psi : VY \to VX$ be quasi-isometries between connected graphs X and Y with constants a, b, c and d as in Definition 2.1.

Lemma 2.9. *Let $\pi = (x_0, \dots, x_n)$ be a path in X and τ_i be a geodetic path in Y from $\phi(x_{i-1})$ to $\phi(x_i)$. Define τ as the walk $\tau_1 \circ \cdots \circ \tau_n$ from $\phi(x_0)$ to $\phi(x_n)$. Then $d_X(\psi(y), \pi) \leqslant ab/2 + c$ for any y in τ.*

Proof. We have $d_Y(\phi(x_{i-1}), \phi(x_i)) = \mathrm{diam}_Y(\tau_i) \leqslant a$, for $1 \leqslant i \leqslant n$. For any vertex y in τ we can find a number $j \in \{1, \dots, n\}$ such that $d_Y(y, \phi(x_j)) \leqslant a/2$. Thus $d_X(\psi(y), \psi\phi(x_j)) \leqslant ab/2$ and

$$\begin{aligned} d_X\big(\psi(y), \pi\big) &\leqslant d_X\big(\psi(y), x_j\big) \\ &\leqslant d_X\big(\psi(y), \psi\phi(x_j)\big) + d_X\big(\psi\phi(x_j), x_j\big) \\ &\leqslant ab/2 + c. \qquad \Box \end{aligned}$$

We set $\kappa = ab/2 + bd + c$ and we will use this notation throughout the following sections.

Lemma 2.10. *Suppose $B_Y(z, r)$ separates the vertices v and w in Y. Then the ball $B_X(\psi(z), br + \kappa)$ separates $\psi(v)$ from $\psi(w)$ in X.*

Proof. Let $\pi = (x_0, \dots, x_n)$ be a path from $\psi(v)$ to $\psi(w)$. Let τ_i be a geodetic path from $\phi(x_{i-1})$ to $\phi(x_i)$ and let τ be the walk $\tau_1 \circ \cdots \circ \tau_n$, as in Lemma 2.9. For any vertex y in τ we have $d_X(\psi(y), \pi) \leqslant ab/2 + c$. Let π_v be a geodetic path from v to $\phi\psi(v)$ and let π_w be a geodetic path from $\phi\psi(w)$ to w. The lengths of π_v and π_w are each less than or equal to d. Because $B_Y(z, r)$ separates v and w, the walk $\pi_v \circ \tau \circ \pi_w$ intersects $B_Y(z, r)$. If $B_Y(z, r)$ contains no vertex of τ then it must contain a vertex either of π_v or of π_w. Then $B_Y(z, r + d)$ contains either v or w. In either case, $B_Y(z, r + d)$ contains a vertex y' from τ which implies $d_X(\psi(y'), \psi(z)) \leqslant b(r + d)$. Thus

$$\begin{aligned} d_X\big(\psi(z), \pi\big) &\leqslant d_X\big(\psi(z), \psi(y')\big) + d_X\big(\psi(y'), \pi\big) \\ &\leqslant b(r + d) + ab/2 + c = br + \kappa \end{aligned}$$

and therefore π has a non-empty intersection with $B_X(\psi(z), br + \kappa)$. Hence $B_X(\psi(z), br + \kappa)$ separates $\psi(v)$ from $\psi(w)$. $\Box$

Lemma 2.11. *Let Y be a tree and let C be a radial cut of X. Then* $\mathrm{diam}_X IC \leqslant 6\kappa + 6c$.

Proof. Suppose there are vertices x_0 and y_0 in IC such that $d_X(x_0, y_0) > 6\kappa + 6c$. Let o be a center of C and let z be a vertex in VY which separates $\phi(x_0)$, $\phi(y_0)$ and $\phi(o)$. (Note that z is one of these three vertices if this vertex lies on a path which connects the other two vertices. Also note that $B_Y(z, 0) = \{z\}$.) By Lemma 2.10, $B_X(\psi(z), \kappa) = B_X(\psi(z), b0 + \kappa)$ separates $\psi\phi(x_0)$, $\psi\phi(y_0)$ and $\psi\phi(o)$. By axiom (Q3), the vertices $\psi\phi(x_0)$, $\psi\phi(y_0)$ and $\psi\phi(o)$ are in $B_X(x_0, c)$, $B_X(y_0, c)$ and $B_X(o, c)$, respectively. If one of the vertices $\psi\phi(x_0)$, $\psi\phi(y_0)$ and $\psi\phi(o)$ is not in the same component of $VX \setminus B_X(\psi(z), \kappa)$ as the corresponding vertex x_0, y_0 or o, then this vertex is an element of $B_X(\psi(z), \kappa + c)$ and is therefore separated by $B_X(\psi(z), \kappa + c)$ from the other two vertices. If it is in the same component, then it is separated from the other two vertices by $B_X(\psi(z), \kappa + c)$ anyway. Hence $B_X(\psi(z), \kappa + c)$ separates x_0, y_0 and o from each other.

Let $\pi_x = (x_0, x_1, \dots, x_m = o)$ and $\pi_y = (y_0, y_1, \dots, y_n = o)$ be geodetic paths and let π_C be a path from x_0 to y_0 which is contained in C. Because $B_X(\psi(z), \kappa + c)$ separates x_0, y_0 and o, there are vertices $x_i \in \pi_x$, $y_j \in \pi_y$ and $p \in \pi_C$ which are elements of $B_X(\psi(z), \kappa + c)$. Since

$d_X(p, x_i) \leqslant 2\kappa + 2c$ and $d_X(p, y_j) \leqslant 2\kappa + 2c$ we have $d_X(x_i, C) \leqslant 2\kappa + 2c$ and $d_X(y_j, C) \leqslant 2\kappa + 2c$. Because C is a radial cut, we have $d_X(x_i, C) = d_X(x_i, x_0)$ and $d_X(y_j, C) = d_X(y_j, y_0)$. Therefore $d_X(x_i, x_0) \leqslant 2\kappa + 2c$ and $d_X(y_j, y_0) \leqslant 2\kappa + 2c$. We have assumed that $d_X(x_0, y_0) > 6\kappa + 6c$. By the triangle inequality, we get

$$\begin{aligned} d_X(x_i, y_j) &\geqslant d_X(x_0, y_0) - d_X(x_0, x_i) - d_X(y_0, y_j) \\ &> 6\kappa + 6c - (2\kappa + 2c) - (2\kappa + 2c) = 2\kappa + 2c \end{aligned}$$

which is a contradiction to $\{x_i, y_j\} \subseteq B_X(\psi(z), \kappa + c)$. □

Proof of Theorem 2.8. Let X be a thin graph. Set $\lambda = \sup\{\operatorname{diam}_X IC \mid C \in \mathcal{C}_o\}$. For any vertices x and y in VX and v and w in VT_o we have:

(Q1) $d_{T_o}(\phi_o(x), \phi_o(y)) \leqslant d_X(x, y)$,
(Q2) $d_X(\psi_o(v), \psi_o(w)) \leqslant \lambda \cdot d_{T_o}(v, w)$,
(Q3) $d_X(x, \psi_o\phi_o(x)) \leqslant \lambda$, and
(Q4) $d_{T_o}(v, \phi_o\psi_o(v)) = 0$,

and therefore (1) ⇒ (2). The implication (2) ⇒ (3) is trivial and (3) ⇒ (1) is a consequence of Lemma 2.11. □

3. Metric ends, fibers and quasi-geodesics

Definition 3.1. *Metrically transient rays* are unbounded rays such that every infinite subset of vertices has infinite diameter. Two metrically transient rays R_1 and R_2 in a graph X are *metrically equivalent* if they cannot be separated by a bounded set of vertices. The corresponding equivalence classes on the set of metrically transient rays are called the *metric ends*. The set of all metric ends of X is denoted by ΩX. A *metric cut* is a connected set of vertices such that NC is bounded. A metric end ω *lies* in C if C contains all but finitely many vertices of every ray in ω.

It is easy to prove that metric equivalence is an equivalence relation on the set of metrically transient rays of X. We restrict our attention to metrically transient rays because they have the property that if T is a bounded set of vertices then there is precisely one component of $VX \setminus T$ that contains infinitely many vertices from our ray. Note that a metric cut C contains all but finitely many vertices of a metrically transient ray of a metric end ω if and only if C contains all but finitely many vertices of every ray in ω. Several results on metric ends and the corresponding topology can be found in [10]. The definition of metric ends can also be found in the Master's thesis of Hien [7].

Definition 3.2. (See [9, Definition 1].) Two rays R_1 and R_2 are *d-equivalent* if there is a number m such that

$$R_1 \subseteq \{x \in VX \mid d_X(x, R_2) \leqslant m\} \quad \text{and} \quad R_2 \subseteq \{x \in VX \mid d_X(x, R_1) \leqslant m\}.$$

This relation, d-equivalence, is an equivalence relation on the set of all rays. The equivalence classes are called *d-fibers*.

The following lemma is easy to prove and can be found in [12, Lemma 1(ii)].

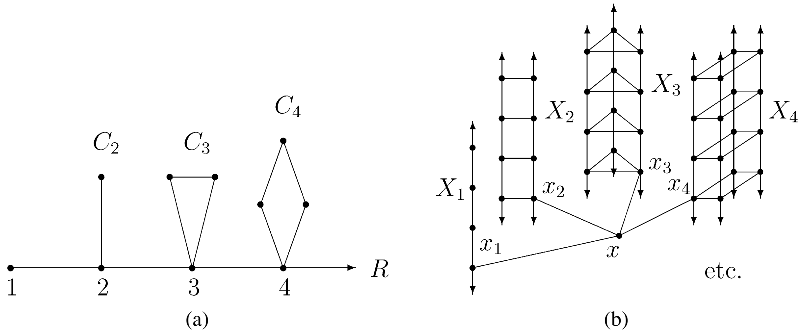

Fig. 1.

Lemma 3.3. *Two metrically transient rays that belong to the same d-fiber also belong to the same metric end.*

Definition 3.4. Let $C_o(n, \omega)$ denote the radial cut with center o and coradius n which contains a metric end ω. We define

$$\mu_o(\omega) = \sup\bigl\{\operatorname{diam}_X IC_o(n, \omega) \bigm| n \geqslant 0\bigr\}.$$

Lemma 3.5. *Let ω be a metric end and let v and w be any two vertices. Then $\mu_w(\omega) < \infty$ implies $\mu_v(\omega) < \infty$.*

Proof. Let D_v be a radial cut with center v and coradius r_v such that $r_v > 2d_X(v, w)$ and such that ω lies in D_v. We can copy the proof of Lemma 2.6 word-for-word and obtain the inequality $\operatorname{diam}_X ID_v \leqslant 3 \operatorname{diam}_X ID_w \leqslant 3\mu_w(\omega)$ for some radial cut D_w with center w which contains ω. Since this inequality holds for all but finitely many positive integers r_v, this implies the statement of the lemma. $\square$

Definition 3.6. An end ω is *thin* if $\mu_o(\omega)$ is finite for some vertex o (equivalently: for every vertex o, see Lemma 3.5). An end is *thick* if it is not thin. A metric end space ΩX is *thin* if

$$\sup\bigl\{\mu_o(\omega) \bigm| o \in VX,\ \omega \in \Omega X\bigr\} < \infty. \tag{1}$$

We call ΩX *thick* if it is not thin.

Note that the supremum in (1) is the same as

$$\sup\{\operatorname{diam}_X IC \mid C \text{ is any radial cut which contains a metric end}\}.$$

Example 3.7.

(i) Let R be the ray such that $VR = \mathbb{N}$ and vertices x and y are adjacent if $|x - y| = 1$. To each of the vertices x, $x \geqslant 2$, we add a cycle of length x. The resulting graph X (see Fig. 1(a)) is not thin. There is only one metric end which is thin. Consequently, the end space of X is thin in the sense of Definition 3.6.

(ii) Let G_n be the group $\langle a, b_n \mid ab_na^{-1}b_n^{-1} = b_n^n = 1\rangle$, which is the direct product of an infinite cyclic group with a cyclic group of order n. Let X_n be the Cayley graph $\operatorname{Cay}(G_n, S_n)$ where

$S_n = \{a, a^{-1}, b_n, b_n^{-1}\}$. Now we take an additional vertex x, choose one vertex x_n from each of the graphs X_n and connect the vertices x_n and x with additional edges. The resulting graph X (see Fig. 1(b)) is not thin. All metric ends are thin, but the metric end space is thick.

Definition 3.8. A ray R in a graph X is called *quasi-geodetic* (or a *quasi-geodesic*) if there is a positive constant σ such that $\sigma \cdot d_R(x, y) \leqslant d_X(x, y)$, for any pair of vertices x and y in R.

If $\sigma = 1$ then the quasi-geodesic is geodetic ray, which is a ray whose finite subpaths are all geodetic.

Note that a ray $R = (x_1, x_2, \ldots)$ is quasi-geodetic if and only if the map $\varphi : R \to \mathbb{N}, x_i \mapsto i$, is a quasi-isometry, where in R we consider the metric of X and in $\mathbb{N}$ we consider the natural metric which is induced be the absolute value of the difference of integers. The ray R is geodetic if and only if φ is an isometry.

Theorem 3.9. *A thin metric end contains a quasi-geodesic. All quasi-geodesics in a thin metric end are d-equivalent. In other words*: *In a thin metric end there is exactly one d-fiber, which contains a quasi-geodesic.*

Proof. Let ω be a thin end of X and let o be any vertex. Our first aim is to construct a quasi-geodesic in ω which originates in o. Set $\mu = \mu_o(\omega)$. Any vertex in $IC_o(n, \omega)$ can be connected with a vertex in $IC_o(n+1, \omega)$ by a path of length less or equal $\mu + 1$. By induction we obtain an infinite walk $\pi = (x_i)_{i \geqslant 0}$ originating in o with a subsequence $\xi = (x_{k_n})_{n \geqslant 1}$, such that $x_{k_n} \in IC_o(n-1, \omega)$ and $d_X(x_{k_n}, x_{k_{n+1}}) \leqslant \mu + 1$. Note that $d_X(o, x_{k_n}) = n$.

Let x_i be a vertex of π. For each $i \geqslant 1$, let $i^- \leqslant i$ be the largest integer such that $x_{i^-} \in \xi$ and let $i^+ > i$ be the smallest integer such that $x_{i^+} \in \xi$. Note that $d_X(o, x_{i^+}) = d_X(o, x_{i^-}) + 1$ and $0 < i^+ - i^- \leqslant \mu + 1$.

Suppose $d_X(o, x_i) \leqslant d_X(o, x_{i^-}) = d_X(o, x_{i^+}) - 1$. Then $0 < i - i^- \leqslant \mu/2$ or $0 < i^+ - i \leqslant \mu/2 + 1$ which implies

$$\begin{aligned} d_X(o, x_i) &\geqslant d_X(o, x_{i^-}) - d_X(x_{i^-}, x_i) \geqslant d_X(o, x_{i^-}) - (i - i^-) \\ &\geqslant d_X(o, x_{i^-}) - \frac{\mu}{2} \quad \text{or} \\ d_X(o, x_i) &\geqslant d_X(o, x_{i^+}) - d_X(x_{i^+}, x_i) \geqslant d_X(o, x_{i^+}) - \left(i^+ - i\right) \\ &\geqslant d_X(o, x_{i^+}) - \frac{\mu}{2} - 1 = d_X(o, x_{i^-}) - \frac{\mu}{2}. \end{aligned}$$

Hence $d_X(o, x_i) \geqslant d_X(o, x_{i^-}) - \mu/2$ for any $i \geqslant 1$.

Suppose $d_X(o, x_i) \geqslant d_X(o, x_{i^-}) + 1 = d_X(o, x_{i^+})$. Then similarly,

$$\begin{aligned} &i - i^- \leqslant \frac{\mu}{2} + 1 \quad \text{or} \quad i^+ - i \leqslant \frac{\mu}{2} \quad \text{and} \\ &d_X(o, x_i) \leqslant d_X(o, x_{i^-}) + \frac{\mu}{2} + 1 \quad \text{or} \quad d_X(o, x_i) \leqslant d_X(o, x_{i^+}) + \frac{\mu}{2}. \end{aligned}$$

Hence $d_X(o, x_i) \leqslant d_X(o, x_{i^+}) + \mu/2$ for any $i \geqslant 1$. We sum up,

$$d_X(o, x_{i^-}) - \frac{\mu}{2} \leqslant d_X(o, x_i) \leqslant d_X(o, x_{i^+}) + \frac{\mu}{2}, \tag{2}$$

for any $i \geqslant 1$.

Let j be an integer such that $j \geqslant (\mu+1)^2 + 2\mu + 1 + i$. Then

$$\big(j^- - i^+\big) \geqslant (\mu+1)^2 + 2\mu + 1 - \big(i^+ - i\big) - (j - j^-) \geqslant (\mu+1)^2,$$

because $i^+ - i \leqslant \mu + 1$ and $j - j^- \leqslant \mu$. The subwalk of π from x_{i^+} to x_{j^-} consists of walks which connect vertices x_{k_r} and $x_{k_{r+1}}$ in ξ. Since the length of each of these walks is at most $\mu+1$, this subwalk from x_{i^+} to x_{j^-} consists of at least $\mu+1$ such walks. This implies $x_{i^+} = x_{k_r}$ and $x_{j^-} = x_{k_s}$, where $s - r \geqslant \mu + 1$. Since $d_X(o, x_{k_r}) = r$ and $d_X(o, x_{k_s}) = s$, we have

$$d_X(o, x_{j^-}) \geqslant d_X(o, x_{i^+}) + \mu + 1.$$

By (2),

$$d_X(o, x_j) \geqslant d_X(o, x_{j^-}) - \frac{\mu}{2} \geqslant d_X(o, x_{i^+}) + \frac{\mu}{2} + 1 \quad \text{and}$$
$$d_X(o, x_i) \leqslant d_X(o, x_{i^+}) + \frac{\mu}{2}.$$

Hence $d_X(o, x_j) \geqslant d_X(o, x_i) + 1$, for any $j \geqslant (\mu+1)^2 + 2\mu + 1 + i$. By induction on m,

$$d_X(o, x_j) - d_X(o, x_i) \geqslant m, \tag{3}$$

for any integers i, j and m such that $j - i \geqslant m((\mu+1)^2 + 2\mu + 1)$.

Let i and j be any positive integers. There is an integer m such that

$$m\big((\mu+1)^2 + 2\mu + 1\big) \leqslant j - i \leqslant (m+1)\big((\mu+1)^2 + 2\mu + 1\big).$$

By (3), the latter term is less or equal

$$\big(d_X(o, x_j) - d_X(o, x_i) + 1\big)\big((\mu+1)^2 + 2\mu + 1\big)$$
$$\leqslant \big(d_X(x_i, x_j) + 1\big)\big((\mu+1)^2 + 2\mu + 1\big).$$

It follows that

$$j - i \leqslant 2\big((\mu+1)^2 + 2\mu + 1\big) d_X(x_i, x_j), \tag{4}$$

for any vertices x_i and x_j in π such that $j - i \geqslant (\mu+1)^2 + 2\mu + 1$.

If π is not a ray (i.e., not all its vertices are distinct) then let i be the minimal integer for which there is an integer j such that $x_i = x_j$ and such that the vertices $x_i, x_{i+1}, \ldots, x_{j-1}$ are distinct. Let π_1 be the walk $(x_1, \ldots, x_i, x_{j+1}, \ldots)$. If π_1 is not a ray, then again there is a minimal integer k for which there is an integer l such that $x_k = x_l$ and such that the vertices $x_k, x_{k+1}, \ldots, x_{l-1}$ are distinct. From π_1 we remove the subpath $(x_{k+1}, x_{k+1}, \ldots, x_l)$ and obtain a path π_2 such that $\pi_2 \subsetneq \pi_1 \subsetneq \pi$. Either this procedure stops after a finite number of steps and we obtain a ray $R = \pi_n$ or we get a sequence of paths $(\pi_n)_{n \geqslant 1}$ whose intersection is a ray R. If $0 \leqslant j - i \leqslant (\mu+1)^2 + 2\mu + 1$ then

$$d_R(x_i, x_j) \leqslant j - i \leqslant \big((\mu+1)^2 + 2\mu + 1\big) d_X(x_i, x_j). \tag{5}$$

Now (4), (5) and the inequality $d_R(x_i, x_j) \leqslant |j - i|$ imply that

$$d_R(x_i, x_j) \leqslant 2\big((\mu+1)^2 + 2\mu + 1\big) d_X(x_i, x_j)$$

for any vertices x_i and x_j of R. Hence R is a quasi-geodesic.

By Lemma 3.3, the end ω contains all metrically transient rays of the d-fiber of R.

Finally, we have to show that a thin end ω does not contain more than one d-fiber which contains a quasi-geodesic.

Let $R = (y_n)_{n\in\mathbb{N}}$ and $S = (z_n)_{n\in\mathbb{N}}$ be two quasi-geodesics in ω. We have to prove that R and S are in the same d-fiber. There is an integer N such that R contains a subsequence $R' = (y_{n_k})_{k\geqslant N}$ and S contains a subsequence $S' = (z_{n_k})_{k\geqslant N}$ where y_{n_k} and z_{n_k} are in $IC_o(k,\omega)$. By the definition of a quasi-geodesic, there is a positive constant σ such that $\sigma \cdot d_S(z_i, z_j) \leqslant d_X(z_i, z_j)$ for any vertices z_i and z_j in S. Since $d_X(z_{n_k}, z_{n_{k+1}}) \leqslant \mu + 1$, we have

$$d_S(z_{n_k}, z_{n_{k+1}}) \leqslant \frac{d_X(z_{n_k}, z_{n_{k+1}})}{\sigma} \leqslant \frac{\mu+1}{\sigma}.$$

For the subpaths S_k of S which go from z_{n_k} to $z_{n_{k+1}}$, $n \geqslant N$, this means

$$\operatorname{diam}_S S_k \leqslant \frac{\mu+1}{\sigma}.$$

We also have

$$S' \subseteq \bigcup_{k\geqslant N} IC_o(k,\omega) \subseteq \bigl\{x \in VX \bigm| d_X(x,S) \leqslant \mu\bigr\}.$$

Any vertex z_n in S with $n \geqslant N$ can be connected to a vertex z' in S' with a path whose length is at most $(\mu+1)/\sigma$ and from there to a vertex in R' with a path whose length is at most μ. This means that from the vertex z_N onwards, the ray S is contained in

$$\bigl\{x \in VX \bigm| d_X(x,R) \leqslant (\mu+1)/\sigma + \mu\bigr\}.$$

By repeating these arguments after transposing R and S we get the second inclusion of Definition 3.2. Hence S is in the same d-fiber as R. □

Let $\mathcal{D}_q X$ be the set of d-fibers of X which contain a quasi-geodesic. Combining Theorems 2.8 and 3.9 we obtain the following.

Corollary 3.10. *Let X be a graph which is quasi-isometric to a tree. Then there is a one-to-one correspondence between ΩX and $\mathcal{D}_q X$ in the way that each d-fiber in $\mathcal{D}_q X$ contains exactly one metric end and each metric end contains the metrically transient rays of exactly one d-fiber in $\mathcal{D}_q X$.*

Example 3.11. Let R be a graph with vertex set $VR = \mathbb{N}$. Two vertices x and y are adjacent if $|x-y| = 1$. We set

$$n_k = \sum_{i=1}^{k} i = \frac{k^2+k}{2}, \quad k \in \mathbb{N},$$

and connect any pair of vertices n_k and n_{k+1} with a path π_k of length $(k+1)^2$ which is disjoint from the rest of the graph except for the vertices n_k and n_{k+1}. Let X denote the resulting graph, see Fig. 2. Then $d_X(n_k, n_{k+1}) = d_R(n_k, n_{k+1}) = k+1$. The ray R is a quasi-geodesic in X. Moreover, it is geodetic. A ray which contains infinitely many of the paths π_k is not quasi-geodetic. This means that every quasi-geodesic must be d-equivalent to R.

Thus there is only one d-fiber which contains a quasi-geodesic, and this d-fiber is contained in the only metric end of X. But the end of X is not thin, and X is not quasi-isometric to a tree. This means that in general the implications in Theorem 3.9 and in Corollary 3.10 do not hold in the other direction. Note that there are infinitely many d-fibers.

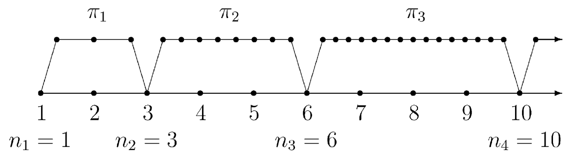

Fig. 2.

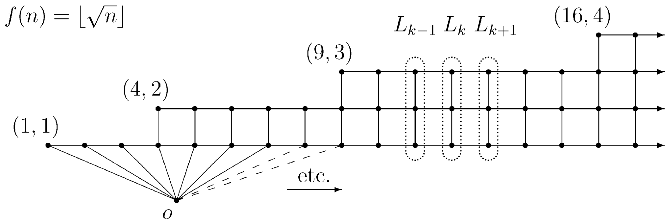

Fig. 3.

The graph X in Example 3.7(i) has also only one d-fiber, and this fiber contains a quasi-geodesic. This d-fiber is contained in the only metric end of X. And this end is thin although X is thick. A d-fiber is defined as an equivalence class of rays (whose vertices are distinct), see Definition 3.2. If we had defined d-fibers as equivalence classes of infinite paths (whose vertices are not necessarily distinct), then this graph X would have infinitely many d-fibers.

The graph in the following example is unbounded but does not contain any quasi-geodesics.

Example 3.12. Let f be a nondecreasing function $\mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} \frac{f(n)}{n} = 0$. Set $A = \{(x, y) \in \mathbb{N}^2 \mid y \leqslant f(x)\}$ and $VX = A \cup \{o\}$ where o is any element which is not in A. Vertices in A are adjacent if their difference is $(1, 0)$, $(-1, 0)$, $(0, 1)$ or $(0, -1)$ and o is adjacent to all vertices $(x, 0)$ for any $x \in \mathbb{N}$. The resulting graph X has infinite diameter, it has one thick metric end and there is no quasi-geodesic. One possible function $f : \mathbb{N} \to \mathbb{N}$ is given by $n \mapsto \lfloor\sqrt{n}\rfloor$, see Fig. 3.

Proof. We define

$$L_n = \bigl\{(n, y) \bigm| 0 \leqslant y \leqslant f(n)\bigr\}.$$

Let $R = (o, x_0, x_1, \ldots)$ be a ray starting in o. If $x_{n-1} \in L_k$ then $x_n \in L_{k-1} \cup L_k \cup L_{k+1}$. Note that x_n cannot be o because a ray consists of distinct vertices. Let x_0 be an element of L_k. Then $x_n \in L_m$, where $m \leqslant n + k$. Thus x_0 and

$$d_X(x_0, x_n) \leqslant f(k) + 2 + f(m) \leqslant f(k) + 2 + f(n + k).$$

Since $\lim_{n\to\infty} f(n + k)/n = 0$ there is no positive σ such that

$$d_X(x_0, x_n) \geqslant \sigma \cdot d_R(x_0, x_n) = \sigma n$$

for every positive integer n. Hence R is not quasi-geodetic. Whether or not a ray is quasi-geodetic, does not depend on a finite number of its vertices. Thus any ray in X is not a quasi-geodesic. $\square$

4. Star balls and metrically almost transitive graphs

A graph automorphism is a map $\alpha : VX \to VX$ such that x is adjacent to y if and only if $\alpha(x)$ is adjacent to $\alpha(y)$, for any vertices x and y. The set $\mathrm{Aut}(X)$ of all graph automorphisms is a group with respect to the composition of functions.

Definition 4.1. A graph X is *metrically almost transitive* if there is an integer r such that

$$\bigcup_{g \in \mathrm{Aut}(X)} g\big(B_X(x,r)\big) = VX,$$

for any vertex x. The smallest integer r with that property is denoted by ρ and we call it the *covering radius* of X. A ball S is a *star ball* if there is no upper bound on the diameters of those components of $X \setminus S$ that have finite diameter.

The concept of a *star ball* is introduced in [11]. For us the most important property of star balls is that they do not exist in metrically almost transitive graphs. The following lemma can be found in [11, Lemma 20] and [12, Lemma 3.2].

Lemma 4.2. *(See [11, Lemma 3.12].) There are no star balls in connected metrically almost transitive graphs.*

Corollary 4.3. *Any unbounded metric cut in a metrically almost transitive graph contains a metrically transient ray.*

For a proof of Corollary 4.3 we refer to [12, Corollary 3.14]. The following can be found in [12, Corollary 3.7].

Corollary 4.4. *Let X be a connected metrically almost transitive graph with covering radius ρ. If there is a bounded and connected set T and components C_1 and C_2 of $VX \setminus T$ which contain vertices x_1 and x_2, respectively, such that*

$$\min\big\{d_X(x_1,T), d_X(x_2,T)\big\} > \mathrm{diam}_X T + \rho$$

then C_1 and C_2 are both unbounded.

Lemma 4.5. *Suppose X is quasi-isometric to Y. Then X contains a star ball if and only if Y contains a star ball.*

Proof. Let ψ be a quasi-isometry from Y to X. Suppose there is a star ball $B_Y(y,r)$ in Y. For any positive integer n there is a bounded component D_n of $VY \setminus B_Y(y,r)$ such that $\mathrm{diam}_X D_n > n$. By Lemma 2.10, the ball $B_X(\psi(y), br+\kappa)$ separates $\psi(D_n)$ from $\psi(VY \setminus D_n)$.

Let $\mathcal{U}_n$ be the union of all components $C_{n,i}$ of $VY \setminus B_X(\psi(y), br+\kappa)$, $i \in I_n$, which are disjoint from $\psi(VY \setminus D_n)$. These components are all bounded. Otherwise we would get a contradiction to axiom (Q3) or to the fact that D_n is bounded. Lemma 2.2 implies $\lim_{n\to\infty} \mathrm{diam}_X \mathcal{U}_n = \infty$. Let M_n be a component $C_{n,i}$, $i \in I_n$, of maximal diameter. Then $\lim_{n\to\infty} \mathrm{diam}_X M_n = \infty$, and $B_X(\psi(y), br+\kappa)$ is a star-ball in X.

We have now proved that if X contains a star-ball then Y also contains a star-ball. After replacing X with Y we see that X contains a star-ball if and only if Y contains a star-ball. $\square$

Theorem 4.6. *(See Proposition 5.6 in [16].) If a connected metrically almost transitive graph has exactly one end then this end is thick.*

Proof. Suppose X has only one thin metric end ω. Then we can find a bounded set T and components C_1 and C_2 of $VX \setminus T$ which contain vertices x_1 and x_2, respectively, such that

$$\min\{d_X(x_1,T), d_X(x_2,T)\} > \operatorname{diam}_X T + \rho.$$

Corollary 4.4 applies and C_1 and C_2 are both unbounded. By Corollary 4.3, both cuts C_1 and C_2 contain a metric end. Hence ω is thick. □

Definition 4.7. Let ΩC be the set of metric ends which lie in some metric cut C (see Definition 3.1). The topology on $VX \cup \Omega X$ generated by

$$\mathcal{B}(X) = \{C \cup \Omega C \mid C \text{ is a metric cut}\}$$

is call the *metric end topology* of X.

It is easy to check that the intersection of two elements of $\mathcal{B}(X)$ is again in $\mathcal{B}(X)$. The open sets of the metric end topology are the unions of elements of $\mathcal{B}(X)$. For more details concerning this topology, see [10, Section 5].

Lemma 4.8. *Let X be a connected graph and let T_o, ϕ_o and ψ_o be defined as in Section 2 for some vertex o. Then there are unique continuous extensions $\Phi_o : VX \cup \Omega X \to VT_o \cup \Omega T_o$ and $\Psi_o : VT_o \cup \Omega T_o \to VX \cup \Omega X$ of ϕ_o and ψ_o such that the restrictions Φ_o and Ψ_o on the end spaces are homeomorphisms of the corresponding relative topologies.*

Proof. Let ω be a metric end of X. Then Φ_o is determined by the sequence $(IC_o(n,\omega))_{n\geqslant 1}$ which is a ray in T_o. Let η be an end of T_o and let $R = (x_0, x_1, \ldots)$ be the ray in T_o which originates in o. Then $(\psi(x_0), \psi(x_1), \ldots)$ is a sequence of vertices in the boundaries of radial cuts in X. There is exactly one metric end lying in all these cuts. This end is the image $\Psi_o(\eta)$. It is easy to check that the maps Φ_o and Ψ_o are homeomorphisms of the relative topologies of the metric end topology on ΩX and ΩT_o. □

Definition 4.9. A metric end ω is called *free* if there is a metric cut C such that ω is the only one metric end lying in C.

Note that an end is free if and only if it is an isolated point in ΩX (with respect to the metric end topology). The following lemma was proved in [12].

Lemma 4.10. *(See [12, Theorem 3.3].) A connected metrically almost transitive graph with more than two ends has no free metric end.*

In [12, Corollary 3.15] it was proved that an unbounded, connected metrically almost transitive graph has 1, 2 or infinitely many ends. We can now formulate a stronger version of this theorem:

Theorem 4.11. *An unbounded connected metrically almost transitive graph has either* 1, 2 *or at least* $2^{\aleph_0}$ *ends.*

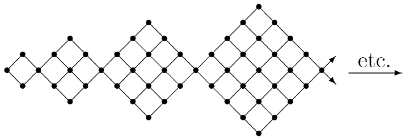

Fig. 4.

For almost transitive locally finite graphs it is well known that if $|\Omega X| > 2$ then ΩX is homeomorphic to the Cantor set. In non-locally finite graphs ΩX may have a cardinality larger than $2^{\aleph_0}$ in which case it is not homeomorphic to the Cantor set.

Proof of Theorem 4.11. Let X be unbounded, connected and metrically almost transitive. By Corollary 4.3, X has at least one metric end. Suppose X has more than two metric ends. Then, by Lemma 4.10, ΩX has no isolated points (free ends). By Lemma 4.8, ΩX is homeomorphic to ΩT_o and consequently, T_o has no free ends. A tree without free ends has at least $2^{\aleph_0}$ ends. Hence X has also at least $2^{\aleph_0}$ ends. □

5. Quasi-isometries between metrically almost transitive graphs and trees

In [6] Halin defined the thickness of a vertex end ω as the maximal number of disjoint rays in ω. Similarly, Woess defined the diameter of an end ω in a locally finite graph as the minimal number k such that there is a descending sequence $(C_n)_{n\in\mathbb{N}}$ of cuts containing ω such that $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$ and $\mathrm{diam}_X\, NC_n \leqslant k$, see Definition 1 in [18].

Example 5.1. The graph shown in Fig. 4 can be found in [18, Fig. 2]. It has one thin end in the sense of Definition 1 in [18]. But this end is thick in the sense of our Definition 3.6.

In [15] Sabidussi has shown how every transitive graph can be obtained as a factor of a Cayley graph. The following theorem is much simpler, but it is related to Sabidussi's results.

Theorem 5.2. *A connected metrically almost transitive graph is quasi-isometric to some connected transitive graph.*

Proof. Let ρ denote the covering radius of X according to Definition 4.1. Let o be some fixed vertex in X. Define Y as the graph whose vertex set is the orbit of o under $\mathrm{Aut}(X)$, and two vertices x and y are adjacent in Y if and only if $d_X(x,y) \leqslant 2\rho + 1$. Note that $\mathrm{Aut}(X)$ acts transitively on Y.

Our first task is to show that Y is connected. From the way the covering radius is defined, we learn that if x is some vertex in X then there is a vertex in the $\mathrm{Aut}(X)$-orbit of o at distance at most ρ from x. Let v and w be vertices in Y. There is a path $(v = x_1, x_2, \dots, x_n = w)$ in X. Let y_i be a vertex in Y such that $d(y_i, x_i) \leqslant \rho$. Then

$$d_X(y_i, y_{i+1}) \leqslant d_X(y_i, x_i) + d_X(x_i, x_{i+1}) + d(x_{i+1}, y_{i+1}) \leqslant 2\rho + 1.$$

Hence the vertices y_i and y_{i+1} are adjacent in Y or identical. The sequence $(y_1, y_2, \dots, y_n)$ may not itself be a path in Y, because there might be loops or consecutive elements which coincide. But this sequence has a subsequence which is a path in Y from v to w.

For any vertices x_1 and x_2 in VY we have

$$d_Y(x_1,x_2) \leqslant d_X(x_1,x_2) \quad \text{and} \quad d_X(x_1,x_2) \leqslant (2\rho+1)d_Y(x_1,x_2).$$

Define $\phi : VY \to VX$ as the identity and $\psi : VX \to VY$ such that $\psi(x)$ is a vertex y in Y such that $d_X(x,y)$ is minimal. It is left to the reader to show that X and Y are quasi-isometric with respect to ϕ and ψ. □

The converse of the above theorem is not true. It is easy to find graphs which are quasi-isometric to a transitive graph without being metrically almost transitive.

The following example shows that graphs as in Fig. 4, which only have thin ends in the sense of Woess but which have thick ends in the sense of Definition 3.6, can also be transitive.

Example 5.3. As in Example 3.7(ii), let G_n be the group

$$\langle a, b_n \mid a b_n a^{-1} b_n^{-1} = b_n^n = 1 \rangle,$$

which is the direct product of an infinite cyclic group with a cyclic group of order n. Let G be the free product

$$\begin{aligned} G_2 * G_3 * G_4 \cdots &= (\mathbb{Z}\times\mathbb{Z}_2) * (\mathbb{Z}\times\mathbb{Z}_3) * (\mathbb{Z}\times\mathbb{Z}_4) * \cdots \\ &= \langle a, b_2, b_3, \ldots \mid a b_n a^{-1} b_n^{-1} = b_n^n = 1 \rangle, \end{aligned}$$

and let X be the Cayley graph $\mathrm{Cay}(G,S)$ where $S = \{a, a^{-1}, b_n, b_n^{-1} \mid n \geqslant 2\}$. Then X is not quasi-isometric to a tree. It is not a thin graph and there exist ends which are thick in the sense of Definition 3.6. Each metric end ω in X has the property that

$$\liminf_{n\to\infty} \mathrm{diam}_X\, IC_o(n,\omega) < \infty$$

which means that all ends are thin in the sense of Woess.

Proof. The subgraphs X_n which are spanned by the subgroups $\mathbb{Z}\times\mathbb{Z}_n$ are (graph theoretic) Cartesian products of a cycle of length n with a double-ray (i.e., a two-way infinite path). They have two metric ends ω_n^+ and ω_n^-. For these ends, we have $\mu_o(\omega_n^+) = \mu_o(\omega_n^-) < \infty$ but $\lim_{n\to\infty} \mu_o(\omega_n^-) = \lim_{n\to\infty} \mu_o(\omega_n^+) = \infty$. Thus X is not thin and, by Theorem 2.8, X is not quasi-isometric to a tree.

The graph X consists of infinitely many isomorphic copies of the graphs X_n which correspond to the left cosets of the subgroups $\mathbb{Z}\times\mathbb{Z}_n$. Since X is constructed as a free product, any pair of distinct left cosets $g_1(\mathbb{Z}\times\mathbb{Z}_{n_1})$ and $g_2(\mathbb{Z}\times\mathbb{Z}_{n_2})$ can be separated by removing a single vertex. Let $A_n = \{\omega_{n,i}^+, \omega_{n,i}^- \mid i \in I\}$ be the set of all ends which belong to the left cosets of $\mathbb{Z}\times\mathbb{Z}_n$, where $\omega_{n,i}^+$ and $\omega_{n,i}^-$ are the ends corresponding to one coset and I is a suitable set of indices. We remark that $A_n \neq \Omega X$ but A_n is dense in ΩX. Let ω be any metric end of X and let o be some vertex. If ω is one of the ends in A_n then ω is thin, no matter which definition of thickness we consider. If ω is not in A_n then there are infinitely many integers n_k, $k \in \mathbb{N}$, such that $NC_o(n_k,\omega)$ consists of a single point. Thus

$$\liminf_{n\to\infty} \mathrm{diam}_X\, IC_o(n,\omega) = 0$$

and all ends are thin in the sense of Definition 1 in [18]. Our next aim is to show that there are ends which are thick in the sense of our Definition 3.6.

Let o be any vertex in VX and let S_1 be an isomorphic copy of a two-way infinite path corresponding to a left coset of the subgroup $\mathbb{Z} \times \mathbb{Z}_1$. Then there is a radial cut C_1 in $\mathcal{C}_o$ such that both $S_1 \cap C_1$ and $S_1 \cap (VX \setminus C_1)$ are infinite. Suppose there are cuts C_i in $\mathcal{C}_o$, $1 \leqslant i \leqslant k$, and subgraphs S_i of X which are isomorphic to X_i such that $S_i \cap C_i$ and $S_i \cap (VX \setminus C_i)$ are infinite. Then $\operatorname{diam}_X NC_k \geqslant \lfloor k/2 \rfloor$ (the diameter of a cycle of length k). Let S_{k+1} be a subgraph which is spanned by a left coset of $\mathbb{Z} \times \mathbb{Z}_{k+1}$ and has infinite intersection with C_k. If $S_{k+1} \cap C_k$ and $S_{k+1} \cap (VX \setminus C_k)$ are infinite then we set $C_{k+1} = C_k$ and we have $\operatorname{diam}_X NC_{k+1} \geqslant \lfloor (k+1)/2 \rfloor$. Otherwise, there is a radius r_{k+1} such that $B_X(o, r_{k+1})$ contains one of the cycles which correspond to a left coset of $\mathbb{Z}_{k+1}$ in S_{k+1}. This implies that the two ends of the rays in S_{k+1} are in different radial cuts with coradius r_{k+1}. These cuts are both subsets of C_k. Let C_{k+1} be one of these cuts. Then again,

$$|S_{k+1} \cap C_{k+1}| = \left|S_{k+1} \cap (VX \setminus C_{k+1})\right| = \infty,$$
$$\operatorname{diam}_X NC_{k+1} \geqslant \left\lfloor (k+1)/2 \right\rfloor \quad \text{and} \quad C_{k+1} \subseteq C_k.$$

By induction we get a descending sequence of cuts in $\mathcal{C}_o$ which define an end ω. We have $\sup\{\operatorname{diam}_X NC_k \mid k \geqslant 1\} = \infty$ which is equivalent to $\sup\{\operatorname{diam}_X IC_k \mid k \geqslant 1\} = \infty$, and the end ω is thick in the sense of Definition 3.6. $\square$

Lemma 5.4. *Let X be a connected metrically almost transitive graph and let k be an integer. Then*

$$\sup\{\operatorname{diam}_X C \mid C \textit{ is a bounded radial cut and } \operatorname{diam}_X NC \leqslant k\} < \infty. \tag{6}$$

Proof. Suppose there is a sequence $(C_n)_{n \in \mathbb{N}}$ of bounded radial cuts such that $\operatorname{diam}_X C_n > 2n$ and $\operatorname{diam}_X NC_n \leqslant k$. Let o be any vertex. For each of the boundaries NC_n there is an automorphism g_n such that $g_n(NC_n) \subseteq B_X(o, k+\rho)$ where ρ is the covering radius of X. Each of the sets $g_n(C_n) \setminus B_X(o, k+\rho)$ is a union of components of $VX \setminus B_X(o, k+\rho)$ which are all bounded. The set C_n cannot be contained in $B_X(o, n)$, because $\operatorname{diam}_X C_n > 2n$. If $n \geqslant k+\rho$ then $g_n(C_n) \setminus B_X(o, k+\rho)$ is not empty and one of the components D_n of $g_n(C_n) \setminus B_X(o, k+\rho)$ contains a vertex x_n such that $d_X(x_n, o) > n$ and $d_X(x_n, B_X(o, k+\rho)) > n-k-\rho$. Hence $\operatorname{diam}_X D_n \geqslant n-k-\rho$ and $B_X(o, k+\rho)$ is a star ball which contradicts Lemma 4.2. $\square$

Theorem 5.5. *Suppose X is a metrically almost transitive connected graph. Then the following statements are equivalent*:

(1) *X is thin.*
(2) *ΩX is thin.*
(3) *Every metric end is thin.*
(4) *X is quasi-isometric to a tree.*
(5) *X is quasi-isometric to a tree without vertices of degree* 1.

We have seen in Example 3.7 that the conditions (1)–(3) are not necessarily equivalent for graphs which are not metrically almost transitive. One important property of metrically almost transitive graphs is that they do not contain any star balls. But assuming that there are no star balls is not enough to get the equivalence of (2)–(4). The graph X in Example 3.7(ii) contains no star ball, it satisfies (2) and (3) but not (4).

Definition 5.6. A metric cut C is called *non-trivial* if both C and the complement $VX \setminus C$ have infinite diameter in X.

Proof of Theorem 5.5. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follow immediately from Definitions 2.4 and 3.6. The implication (5) $\Rightarrow$ (4) is obvious and (1) $\Leftrightarrow$ (4) is part of Theorem 2.8. We will prove (3) $\Rightarrow$ (2), (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5).

Let us start with (3) $\Rightarrow$ (2). Let X be a metrically almost transitive graph with covering radius ρ. Suppose all metric ends are thin but ΩX is not thin. Then there are non-trivial radial cuts with arbitrary large diameters of their inner boundaries. In order to show that the assumptions above lead to a contradiction we will show that there exists a thick end. We will do this by showing that there are cuts with arbitrarily large diameters of their inner boundaries which all have the same center.

Suppose there are non-trivial radial cuts $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_{n-1}$ with center o such that $\operatorname{diam}_X IC_k > k$ for $0 \leqslant k \leqslant n-1$. Since X is metrically almost transitive and since there are non-trivial radial cuts with arbitrary large diameters of their inner boundaries, there is a non-trivial radial cut A with center w such that $\operatorname{diam}_X IA > 3n$ and such that $IA \subseteq C_{n-1}$ and $w \in C_{n-1}$.

Case 1. There is a radial cut C_n with center o such that $C_n \subseteq C_{n-1}$, $IC_n \cap IA \neq \emptyset$ and such that IC_n separates w from A. Then, by Lemma 2.5, $3 \operatorname{diam}_X IC_n \geqslant \operatorname{diam}_X IA > 3n$ and $\operatorname{diam}_X IC_n > n$.

Case 2. Suppose we are not in Case 1. Since $w \in C_{n-1}$, there is a radial cut C_n with center o, $C_n \subseteq C_{n-1}$, such that $w \in IC_n$. If $A \cap C_n = \emptyset$ then by reducing the coradius of C_n we can find a cut which satisfies Case 1. If $A \subseteq C_n$ then we can find a cut which satisfies Case 1 by increasing the coradius of C_n. Hence there are vertices of A in both, C_n and $VX \setminus C_n$. Since A is connected there is a vertex x in $A \cap IC_n$. Because A is a radial cut such that $\operatorname{diam}_X IA > 3n$ we have $d_X(w, A) > 3/2n$ and therefore $d_X(w, x) > 3/2n$ which implies $\operatorname{diam}_X IC_n > 3/2n$.

We conclude that there is a radial cut C_n with center o such that $C_n \subseteq C_{n-1}$ and $\operatorname{diam}_X IC_n > n$. By induction we get a sequence $(C_n)_{n\in\mathbb{N}}$ of non-trivial radial cuts which define a thick metric end. Hence (3) $\Rightarrow$ (2).

Next we prove (2) $\Rightarrow$ (1). Suppose ΩX is thin. There is nothing to prove if X has finite diameter. If X has one metric end, then, by Theorem 4.6, this end is thick and therefore ΩX is not thin. If X has more then one metric end, then there is a non-trivial metric cut C. Suppose X is not thin. Then there exist in X radial cuts whose boundaries have arbitrarily large diameters and that contain no metric end. For if there were non-trivial radial cuts whose boundaries have arbitrarily large diameters then we could proceed as in the proof of the implication (3) $\Rightarrow$ (2) and construct a thick metric end. Corollary 4.3 says that in almost transitive graphs the metric cuts are bounded when they contain no metrically transient ray. Thus there exist bounded radial cuts in X such that the diameters of their boundaries are arbitrarily large.

Let x be any vertex. We set $\mu = \sup\{\mu_o(\omega) \mid \omega \in \Omega X\}$. If the supremum of the diameters of the bounded components of $VX \setminus B_X(x, \rho + \mu + 2)$ is infinite then, by Definition 4.1, $B_X(x, \rho + \mu + 2)$ is a star ball. By Lemma 4.2, there are no star balls in metrically almost transitive graphs. Hence all the bounded components of $VX \setminus B_X(x, \rho + \mu + 2)$ have diameters smaller than some constant p. Let C be a bounded radial cut with center o and coradius n such that $\operatorname{diam}_X IC > 4\rho + 5\mu + 10 + 2p$. Then $n > 2\rho + 5/2\mu + 5 + p$. Let C' be the radial cut with center o and minimal coradius m such that $C \subseteq C'$ and C' is bounded. Let C'' be the radial cut with center o and coradius $m - 1$ which contains C'. Note that $m - 1 \geqslant 0$ which means that

$C'' \neq VX$. Since C'' is unbounded, we have $\operatorname{diam}_X IC'' \leqslant \mu$. Consequently $\operatorname{diam}_X IC' \leqslant \mu + 2$ and

$$\operatorname{diam}_X IC - \operatorname{diam}_X IC' > 4\rho + 4\mu + 8 + 2p.$$

Any vertices v and w in IC can be connected to a vertex in IC' by paths of length $d_X(IC, IC')$, because C and C' are radial cuts with the same center. Hence we can find a path from v to w whose length is less or equal to

$$d_X(IC, IC') + \operatorname{diam}_X IC' + d_X(IC, IC')$$

which implies

$$\begin{aligned} &\operatorname{diam}_X IC \leqslant 2d_X(IC, IC') + \mu + 2 \quad \text{and} \\ &d_X(IC, IC') \geqslant (\operatorname{diam}_X IC - \mu - 2)/2 \\ &\qquad\qquad > (4\rho + 5\mu + 10 + 2p - \mu - 2)/2 = 2\rho + 2\mu + 4 + p. \end{aligned}$$

Let g be an automorphism of X such that $d_X(x, g(IC')) \leqslant \rho$. Then $g(IC')$ is contained in $B_X(x, \rho + \mu + 2)$. The set $g(C') \setminus B_X(x, \rho + \mu + 2)$ is one of the bounded components of $VX \setminus B_X(x, \rho + \mu + 2)$. Since $g(IC') \cup g(IC) \subseteq g(C')$ we have

$$\begin{aligned} &\operatorname{diam}_X g(C') \geqslant d_X\big(g(IC'), g(IC)\big) > 2\rho + 2\mu + 4 + p \quad \text{and} \\ &\operatorname{diam}_X g(C') \setminus B_X(x, \rho + \mu + 2) \\ &\quad \geqslant \operatorname{diam}_X g(C') - \operatorname{diam}_X B_X(x, \rho + \mu + 2) > p. \end{aligned}$$

This is a contradiction which proves (2) $\Rightarrow$ (1).

Finally we prove the implication (4) $\Rightarrow$ (5). Suppose X is a graph which is metrically almost transitive and quasi-isometric to a tree T.

Note that if T has a vertex v of degree 1 then there is a vertex w such that $VT \setminus \{w\}$ has a bounded component which contains v. Conversely, if $VT \setminus \{v\}$ has only unbounded components for every vertex v then T cannot have any vertices of degree 1.

(a) If the supremum of the diameters of the bounded components of the complements of single vertices in T is finite then we remove all these bounded components and obtain a graph T'. The trees T and T' are quasi-isometric and T' has no vertices of degree one. Since quasi-isometry is an equivalence relation, X is quasi-isometric to T'.
(b) If this supremum is infinite then we find vertices v_n and bounded components D_n in $VT \setminus \{v_n\}$ such that

$$\lim_{n\to\infty} \operatorname{diam}_T D_n = \infty.$$

Let ψ be a quasi-isometry from T to X. By Lemma 2.10, any ball $B_X(\psi(v_n), \kappa)$ separates $\psi(D_n)$ from $\psi(VT \setminus (D_n \cup \{v_n\}))$. Let C_n denote the component of $VX \setminus B_X(\psi(v_n), \kappa)$ such that $C_n \cup B_X(\psi(v_n), \kappa)$ contains $\psi(D_n)$. Then $\operatorname{diam}_X NC_n \leqslant 2\kappa$. By Lemma 2.2, we get

$$\lim_{n\to\infty} \operatorname{diam}_X \psi(D_n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \operatorname{diam}_X C_n = \infty.$$

This is a contradiction to Lemma 5.4. Hence case (b) is impossible which proves (4) $\Rightarrow$ (5). $\square$

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