Math. Nachr. **281,** No. 1, 62 – 74 (2008) / **DOI** 10.1002/mana.200510587

Metric ends, fibers and automorphisms of graphs

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Received 17 February 2005, revised 20 March 2006, accepted 2 May 2006
Published online 14 December 2007

Key words Ends of graphs, metric ends, fibers, graph automorphisms
MSC (2000) 05C12, 05C25

Several results on the action of graph automorphisms on ends and fibers are generalized for the case of metric ends. This includes results on the action of the automorphisms on the end space, directions of automorphisms, double rays which are invariant under a power of an automorphism and metrically almost transitive automorphism groups. It is proved that the bounded automorphisms of a metrically almost transitive graph with more than one end are precisely the kernel of the action on the space of metric ends.

Introduction

The basic idea behind all the various *end* concepts is to distinguish between different ways of going to infinity. Ends are used in different topological settings and they carry a natural topology which is often not mentioned explicitly. Ends of graphs were introduced by Freudenthal in 1945 (see [8]). His "abzählbare, diskrete Räume" (countable, discrete spaces) with "Nachbarschaft zweier Punkte" (adjacency of two points) are what we nowadays call "graphs". Freudenthal only considered locally finite graphs. His work goes back to his thesis [9] which was also published in [10]. Halin was the first to consider ends of non-locally finite graphs in [11]. He defined ends as equivalence classes of rays where two rays are equivalent if they have infinitely many vertices in common with a third ray. We will refer to this type of ends as *vertex ends*. Vertex ends are the most common type of ends in graph theory, see e.g. Diestel [4], Diestel and Kühn [6], Halin [11] and Jung and Niemeyer [19].

Others who have linked ends of graphs with group theory or random walks have used *edge ends*, see Definition 1.1, or related concepts like structure trees and D-cuts. For examples see Cartwright, Soardi and Woess [2], Dicks and Dunwoody [3], Dunwoody [7] and Stallings [27].

Ends of graphs are defined as equivalence classes of rays. Two rays R_1 and R_2 are equivalent whenever it is impossible to find a "small" set of vertices or edges F such that any path from R_1 to R_2 contains an element of F. In the case of vertex ends these small sets are finite sets of vertices, in the case of edge ends they are finite sets of edges. In [21], the first author studied *metric ends* for non-locally finite graphs where these small sets are sets of vertices which are bounded with respect to the natural metric of the graph. This natural distance of two vertices x and y is the length of the shortest path from x to y. Metric ends are defined as equivalence classes of *metrically transient rays*, which are rays without infinite, bounded subsets. Precise definitions are given in Section 1. It makes sense to use metric ends whenever the metric of the graph plays an important role. For example, when quasi-isometries between graphs are extended to the metric ends they become homeomorphisms of the metric end topologies, see [21, Theorem 6]. In [23], metric ends are used to classify graphs which are quasi-isometric to trees. One aim of the present paper is to lay a basis for the study of metric ends in the relation to the action of the group of automorphisms.

In locally finite graphs we do not have to distinguish between different types of ends, because in this case the three end concepts (vertex ends, edge ends and metric ends) coincide.

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Jung and Niemeyer (see [17] and [19]) introduced and studied two different equivalence relations on the rays in a given graph. The corresponding equivalence classes are called fibers. Informally, one can say the idea is that two rays are equivalent if they lie close together, see Definition 1.2.

Halin proves in his fundamental paper [12] that every automorphism σ of a graph either leaves a finite nonempty set of vertices invariant or there is a double ray L in the graph that some power σ^n of the automorphism acts on as a nontrivial translation (see Theorem 2.2). Such double rays are called *periodic double rays*. If L contains a metrically transient subray R such that $\sigma^n(R) \subsetneq R$ then we call the metric end which contains R the *direction* (or *metric direction*) of σ and σ is called a *metric translation*. In Section 2 we prove results on periodic double rays, metric translations and their directions which are analogues to results in [12] and [16]. These results will turn out to be partly different from the case of vertex ends.

When considering metric ends, the natural symmetry condition on the graph is that of *metrically almost transitivity* which was introduced in [22]. A graph is said to be *almost transitive* if the automorphism group has only finitely many orbits on the set of vertices. The graph is said to be *metrically almost transitive* if there is a vertex v and a constant c such that the distance of any vertex to the orbit of v under the automorphism group is at most c. In Section 3 we prove results which are similar to results from [1], [5], [15] and [16] concerning bisections of the graph, denseness of the directions in the end space and free ends for the case of metric ends in metrically almost transitive graphs. It is also shown that every unbounded Cayley graph of a torsion group has exactly one metric end, see Corollary 3.9.

An automorphism is bounded if the distances between the vertices and their images under this automorphism are bounded, see Definiton 4.1. There is a series of results where the subgroup of bounded automorphisms are identified as the kernel of an action of the group of automorphisms. In [24, Theorem 6], the second author has shown that if X is a locally finite graph with infinitely many ends and the automorphism group of X acts transitively then the bounded automorphisms are the kernel of the action of automorphism group on the end space. This conclusion is also true if the graph has only two ends, see Jung and Watkins [18, Theorem 5.10]. This result in [24] was extended to vertex ends of non-locally finite graphs with a transitive automorphism group in [5, Theorem 6]. Niemeyer [25] has continued with this line of investigations and has shown that the subgroup of bounded automorphisms of an almost transitive locally finite graph is the kernel of the action of the automorphism group on the so called d-fibers, see Definition 1.2. Note that this result also applies to graphs with only one end. Results in this spirit which consider ends instead of fibers only hold for graphs with at least two ends. For the case of non-locally finite metrically almost transitive graphs with more than one end, it is proved in Section 4 of the present paper that the bounded automorphisms are precisely the kernel of the action on the space of metric ends, see Theorem 4.2.

As mentioned above, many of the results in this paper about group actions and metric ends have close analogues in known results about group actions and vertex ends. Even if the statements are similar the shift of emphasizes from the combinatorial structure of the graph to the metric on the graph calls for new proofs. There are also several aspects and concepts (e.g. the concepts of a *star ball* and a *metrically almost transitive graph*) in the new theory that are fundamentally different from the existing theory of group actions on vertex ends.

1 Preliminaries

Graphs $X = (VX, EX)$ with vertex set VX and edge set EX are always undirected and without loops or multiple edges. Edges are regarded as two element subsets of VX. Two vertices x and y are said to be *adjacent*, or *neighbours*, if $\{x, y\}$ is an edge in X. A graph is *locally finite* if each vertex has only finitely many neighbours. A *path* of length n from x to y is an $n+1$-tuple $(x = x_0, x_1, \dots, y = x_m)$ of distinct vertices, such that x_i and x_{i+1} are adjacent for $i = 0, 1, \dots, m-1$. Let $\pi_1 = (v_0, v_1, \dots, v_m)$ and $\pi_2 = (w_0, w_1, \dots, w_n)$ be paths such that $v_m = w_0$. Then the *concatenation* of π_1 and π_2 is the path $(v_0, v_1, \dots, v_m = w_0, w_1, \dots, w_n)$. A *ray* is a sequence $(x_0, x_1, \dots)$ of distinct vertices such that x_i and x_{i+1} are adjacent for $i \geq 0$. A two-sided infinite sequence $(\dots, x_{-1}, x_0, x_1, \dots)$ of distinct vertices is called a *double ray* if x_i and x_{i+1} are adjacent for any integer i. The *distance* $d_X(x, y)$ between two vertices x and y is the length of a shortest path from x to y. Let A be a set of vertices. We set $d_X(x, A) = \min\{d_X(x, y) \mid y \in A\}$. The set A is *connected* if any two vertices in A can be connected by a path which is contained A. The *components* of A are the maximal connected subsets of A. If the graph is connected (i.e., VX is connected) then d_X is a metric on VX. We write $\operatorname{diam}_X A$ for the diameter of A with respect to this metric. A *ball* with *center* $x \in VX$ and *radius* r is the set

$$B_X(x, r) = \{y \in VX \mid d_X(x, y) \leq r\}.$$

A graph automorphism is a map $\alpha : VX \to VX$ such that x is adjacent to y if and only if $\alpha(x)$ is adjacent to $\alpha(y)$, for any vertices x and y. The set $\mathrm{Aut}(X)$ of all graph automorphisms is a group with respect to the concatenation of functions.

Definition 1.1 A set of vertices F *separates* vertices x and y if every path from x to y contains a vertex of F. A set of edges F *separates* x from y if every path $(x_0 = x, x_1, \ldots, x_n = y)$ from x to y has an edge $\{x_{i-1}, x_i\}$ in F, $1 \leq i \leq n$. The set F (of vertices or edges) *separates* sets of vertices A and B if it separates any vertex in A from any vertex in B.

Two rays R_1 and R_2 belong to the same *vertex end* if they cannot be separated by a finite set of vertices.

Two rays R_1 and R_2 belong to the same *edge end* if they cannot be separated by a finite set of edges.

Note that any vertex in F is separated by F from any other vertex.

It is easy to see that the relation of belonging to the same vertex end, or edge end, is an equivalence relation on the set of all rays. The classes of this relation are called the vertex ends or edge ends of the graph X, respectively.

There are various ways to rephrase the definitions of vertex ends or edge ends. In [11], Halin defined vertex ends by saying that two rays are in the same vertex end if there are infinitely many disjoint paths connecting vertices in one ray to vertices in the other. If we want to consider edge ends then we relax the condition that the paths are disjoint to the condition that no two of the paths have a common edge.

If two rays are in the same vertex end then they are in the same edge end. The number of vertex ends of a graph is not necessarily the same as the number of edge ends: a graph consisting of two infinite complete graphs with a single common vertex has two vertex ends but only one edge end. Thus, in general, the vertex ends give a finer partition on the set of rays.

If the graph is locally finite then two rays belong to the same vertex end if and only if they belong to the same edge end. In this case, also the metric ends (see Definition 1.3) give the same equivalence relation on the rays as the vertex ends and the edge ends.

Various ideas have been proposed for similar concepts that can distinguish between more possible ways of going towards infinity. The following ideas stem from Jung and Niemeyer, see [17] and [19].

Definition 1.2 Let X be a connected graph. Suppose R_1 and R_2 are two rays in X.

(i) ([19, Definition 1]) We say that the rays R_1 and R_2 are *d-equivalent* if there is a number m such that

$$R_1 \subseteq \{v \in VX \mid d_X(v, R_2) \leq m\},$$

and

$$R_2 \subseteq \{v \in VX \mid d_X(v, R_1) \leq m\}.$$

(ii) ([19, Definition 2]) We say that the rays R_1 and R_2 are *b-equivalent* if there are sequences $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$ of vertices belonging to R_1 and R_2, respectively, such that the sets $\{d_X(x_i, y_i) \mid i = 0, 1, 2, \ldots\}$, $\{d_{R_1}(x_i, x_{i+1}) \mid i = 0, 1, 2, \ldots\}$ and $\{d_{R_2}(y_i, y_{i+1}) \mid i = 0, 1, 2, \ldots\}$ are all finite.

Both these relations define an equivalence relation on the set of rays in X. The equivalence classes are called b- and d-*fibers*, respectively.

Two b-equivalent rays are also d-equivalent. If we restrict the attention to locally finite graphs then it is easy to see that two d-equivalent rays belong to the same end. In general, this does not hold if the graph is not locally finite. For instance, let us again consider the graph consisting of two copies of an infinite complete graph with one common vertex. This graph has two vertex ends. But since it has finite diameter it has only one b-fiber and only one d-fiber. Thus, if we drop the assumption of local finiteness imposed in [17] and [19] we lose the connections between fibers and ends.

If we restrict the attention to metrically transient rays then the situation is different.

Definition 1.3 *Metrically transient rays* are rays such that every infinite subset of vertices has infinite diameter. *Metrically transient double rays* are defined similary. Two metrically transient rays R_1 and R_2 in a graph X are *metrically equivalent* if they cannot be separated by a bounded set of vertices. The corresponding equivalence classes on the set of metrically transient rays are called the *metric ends*. The set of all metric ends of X is denoted by ΩX.

Note that in [21] and [22] the term *metric ray* is used for what we now call *metrically transient rays*. The change is to avoid confusion because the term *metric ray* has been used for a different, but related concept by other authors, see [19].

It is easy to prove that metric equivalence is an equivalence relation on the set of metrically transient rays of X. We restrict our attention to metrically transient rays because they have the property that if T is a bounded set of vertices then there is precisely one component of $VX \setminus T$ that contains infinitely many vertices from any metrically transient ray. Several results on metric ends and the corresponding topology can be found in [21]. A definition of metric ends can also be found in the Master's thesis of Hien [13].

Lemma 1.4 *Let X be a connected graph.*

(i) *If one ray in a b-fiber is metrically transient then every ray in that b-fiber is metrically transient.*

(ii) *Two metrically transient rays that belong to the same d-fiber also belong to the same metric end.*

P r o o f. (i) Suppose R_1 and R_2 are rays that belong to the same b-fiber. Let $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$ be sequences of vertices of the rays R_1 and R_2, respectively, satisfying the conditions in Definition 1.2. Let R_1 be a metrically transient ray. Then every infinite subset of $\{x_0, x_1, \ldots\}$ is unbounded. There is a constant C such that $d_X(x_i, y_i) \le C$ for all $i \ge 0$. Hence every infinite subset of $\{y_0, y_1, \ldots\}$ is also unbounded. Since $d_{R_2}(y_i, y_{i+1}) \le D$ for some constant D, we see that every infinite set of vertices in R_2 must be unbounded. Hence R_2 is a metrically transient ray.

(ii) Suppose that R_1 and R_2 are two metrically transient rays which are in the same d-fiber but not in the same metric end. Then there is a bounded set of vertices S and there are components C_1 and C_2 of $VX \setminus S$ such that C_1 contains all but finitely many of the vertices of R_1 and C_2 contains all but finitely many of the vertices of R_2. Since the two rays are metrically transient, we see that C_1 contains vertices from R_1 which are arbitrarily far away from S and C_2 contains vertices from R_2 which are arbitrarily far away from S. This contradicts the assumption that R_1 and R_2 are in the same d-fiber. □

Since b-equivalent rays are also d-equivalent, the above lemma implies that if some metrically transient ray of a b-fiber β is in some metric end ω then $\beta \subseteq \omega$.

The following example shows that in general it is not true that if one ray in a d-fiber is metrically transient that then every ray in this d-fiber has to be metrically transient.

Example 1.5 Let X be a graph with vertex set $\mathbb{Z}^2$ and let (i, j) and (i, j') be adjacent for all i, j, j' and (i, j) and $(i + 1, j)$ are adjacent for all i, j. Define two rays $R = ((0, 0), (1, 0), (2, 0), \ldots)$ and

$$R' = ((0,0),(0,1),(1,1),(1,2),(0,2),(0,3),(1,3),\ldots,(3,3),(3,4),(2,4),\ldots(0,4),(0,5),\ldots(5,5),\ldots).$$

These two rays are in the same d-fiber. The ray R is metrically transient but R' is not metrically transient. Figure 1 shows these two rays and the graph X, where the edges connecting (i, j) and (i, j') are only drawn for $|j - j'| = 1$. All pairs of vertices in Figure 1 which are on the same vertical line are adjacent in X.

The graph X has two metric ends but only one vertex end and only one edge end.

2 Automorphisms, periodic double rays and metric ends

Definition 2.1 Let σ be an automorphism of a graph X. A double ray L in X is called *σ-periodic* with period n if $\sigma^n(L) = L$ and σ^n acts as a nontrivial translation on L.

Note that the period of L with respect to σ is not unique. The starting point of the investigations in this section is the following [12, Theorem 7].

Theorem 2.2 *Every automorphism of a connected graph which leaves no finite set of vertices invariant has a periodic double ray.*

In [12], this result is stated with the additional assumption that the graph X is locally finite but going through the proof given in [12] one sees that this assumption is not used.

Definition 2.3 An automorphism σ of a graph X is called (*metrically*) *elliptic* if there is a nonempty bounded subset F of X such that $\sigma(F) = F$. An automorphism which has a periodic metrically transient double ray is called *metric translation*. The *metric direction* $\mathcal{M}(\sigma)$ of a metric translation σ is the metric end which contains a ray R such that there is an integer $s > 0$ for which $\sigma^s(R) \subsetneq R$.

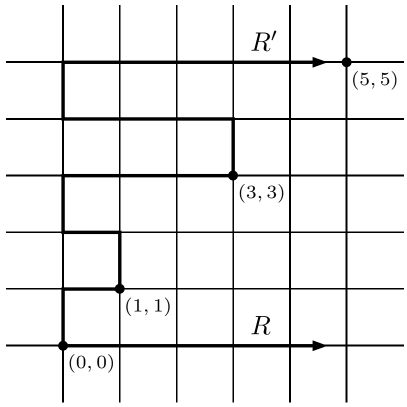

Fig. 1

Note that metric translations are non-elliptic. From Theorem 2.2 we see that if σ is not metrically elliptic then there is some σ-periodic double ray. By Lemma 2.4 below, the metric direction of σ does not depend on the choice of the ray R and is therefore well defined.

One might hope that Theorem 2.2 has a metric analogue, such that every automorphism which leaves no bounded set invariant has a metrically transient periodic double ray. In Example 3.16 we will see that in general, this is not necessarily true.

Lemma 2.4 *Let σ be a metric translation of a connected graph X. Then every σ-periodic double ray is metrically transient. Furthermore, if R and R' are two metrically transient rays, such that there are integers s and s' such that $\sigma^s(R) \subseteq R$ and $\sigma^{s'}(R') \subseteq R'$ then R and R' are in the same metric end.*

Proof. Suppose L and L' are some σ-periodic double rays and that L is metrically transient. If L is invariant under σ^s and L' is invariant under $\sigma^{s'}$ then both L and L' are invariant under $\tau = \sigma^{ss'}$.

Let A be an infinite set of vertices from L'. Take some path P from a vertex v in L to a vertex w in L'. Define k as the length of P. Because A is infinite, there are infinitely many vertices in the orbit of w under τ that are in a distance from A which is bounded by some constant. Hence we see that if the set A is bounded then there would be an infinite bounded subset of the orbit of w under τ. But we also see that $\tau^i(w)$ is in distance k from $\tau^i(v)$, and every infinite subset of L is unbounded, because we have assumed that L is metrically transient. Hence every infinite subset of the orbit of w is unbounded and A must be unbounded too. This means that L' is metrically transient. (This can also be inferred from Polat and Watkins study of *quasi-axes* in graphs, see [26, Theorem 3.1 and Proposition 3.4].)

Continuing with the above setup, let R denote the ray in L that starts in v and contains all vertices $\tau^i(v)$ for $i \geq 0$ and let R' denote the ray in L' that starts in w and contains all vertices $\tau^i(w)$ for $i \geq 0$. The paths $\tau^i(P)$, for $i \geq 0$, connect R with R'. Let T be a set of vertices which separates R from R'. Then T must contain a vertex from each of the paths τ^i. Since R and R' are metrically transient, T must be unbounded. Hence R and R' are metrically equivalent. □

The proof of the following lemma is easy and is left to the reader.

Lemma 2.5 *Let X be a connected graph and let φ be an automorphism of X. Then the following are equivalent:*

(i) *φ is metrically elliptic;*

(ii) *for every vertex v in X the orbit of v under φ is bounded;*

(iii) *there is some vertex v in X such that the orbit of v under φ is bounded.*

As a consequence of Lemmas 2.4 and 2.5 we obtain the following:

Corollary 2.6 *Let σ be an automorphism which has an invariant double ray. Then either* (i) *all σ-periodic double rays are bounded or* (ii) *all σ-periodic double rays are metrically transient or* (iii) *all σ-periodic double rays are unbounded and not metrically transient.*

In locally finite graphs, a set of vertices is bounded if and only if it is finite. This implies that there are no bounded periodic double rays and there are no unbounded double rays which are not metrically transient. Thus in locally finite graphs, the cases (i) and (iii) cannot occur. The metric translations of locally finite graphs are precisely the automorphism that do not fix any finite set of vertices.

Definition 2.7 The *boundary* of C is defined as

$$\theta C = \{x \in VX \setminus C \mid x \text{ is adjacent to some vertex } y \text{ in } C\}.$$

Lemma 2.8 *Let C be a set of vertices such that $VX \setminus C$ is connected and such that C contains the boundary of a set of vertices D. Then either $D \subseteq C$ or $VX \setminus D \subseteq C$.*

Proof. Suppose there are vertices $x \in D \setminus C$ and $y \in (VX \setminus D) \setminus C$. Since the complement of C is connected there is a path π from x to y which is completely contained in $VX \setminus C$. This path contains vertices of D and vertices of $VX \setminus D$. Thus π must contain a vertex of θD which is a contradiction to the assumption $\theta D \subseteq C$. □

Corollary 2.9 *Let C be a set of vertices such that $VX \setminus C$ is connected and let σ be an automorphism such that $\sigma(\theta C) \subseteq C$. Then either $\sigma(C) \subseteq C$ or $\sigma(VX \setminus C) \subseteq C$.*

Definition 2.10 We say that an end ω *is in* or *lies in* a set of vertices C if $|R \setminus C| < \infty$ for all rays R in ω.

This definition is used for edge ends, vertex ends and metric ends. It can happen that an end lies neither in C nor in $VX \setminus C$, but it cannot happen that an end lies in C and in $VX \setminus C$ at the same time.

Definition 2.11 A *vertex cut* is a set of vertices C such that θC is finite, and an *edge cut* is a set of vertices C such that there are only finitely many edges connecting vertices in C to vertices in $VX \setminus C$. A *metric cut* is a set of vertices C such that $\operatorname{diam}_X \theta C < \infty$.

A set of vertices F separates an end ω from a set of vertices A if there is a ray in ω which is separated by F from A. The set F separates an end ω_1 from an end ω_2 if there are rays R_1 in ω_1 and R_2 in ω_2 which are separated by A from each other.

If C is (i) an edge cut, (ii) a vertex cut or (iii) a metric cut, then (i) any edge end (ii) any vertex end and (iii) any metric end lies either in C or in $VX \setminus C$, respectively.

The following result gives a further property that characterizes metric translations. A similar theorem was proved by Jung for vertex ends, see [16, Theorem 2.7].

Theorem 2.12 *Let X be a connected graph and let σ be an automorphism of X. Suppose there is a nonempty bounded set of vertices T, a connected component C of $VX \setminus T$ and a number n such that $\sigma^n(T \cup C) \subseteq C$. Then σ is a metric translation. Furthermore, $\mathcal{M}(\sigma) \neq \mathcal{M}(\sigma^{-1})$ and $\mathcal{M}(\sigma)$ is in C.*

Conversely, if σ is a metric translation such that $\mathcal{M}(\sigma) \neq \mathcal{M}(\sigma^{-1})$ then there is a nonempty bounded set T, a connected component C of $VX \setminus T$ and a number n such that $\sigma^n(T \cup C) \subseteq C$.

Proof. Suppose $\sigma^n(T \cup C) \subseteq C$. We set $\tau = \sigma^n$. Then $T \uplus C \supseteq C \supseteq \tau(T) \uplus \tau(C)$, where $\uplus$ denotes a disjoint union. By induction,

$$T \uplus C \supseteq C \supseteq \tau(T) \uplus \tau(C) \supseteq \tau(C) \supseteq \tau^2(T) \uplus \tau^2(C) \supseteq \tau^2(C) \supseteq \dots \tag{2.1}$$

and the sets T, $\tau(T)$, $\tau^2(T), \dots$ are pairwise disjoint. Let v be a vertex in T. Then $d_X(v, \tau^m(v))$ is greater or equal to m, because a path from v to $\tau^m(v)$ has to contain vertices of $\tau^i(T)$ for $i = 0, \dots, m$. By Lemma 2.5, τ is non-elliptic. Let L be a τ-periodic double ray. No infinite subset of the τ-orbit of v (i.e., the set $\{\tau^i(v) \mid i \in \mathbb{Z}\}$) is bounded. Let A be an infinite subset of L and let w be a vertex in L. There is an infinite subset W of the τ-orbit of w such that the distance of each vertex in W to the set A is less than $d_L(w, \tau(w))$, which denotes the distance with respect to the subgraph which is spanned by L. Since every infinite subset of the τ-orbit of v is unbounded, the set W is also unbounded. Thus A is unbounded and every τ-periodic double ray L is metrically transient. Since every σ-periodic double ray is also τ-periodic, it follows that every σ-periodic double ray is metrically transient. Hence σ and σ^{-1} both have metric directions. The inclusions in (2.1) and the fact that $d_X(v, \tau^m(v)) \geq m$, for $m \geq 1$, imply that $\mathcal{M}(\sigma)$ lies in C. The inclusion $C \supseteq \tau(T) \uplus \tau(C)$ implies $\tau^{-1}(C) \supseteq T \uplus C$, $\tau^{-1}(VX \setminus C) \subseteq (VX \setminus T) \uplus (VX \setminus C)$ and

$$T \uplus (VX \setminus C) \supseteq VX \setminus C \supseteq \tau^{-1}(VX \setminus T) \uplus \tau^{-1}(VX \setminus C) \supseteq \tau^{-1}(VX \setminus C) \supseteq \dots$$

which implies that $\mathcal{M}(\sigma^{-1})$ lies in $VX \setminus C$. Hence $\mathcal{M}(\sigma) \neq \mathcal{M}(\sigma^{-1})$.

Suppose now that we have a metric translation σ such that $\mathcal{M}(\sigma) \neq \mathcal{M}(\sigma^{-1})$. There is a connected bounded set T separating $\mathcal{M}(\sigma)$ and $\mathcal{M}(\sigma^{-1})$. Let C be the component of $VX \setminus T$ which contains $\mathcal{M}(\sigma)$ and let L be a σ-periodic double ray. Then T intersects L and there is a number n such that σ^n acts as a translation along L and $\sigma^n(T) \subseteq C$. Thus $\sigma^n(\theta C) \subseteq C$. The complement $VX \setminus C$ is connected. By Corollary 2.9, either $\sigma^n(C) \subseteq C$ or $\sigma^n(VX \setminus C) \subseteq C$. The latter is impossible because then σ^n would map the end $\mathcal{M}(\sigma^{-1})$ to an end in C, but σ^n fixes $\mathcal{M}(\sigma^{-1})$. Thus $\sigma^n(C) \subseteq C$ and consequently $\sigma^n(C \cup T) \subseteq C$. □

The following result is an analogue of [12, Theorem 9].

Theorem 2.13 *Let X be a connected graph and let σ be an automorphism of X. If σ is a metric translation then the metric ends $\mathcal{M}(\sigma)$ and $\mathcal{M}(\sigma^{-1})$ are fixed by σ and these are the only metric ends fixed by σ.*

Proof. Let S be a ray in $\mathcal{M}(\sigma)$. By Definition 2.3, there is a ray R in $\mathcal{M}(\sigma)$ such that $\sigma^n(R) \subsetneq R$ for some positive integer n. The ray $\sigma(R)$ has again the property that $\sigma^n(\sigma(R)) \subsetneq \sigma(R)$. Hence Lemma 2.4 implies that $\sigma(R)$ is also in $\mathcal{M}(\sigma)$. Because R and S are in the same end and because σ is an automorphism, the rays $\sigma(R)$ and $\sigma(S)$ are again in the same end. Hence S and $\sigma(S)$ are in the same end $\mathcal{M}(\sigma)$ which means that σ leaves $\mathcal{M}(\sigma)$ invariant.

Suppose ω is a metric end of X which is different from both $\mathcal{M}(\sigma)$ and $\mathcal{M}(\sigma^{-1})$. Let L be a σ-periodic double ray. As a consequence of Lemma 2.4, the rays in L either belong to $\mathcal{M}(\sigma)$ or $\mathcal{M}(\sigma^{-1})$. There is a bounded and connected set T such that the component D of $VX \setminus T$ which contains ω contains neither $\mathcal{M}(\sigma)$ nor $\mathcal{M}(\sigma^{-1})$. Since the subrays of L do not lie in D, there are only finitely many elements of L contained in D. Therefore we can choose T such that D and L are disjoint. We set $C = VX \setminus (D \cup T)$. Because σ is non-elliptic, T is bounded and L is σ-periodic, there is some number s such that

$$\sigma^s(T) \cap T = \emptyset$$

and such that σ^s acts as a translation on L. Since $\sigma^s(T)$ is connected and contains elements of L, which are elements of $VX \setminus D$, and since $\sigma^s(T)$ is disjoint from θD we have $\sigma^s(T) \subseteq VX \setminus D$ and $\sigma^s(T) \subseteq C$. The complement $VX \setminus C$ of C is connected and because

$$\sigma^s(\theta C) \subseteq \sigma^s(T) \subseteq C$$

we can apply Corollary 2.9. Hence (i) $\sigma^s(C) \subseteq C$ or (ii) $\sigma^s(VX \setminus C) \subseteq C$. The set $\sigma^{-s}(T)$ is contained in C and $\sigma^s \sigma^{-s}(T)$ is contained in $VX \setminus C$. This means that there are elements of C which are mapped to $VX \setminus C$ by σ^s. Thus case (i) can be excluded. Case (ii) is equivalent to $\sigma^s(D \cup T) \subseteq VX \setminus (D \cup T)$ which implies that the end ω, which lies in D, is not fixed by σ^s. Thus σ does not fix ω. □

3 Groups acting metrically almost transitively on graphs

Definition 3.1 A group G acts *metrically almost transitively* on a graph X if there is an integer r such that

$$\bigcup_{g \in G} g B_X(x, r) = VX,$$

for any vertex x. The smallest integer with that property is denoted by $\rho(G, X)$ and we call it the *covering radius* of the action of G on X. A graph is called metrically almost transitive if its group of automorphisms acts metrically almost transitively. The *covering radius of the graph X* is defined as $\rho(\mathrm{Aut}(X), X)$.

A metric end ξ of a graph X is called *free* if there is a metric cut C which contains no other metric end than ξ.

The following results all have direct counterparts for vertex ends, see [5].

Lemma 3.2 (Confer [5, Theorem 3].) *Let G act metrically almost transitively on a connected graph X with covering radius $\rho = \rho(G, X)$. If T is a bounded set of vertices and C is a metric cut which contains a vertex y such that $d_X(y, \theta C) > \operatorname{diam}_X T + \rho$ then there is an element g of G such that $g(T) \subseteq C$.*

Proof. Let x be a vertex in T. Then there is a g in G such that $g(x) \in B_X(y, \rho)$. This implies $g(T) \subseteq C$. □

Theorem 3.3 (Confer [5, Theorem 5].) *A connected metrically almost transitive graph with more than two ends has no free metric ends.*

Proof. Let X be a connected metrically almost transitive graph with more than two metric ends. Then there is a bounded set T such that $VX \setminus T$ contains at least three unbounded components D_1, D_2 and D_3 each of which contains a metric end. Let C be any metric cut which contains a metric end and let U be a bounded connected set which contains θC. Let C' be a component of $VX \setminus U$ which is contained in C and which contains a metric end. The complement $VX \setminus C'$ of C' is connected. By Lemma 3.2, there is an automorphism α such that $\alpha(T) \subseteq C'$. Now Lemma 2.8 applies and we have $\alpha(D_i) \subseteq C'$ or $VX \setminus \alpha(D_i) \subseteq C'$ for $i = 1, 2, 3$. The latter inclusion is equivalent to $VX \setminus C' \subseteq \alpha(D_i)$. Since the sets $\alpha(D_i)$ are disjoint, the inclusion $VX \setminus C' \subseteq \alpha(D_i)$ can hold for at most one i. Thus $\alpha(D_i) \subseteq C'$ holds for at least two indices i. Hence C' contains at least two metric ends. Consequently, C also contains at least two metric ends. This means that there are no free ends. □

Definition 3.4 A *metric bisection* of a connected graph X is a triple (C_2, T, C_1) where T, C_1, C_2 form a disjoint partition of the vertex set of X such that C_1 and C_2 both contain connected components with infinite diameter, T is bounded and θC_1 and θC_2 are contained in T. An automorphism σ is said to *shift* a metric bisection (C_2, T, C_1) if $\sigma(T \cup C_1) \subseteq C_1$.

Theorem 3.5 (Confer [15, Theorem 1] and [5, Theorem 4].) *Let G act metrically almost transitively on a graph X. Suppose T, C_1, C_2 is a disjoint partition of the vertex set such that $T \cup C_1$ and $T \cup C_2$ are connected and T is bounded. Furthermore, suppose that there are vertices $x_1 \in C_1$ and $x_2 \in C_2$ such that $d_X(x_1, \theta C_1) > \operatorname{diam}_X T + \rho(G, X)$ and $d_X(x_2, \theta C_2) > \operatorname{diam}_X T + \rho(G, X)$. Then (C_2, T, C_1) is metric bisection which is shifted by an element of G.*

Proof. By Lemma 3.2, we can find elements g_1 and g_2 in G such that $g_1(T) \subseteq C_1$ and $g_2(T) \subseteq C_2$. From Lemma 2.8 it follows that either $g_1(C_1) \subseteq C_1$ or $g_1(VX \setminus C_1) \subseteq C_1$ and either $g_2(C_2) \subseteq C_2$ or $g_2(VX \setminus C_2) \subseteq C_2$. If $g_1(C_1) \subseteq C_1$ then g_1 shifts (C_2, T, C_1), if $g_2(C_2) \subseteq C_2$ then g_2^{-1} shifts (C_2, T, C_1). If none of these inclusions holds then $g_1(VX \setminus C_1) \subseteq C_1$ and $g_2(VX \setminus C_2) = g_2(C_1 \cup T) \subseteq C_2$. This implies $g_2(C_1 \cup T) \subseteq C_2 \subseteq VX \setminus C_1 \subseteq g_1^{-1}(C_1)$ and $g_1 g_2(C_1 \cup T) \subseteq C_1$. Hence $g_1 g_2$ shifts (C_2, T, C_1). □

Corollary 3.6 *If a group acts metrically almost transitively on a graph then every metric bisection of the graph is shifted by some element of the group.*

Proof. Let (C_1, T, C_2) be a metric bisection of X. If necessary we can enlarge T to a bounded set T' such that T' is connected. Then Theorem 3.5 applies for (C_1, T', C_2). Hence (C_1, T', C_2) is shifted by an element of the group and this element also shifts (C_1, T, C_2). □

Corollary 3.7 *Let X be a connected metrically almost transitive graph with covering radius $\rho = \rho(\operatorname{Aut}(X), X)$. If there is a bounded and connected set T and components C_1 and C_2 of $VX \setminus T$ which contain vertices x_1 and x_2, respectively, such that*

$$\min\{d_X(x_1, T), d_X(x_2, T)\} > \operatorname{diam}_X T + \rho$$

then C_1 and C_2 are both unbounded.

Proof. By Theorem 3.5, $(VX \setminus (T \cup C_1), T, C_1)$ and $(VX \setminus (T \cup C_2), T, C_2)$ are both shifted by automorphisms. Theorem 2.12 implies that these automorphisms are metric translations and consequently C_1 and C_2 are both unbounded. □

Theorem 3.8 (Confer [1], [8], [16] and [20].) *Let X be a connected graph. Suppose there is a torsion group G acting metrically almost transitively on X. Then X has at most one metric end.*

P r o o f. If X had more than one metric end then it would have a metric bisection. By Corollary 3.6, there would be an element in G that shifts this metric bisection and this element would necessarily have infinite order. □

Corollary 3.9 *Any connected Cayley graph of a torsion group has at most one metric end.*

We write ΩC for the set of metric ends in a set of vertices C. Then define

$$\mathcal{B}(X) = \{C \cup \Omega C \mid C \text{ is a metric cut}\}.$$

The intersection of finitely many sets from $\mathcal{B}(X)$ is again an element of $\mathcal{B}(X)$, see Lemma 8 in [21]. Thus $\mathcal{B}(X)$ is basis of a topology on $VX \cup \Omega X$. Any open set is the union of elements in $\mathcal{B}(X)$. The *space of metric ends* is the restriction of this topology to ΩX.

Theorem 3.10 (Confer [5, Corollary 3].) *Let G act metrically almost transitively on a connected graph with more than one metric end. Then the metric directions of G are dense in the topological space of metric ends.*

P r o o f. Let C be a metric cut such that ΩC is a nonempty open set in the space of metric ends. Since X has more than one end, there is a metric bisection (C_1, T, C_2). Because C is an unbounded metric cut and G acts metrically almost transitive, there is an element g of G such that $g(T) \subseteq C$. By Lemma 2.8, either $g(C_1) \subseteq C$ or $g(C_2) \subseteq C$. By applying Corollary 3.6 to the metric bisection $(g(C_1), g(T), g(C_2))$, we see that both sets $g(C_1)$ and $g(C_2)$ contain a metric direction of an automorphism. Thus ΩC contains a metric direction. □

The concept of a *star ball* is introduced in [22].

Definition 3.11 A ball B is a called a *star ball* if there is no upper bound on the diameters of those components of $VX \setminus B$ which have finite diameter.

Lemma 3.12 ([22, Lemma 20]) *There are no star balls in connected metrically almost transitive graphs.*

P r o o f. Suppose there is a star ball $B_X(z, n)$. Let ρ be the covering radius of X according to Definition 3.1. In the complement of $B_X(z, n)$, there is a bounded component C containing a vertex y whose distance to $B_X(z, n)$ is greater than $2n + \rho$. Let $g(z)$ denote a vertex in the orbit $\mathrm{Aut}(X)z$ such that $d_X(y, g(z)) \leq \rho$. Then $B_X(g(z), n)$ is a star ball which is contained in C. Since the complement $VX \setminus C$ is connected, it is completely contained in one of the components of $VX \setminus B_X(g(z), n)$. All other components of $VX \setminus B_X(g(z), n)$ are contained in the bounded set C which is impossible. Thus, $B_X(g(z), n)$ as well as $B_X(z, n)$ cannot be star balls. □

The following lemma is a slight modification of Lemma 18 in [22].

Lemma 3.13 *If a graph contains no star ball then every unbounded metric cut contains a metrically transient ray.*

P r o o f. Let C be an unbounded metric cut and let $B_X(o, n)$ be a ball which contains θC. There is a component C_n of $C \setminus B_X(o, n)$ which is unbounded. Otherwise, $B_X(o, n)$ would be a star ball. The set $C_n \setminus B_X(o, n+1)$ must contain an unbounded component C_{n+1}. Otherwise, $B_X(o, n + 1)$ would be a star ball. By induction, we obtain a sequence

$$C_n \supseteq C_{n+1} \supseteq C_{n+2} \supseteq \dots$$

where C_i is an unbounded component of $VX \setminus B_X(o, i)$, $i \geq n$.

Let x_n be a vertex in θC_n and let π_n be a path from x_n to a vertex x_{n+1} in θC_{n+1} such that π_n is contained in $C_n \setminus C_{n+1}$ except for the initial vertex x_n. Such a path π_n exists because C_n is connected. By induction we obtain a sequence of paths π_i, $i \geq n$, whose concatenation is a metrically transient ray in C. □

Lemmas 3.12 and 3.13 imply the following corollary.

Corollary 3.14 *Any unbounded metric cut in a metrically almost transitive graph contains a metrically transient ray.*

Corollary 3.15 (Confer [12, Corollary 15].) *Let X be a metrically almost transitive graph with infinite diameter. Then X has either 1, 2 or infinitely many metric ends.*

P r o o f. By Corollary 3.14, the graph X has at least one metric end. Let X have more then two metric ends. Then there exists a metric bisection. By Corollary 3.6, this bisection is shifted by an automorphism σ. This automorphism has two distinct directions $\mathcal{M}(\sigma)$ and $\mathcal{M}(\sigma^{-1})$. Theorem 2.13 says that these two ends are the only ends which are fixed by σ. The same holds for any power of σ. This means that there is no integer $n \neq 0$ and no end $\omega \in \Omega X \setminus \{\mathcal{M}(\sigma), \mathcal{M}(\sigma^{-1})\}$ such that $\sigma^n(\omega) = \omega$. Thus any orbit of σ on $\Omega X \setminus \{\mathcal{M}(\sigma), \mathcal{M}(\sigma^{-1})\}$ is infinite and X has infinitely many ends. □

Example 3.16 Let S be any infinite symmetric set of integers (i.e., $S = \{-s \mid s \in S\}$) such that $\gcd(S) = 1$ and $0 \notin S$. Let X be the Cayley graph of $\mathbb{Z}$ with respect to the generating set S. Hence $VX = \mathbb{Z}$ and vertices x and y are adjacent whenever $|x - y| \in S$.

We will show that if the graph X is unbounded then it has exactly one metric end. In other words: any complement of a ball has exactly one unbounded component.

In Figure 2 we set $S = \{\pm 2^n \mid n \in \mathbb{Z}, n \geq 0\}$. Two vertices x and y are adjacent whenever $|x - y|$ is a power of 2. The ball $B_X(0, n)$ is the set of integers which can be written as a sum of at most n elements of S.

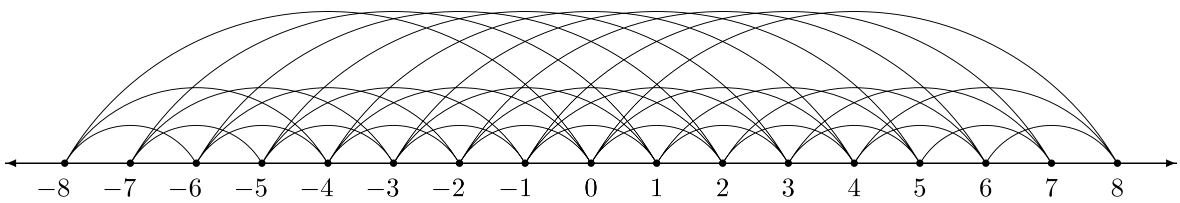

Fig. 2

P r o o f. Since $\gcd(S) = 1$, the set S generates $\mathbb{Z}$ and X is connected. Let σ be an element of S. Then the cyclic group $G = \langle\sigma\rangle$ acts transitively on X by addition and $L = (\ldots, -\sigma, 0, \sigma, 2\sigma, \ldots)$ is a σ-invariant double ray. Let σ^k be any element of $G \setminus \{1\}$, $k \neq 0$. Note that the left multiplication by the k-th power of σ corresponds to an addition of $k \cdot \sigma$ in $\mathbb{Z}$. Any σ^k-periodic double ray is also σ-periodic. For if M is an σ^k-periodic double ray with period $m \neq 0$ then $(\sigma^k)^m(M) = \sigma^{km}(M) = M$ and M is σ-periodic with period km. Because S is infinite, the double ray L contains an infinite bounded subset, and because X is unbounded, L is also unbounded. Hence L is not metrically transient but it is unbounded. By Corollary 2.6, all σ-invariant double rays are unbounded and not metrically transient. Hence, any double ray which is periodic for some element of $G \setminus \{1\}$ is unbounded and not metrically transient. By Definition 2.3, no element of G has a metric direction.

If X had more than one end, then Theorem 3.10 would apply and the directions of elements of G would be dense in ΩX. But since an empty set cannot be dense in a nonempty set, this is impossible. Hence X does not have more than one end. By Corollary 3.14, any unbounded metrically almost transitive graph contains at least one metric end. Thus X has exactly one metric end. □

4 Bounded automorphisms

Definition 4.1 An automorphism σ of a connected graph X is *bounded* if there is a constant c such that $d_X(v, \sigma(v)) \leq c$ for all vertices v in X.

Theorem 4.2 (Confer [5], [18], [24], [25].) *Suppose X is a connected metrically almost transitive graph with more than one metric end. Then the kernel of the action of $\mathrm{Aut}(X)$ on the metric ends is the set of bounded automorphisms. That is, an automorphism fixes all the metric ends of X if and only if it is bounded.*

The part about graphs with more than two ends in the following proof is similar to the proof of Proposition 1 in [25].

Proof. Let σ be a bounded automorphism of X such that $d_X(y, \sigma(y)) \leq c$ for some integer c and every vertex y in X. Assume that R is some metrically transient ray in X and that $\sigma(R)$ is not in the same metric end as R. Let T be a bounded subset of X separating the metric ends that R and $\sigma(R)$ belong to. We may assume that neither R nor $\sigma(R)$ intersects T. Since R is metrically transient, there is a vertex y in R such that $d_X(y, T) > c$. Hence $d_X(y, \sigma(y)) > c$, a contradiction. Hence the image of a metrically transient ray under a bounded automorphism is in the same metric end as the ray itself and σ is in the kernel of the action of $\mathrm{Aut}(X)$ on the metric ends.

Let σ be an automorphism which fixes all metric ends. First we suppose that X has at least three ends. There is a bounded connected set of vertices T which separates three metric ends ω_1, ω_2 and ω_3 from each other, i.e., the ends ω_1, ω_2 and ω_3 lie in different components of $VX \setminus T$. We may also assume that T contains some ball of radius ρ where ρ is the covering radius of X according to Definition 3.1. Suppose $T \cap \sigma(T) = \emptyset$. Then T is contained in some component C of $VX \setminus \sigma(T)$. At least two of the metric ends ω_1, ω_2 and ω_3 lie in C. But C contains at most one of the ends $\sigma(\omega_1)$, $\sigma(\omega_2)$ and $\sigma(\omega_3)$, which means that σ would not leave all metric ends invariant. Hence we see that if σ fixes all metric ends of X then $T \cap \sigma(T) \neq \emptyset$. Thus $d_X(v, \sigma(v)) \leq 2 \operatorname{diam}_X T$ for any vertex v in T. This also holds for any vertex in any image of T under the action of $\mathrm{Aut}(X)$. Since these translates cover the whole graph we have $d_X(v, \sigma(v)) \leq 2 \operatorname{diam}_X T$ for all vertices v. Hence σ is a bounded automorphism.

Finally, let X be a metrically almost transitive graph with exactly two metric ends ω_1 and ω_2. Let σ be an automorphism which fixes both ends. Let y be any vertex and let r' be an integer such that $B_X(y, r')$ separates the two ends. Then for any vertex x there is an automorphism g such that $g(y) \in B_X(x, \rho)$. Because $B_X(g(y), r') = g(B_X(y, r'))$ separates ω_1 from ω_2 and $B_X(g(y), r') \subset B_X(x, r' + \rho)$, we see that any ball with radius $r = r' + \rho$ separates ω_1 from ω_2. If $d_X(x, \sigma(x)) \leq 2r$ for any vertex x, then σ is bounded. Otherwise, let z be a vertex such that $d_X(z, \sigma(z)) > 2r$. Let C_i be the component of $VX \setminus B_X(z, r)$ which contains ω_i, for $i = 1, 2$. Since $\sigma(B_X(z, r)) \cap B_X(z, r) = \emptyset$, Theorem 2.12 implies that σ is a metric translation. The direction of σ is either ω_1 or ω_2. Hence $\sigma(B_X(z, r) \cup C_1) \subset C_1$ or $\sigma(B_X(z, r) \cup C_2) \subset C_2$. Without loss of generality, we may assume that $\sigma(B_X(z, r) \cup C_1) \subset C_1$. The set $T = C_1 \setminus \sigma(C_1)$ is a metric cut. Neither of the two ends lie in T, because ω_1 lies in $\sigma(C_1)$ and ω_2 does not lie in C_1. Hence T contains no metric end. By Corollary 3.14, T is bounded.

By induction, we see that $d_X(z, \sigma^n(C_1)) \geq n$. Hence $(\sigma^n(C_1))_{n \in \mathbb{N}}$ is a descending sequence such that

$$\bigcap_{n \geq 1} \sigma^n(C_1) = \emptyset.$$

Similarly, we get

$$d_X(z, \sigma^{-n}(C_2)) \geq n, \qquad \bigcap_{n \geq 1} \sigma^{-n}(C_2) = \emptyset \quad \text{and} \quad \bigcup_{n \geq 1} \sigma^{-n}(C_1) = VX.$$

Thus

$$\begin{aligned}
\bigcup_{n \in \mathbb{Z}} \sigma^n(T) &= \bigcup_{n \geq 1} (\sigma^{-n}(T) \cup \ldots \cup \sigma^n(T)) \\
&= \bigcup_{n \geq 1} \big((\sigma^{-n}(C_1) \setminus \sigma^{-n+1}(C_1)) \cup \ldots \cup (\sigma^n(C_1) \setminus \sigma^{n+1}(C_1)) \big) \\
&= \bigcup_{n \geq 1} \big(\sigma^{-n}(C_1) \setminus \sigma^{n+1}(C_1) \big) \\
&= VX.
\end{aligned}$$

Hence $\{\sigma^n(T) \mid n \in \mathbb{Z}\}$ is a partition of VX. Let x be any vertex in any of the sets $\sigma^n(T)$. Then $\sigma(x)$ is a vertex in $\sigma^{n+1}(T)$, $d_X(x, \sigma(x)) \leq 2 \operatorname{diam}_X T$ and σ is bounded. □

If X is a metrically almost transitive graph then it need not be true that the kernel of the action of $\mathrm{Aut}(X)$ on the vertex ends or the edge ends is the group of bounded automorphisms.

Example 4.3 Let $L = (\ldots, x_{-1}, x_0, x_1, \ldots)$ be a double ray. For each vertex x_i let Y_i be a complete infinite graph such that Y_i and Y_j have the same cardinality for any i and j in $\mathbb{Z}$, which means that Y_i and Y_j are isomorphic. Each vertex x_i is now connected with one vertex of Y_i with an additional edge. The resulting graph X has only two metric ends. They correspond to the two ends of L. But each graph Y_i carries an edge end and a vertex end. A bounded automorphism must fix the two metric ends of X but it can shift the double ray L such that Y_i is mapped to Y_{i+k} for some fixed k. Hence the bounded automorphisms are not all in the kernel of the action $\mathrm{Aut}(X)$ on the vertex ends or edge ends. In Figure 3, the edges of L are shown with dashed lines.

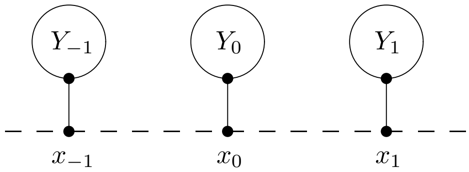

Fig. 3

Acknowledgements Bernhard Krön was supported by the projects P16004-N05-MAT and J2245 of the Austrian Science Fund (FWF) and by a Marie Curie Fellowship (IEF) of the European Union at the University of Hamburg.

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