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# End compactifications in non-locally-finite graphs

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*Abstract*

There are different definitions of ends in non-locally-finite graphs which are all equivalent in the locally finite case. We prove the compactness of the end-topology that is based on the principle of removing finite sets of vertices and give a proof of the compactness of the end-topology that is constructed by the principle of removing finite sets of edges. For the latter case there exists already a proof in [**1**], which only works on graphs with countably infinite vertex sets and in contrast to which we do not use the Theorem of Tychonoff. We also construct a new topology of ends that arises from the principle of removing sets of vertices with finite diameter and give applications that underline the advantages of this new definition.

## 1. *Introduction*

Ends of graphs can be seen as the directions along which sequences of vertices can tend to infinity. Freudenthal [**5**] was the first who considered ends on a class of topological spaces, which we nowadays call locally finite graphs. Usually ends are defined as equivalence classes of rays. Although one can find various literature on ends of locally finite graphs, for an introduction to this topic (see [**11**] or [**13**]), there does not even exist a standard definition of ends in the non-locally-finite case. Halin [**6**] calls two rays equivalent, when both have infinitely many vertices in common with a third ray. We call the arising equivalence classes *vertex-ends*, on which the topological considerations of Polat in [**14–18**] are based. Cartwright, Soardi, Woess [**1**], Dicks and Dunwoody [**2, 4**] and Stallings [**19**] prefer ends that are constructed by the principle of removing finite sets of edges. We call them *edge-ends*. For the latter approach to the subject Dunwoody has proved the existence of so-called structure trees for graphs with more than one end, which can be powerful tools in describing structures of infinite graphs with a strong action of its automorphism group [**4**]. An improved proof can be found in [**2**]. At the other hand each vertex-end is contained in an edge-end, which means that vertex-ends can describe structures that cannot be seen by using edge-ends. Cf. Example 3, Example 5 and Lemma 7. But none of these two approaches could yield convincing arguments for taking one of them as the standard. Our new construction of so-called *metric ends*, that is only based on the principle of distinguishing between sets of finite and sets of infinite diameters, enables us to describe structures that cannot even be seen by using vertex-ends (cf.

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Example 4, Example 5 and Section 8), but we do not claim to have reached the end of this discussion.

We want to see ends as points in the boundaries of the compactification of topologies on the set of vertices of a graph, first of all to obtain a concept for the convergence of sequences of vertices.

Throughout this article we will only use rather simple theorems of General Topology. For the most part we will use pure graphtheoretic arguments. We could see end compactifications in a more general topological context, but in our experience this is in most of the cases not too useful.

The *edge-compactification* with corresponding topology $\mathcal{T}_{\mathrm{E}}$, which is the compactification that arises from the definition of edge-ends, is the only one from which we can construct a Hausdorff topology $\mathcal{T}_{\mathrm{E}_2}$ in a natural way, cf. [**1**], further explanations thereto can be found at the end of Section 4. Although it is a rather weak topology, it is the concept that is used by the majority of the authors.

Into this topology we can embed the topology $\mathcal{T}_{\mathrm{V}}$ of the *vertex-compactification*, the counterpart of the vertex-ends.

The *proper metric topology* $\mathcal{T}_{\mathrm{P}}$, arising from the definition of metric ends, is the strongest of the three topologies and has nearly all 'good' topological properties but when we want it to be compact we have to add two artificial extra points (improper ends) and the resulting *metric topology* $\mathcal{T}_{\mathrm{M}}$ is not even $T_0$ although every sequence has a convergent subsequence with a unique limit. The reason for that is that there is no natural concept of local convergence in compactifications of discrete metric spaces which allows all open balls with positive real radius to be open. We can embed the other end compactifications into the proper metric topology but not into the metric compactification with its two extra points.

There do not arise any problems from these two extra points, when we want to use this topology for example to describe convergence of a transient random walk to a boundary (cf. Section 8), because the probability of the event that they occur as a limit is usually zero. The proper metric topology has furthermore some pretty topological properties concerning the study of quasi-isometries (cf. Theorem 6 and Theorem 7).

Random walks on graphs can tend to infinity along directions that cannot be described in a pure graph theoretical way without using the properties of the transition densities. It would be a subject of further researches to find homeomorphisms between graphtheoretical end compactifications and probability theoretical compactifications like the Martin compactification under corresponding preliminary restrictions to the random walk. In [**1**] the Martin boundary is embedded into the space of edge-ends.

Fig. 1 shows a synopsis of some topological properties of the end topologies that will be discussed in this article.

## 2. *A simple property of non-locally-finite graphs*

To give a slight impression of the properties of non-locally-finite graphs we first of all want to state a simple lemma which characterises graphs with infinite diameter. Throughout this article let $X = (VX, EX)$ be a connected graph without loops and

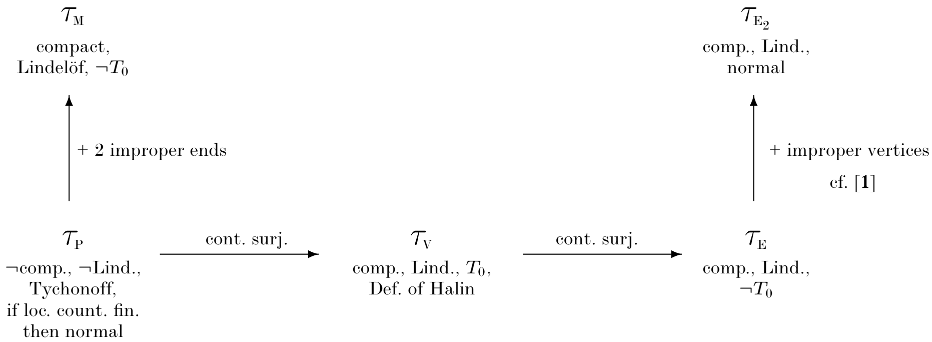


Fig. 1.

multiple edges. A set of vertices $e$ is called connected, if any two vertices in $e$ can be connected by a path in $X$ that does not leave $e$. When we consider topological connectedness we will mention it explicitly if it is not clear from the context. We write $e^*$ for the complement $VX \backslash e$ of $e$ and $\operatorname{diam}_X$ for the diameter with respect to the natural graph metric $\mathrm{d}_X$ of $X$. The set of all connected components of $e^*$ is denoted by $\mathscr{C}(e)$. For the set of those connected components in $e^*$ that have a finite diameter we write $\mathscr{C}_0(e)$. Let $B(x,r)$ be the ball $\{Y \in VX \mid d_X(X,Y) \leqslant r\}$. A *star ball* is a ball $B$ for which

$$\sup\{\operatorname{diam} C \mid C \in \mathscr{C}_0(B)\} = \infty.$$

A *ray* is a sequence $(x_n)_{n\in\mathbb{N}}$ of pairwise disjoint vertices such that $x_n \sim x_{n+1}$ for all $n$.

Lemma 1.
(i) *A locally finite graph has infinite diameter if and only if it contains a ray.*
(ii) *The diameter of a non-locally-finite graph is infinite if and only if it contains a ray with infinite diameter or a star ball.*

*Example* 1. Let $\{P_n \mid n \in \mathbb{N}, \operatorname{diam} P_n = n\}$ be a set of disjoint paths. By joining together the initial vertices of these paths we obtain a graph $X$ with a vertex $x$ of infinite degree. The diameter of $X$ is infinite. There does not exist a ray with infinite diameter but for any natural radius $r$ the ball $B(x,r)$ is a star ball. See Fig. 2.

*Proof of Lemma* 1. The first statement is well known. To prove the second part of the lemma we have to show that there exists a ray with infinite diameter in any graph $X$ that does not contain a star ball.

If there is no star ball in $X$ then the complement of any vertex $x$ in $X$ must contain a connected component $e_0$ with infinite diameter. At least one of the components in the ball $B(x,1)^*$ which are contained in $e_0$ must have infinite diameter. Otherwise $B(x,1)$ would be a star ball. We call this component $e_1$. By induction we obtain a strictly decreasing sequence $(e_r)_{r\in\mathbb{N}}$ of components of $B(x,r)^*$ with infinite diameter. In the proof of Lemma 3 we will see that there must exist a ray that has infinitely many vertices in common with every set $e_r$. Since the ray is not contained in any ball it must have infinite diameter.

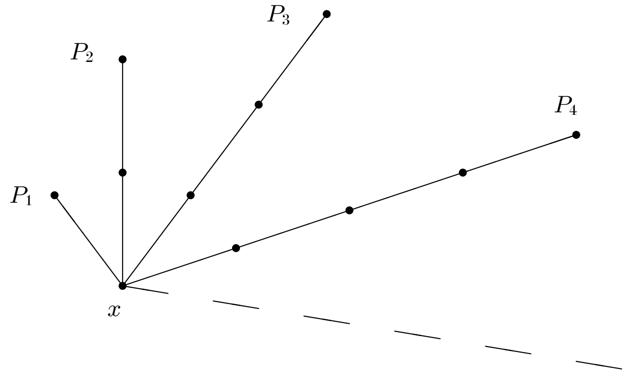


Fig. 2.

## 3. *The construction of vertex- and edge-topology*

We define the *vertex-boundary* $\theta e$ as the set of vertices in $e^*$ which are adjacent to a vertex in $e$ and $I\theta e := \theta e^*$ is called the *inner vertex-boundary* of $e$. The *edge-boundary* $\delta e$ is defined as the set of edges connecting vertices in $e$ with vertices in $e^*$. We call a nonempty set of vertices $e$ *vertex-* or *edge-cut* if $\theta e$ or $\delta e$ are finite, respectively.

Note that every *edge-cut* is also a *vertex-cut*. The reversal of this statement is not true in the general case, but in locally finite graphs *vertex-cut* and *edge-cut* are equivalent terms. It will turn out that this is the reason why the two topologies we are interested in are identical in the locally finite case.

A ray *lies* in a set of vertices $e$ or is *contained* in $e$, if $e$ contains all but finitely many elements of the ray. Sometimes we will use the terms *contain* and *lie* at the same time in the above sense as well as in the sense of set theoretic inclusion. A set of vertices $e$ *separates* two rays, if one of them lies in $e$ and the other lies in $e^*$.

Two rays are called *vertex-equivalent* or *edge-equivalent* if they cannot be separated by *vertex-* or *edge-cuts*, respectively. It is easy to see that these relations are equivalence relations. Their equivalence classes are called *vertex-* and *edge-ends* of $X$, respectively. As every edge-cut is a vertex-cut, all vertex-ends are subsets of edge-ends. An end *lies* in a set of vertices $e$ or is *contained* in $e$, if all of its rays lie in $e$. The set of vertex-ends that lie in $e$ is denoted by $\mathbf{\Omega}_{\mathrm{V}}e$, the set of edge-ends in $e$ by $\mathbf{\Omega}_{\mathrm{E}}e$. We write $\mathbf{\Omega}_{\mathrm{V/E}}X$ instead of $\mathbf{\Omega}_{\mathrm{V/E}}VX$. Indeed, a vertex-end or edge-end $\omega$ lies in a vertex-cut or edge-cut $e$ if and only if one of its rays lies in $e$, respectively. $\omega$ lies either in $e$ or in $e^*$. We say that a finite set of vertices $e$ *separates* two ends, if they lie in different connected components of $e^*$.

*Example* 2. The two sided infinite 'ladder graph' has two ends. In locally finite graphs we do not have to distinguish between different types of ends.

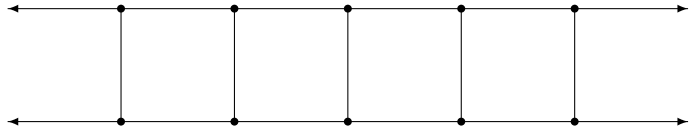

Fig. 3.

LEMMA 2. *For a graph $X$ the set $\mathscr{B}_{\mathrm{V}}X = \{e \cup \Omega_{\mathrm{V}}e \mid e \subset VX$ and $|\theta e| < \infty\}$ is closed under finite intersection.*

*Proof.* For two sets $e_1 \cup \Omega_{\mathrm{V}}e_1$ and $e_2 \cup \Omega_{\mathrm{V}}e_2$ in $\mathscr{B}_{\mathrm{V}}X$ we have

$$(e_1 \cup \Omega_{\mathrm{V}}e_1) \cap (e_2 \cup \Omega_{\mathrm{V}}e_2) = (e_1 \cap e_2) \cup (\Omega_{\mathrm{V}}e_1 \cap \Omega_{\mathrm{V}}e_2),$$

$e_1 \cap e_2$ is a vertex-cut and $\Omega_{\mathrm{V}}e_1 \cap \Omega_{\mathrm{V}}e_2$ the set of ends whose rays are completely contained in $e_1$ and $e_2$ from some index onwards. Thus $\Omega_{\mathrm{V}}e_1 \cap \Omega_{\mathrm{V}}e_2 = \Omega_{\mathrm{V}}(e_1 \cap e_2)$, and finally

$$(e_1 \cup \Omega_{\mathrm{V}}e_1) \cap (e_2 \cup \Omega_{\mathrm{V}}e_2) = (e_1 \cap e_2) \cup \Omega_{\mathrm{V}}(e_1 \cap e_2).$$

We define the set $\mathscr{B}_{\mathrm{E}}X = \{e \cup \Omega_{\mathrm{E}}e \mid e \subset VX$ and $|\delta e| < \infty\}$, for which the proof of Lemma 2 can be copied word by word. Hence $\mathscr{B}_{\mathrm{V}}X$ and $\mathscr{B}_{\mathrm{E}}X$ are bases of topological spaces $(VX \cup \Omega_{\mathrm{V}}X, \mathcal{T}_{\mathrm{V}}X)$ and $(VX \cup \Omega_{\mathrm{E}}X, \mathcal{T}_{\mathrm{E}}X)$, whose topologies $\mathcal{T}_{\mathrm{V}}X$ and $\mathcal{T}_{\mathrm{E}}X$ are called *vertex-topology* and *edge-topology*, respectively.

## 4. *Compactness of the vertex- and edge-topologies*

Let $B(z,n)$ denote the ball $\{x \in VX \mid \mathrm{d}_X(x,z) \leqslant n\}$ where $\mathrm{d}_X$ is the natural geodesic metric of the graph $X$.

LEMMA 3. *If for every strictly decreasing sequence $(e_n)_{n\in\mathbb{N}}$ of connected edge- or vertex-cuts, there is a vertex $z$ such that $B(z,n)$ is a subset of $e_n^*$, then there exists a ray $L$, which has infinitely many vertices in common with all cuts $e_n$. In other words: the end of $L$ lies in all cuts $e_n$.*

*Proof.* As $e_1$ is connected and a superior set of $e_2$ any vertex $x_1$ in $I\theta e_1$ can be connected by a path $\pi_1$ of vertices in $e_1 \backslash e_2$ with a vertex $x_2$ in $I\theta e_2$. By induction we get a path $\pi_n$ such that the initial vertex of $\pi_n$ is adjacent to the last vertex of $\pi_{n-1}$ and $\pi_n$ is contained in $e_n \backslash e_{n+1}$ for every natural $n$ greater than one. The union $L$ of these paths must have finite intersection with every ball with centre $z$ (equivalently, every bounded subset of $VX$). Thus $L$ is infinite and therefore it constitutes a ray. The corresponding end lies in all cuts $e_n$.

LEMMA 4. *Let $e$ be a vertex-cut and $\xi$ an infinite sequence of pairwise different elements of $e \cup \Omega_{\mathrm{V}}e$. If there is no connected component of $e$ containing infinitely many elements of the sequence then $\xi$ has an accumulation point in $\theta e$.*

*Proof.* Under the given preliminaries there exists a vertex $x$ in $\theta e$ with infinite degree, which is adjacent to infinitely many connected components of $e$ containing elements of $\xi$. Every element $f \cup \Omega_{\mathrm{V}} f$ of the base $\mathscr{B}_{\mathrm{V}}X$ which contains $x$ must contain all but finitely many of these components, since the vertex-boundary of $e$ is finite. The same holds for every neighbourhood of $x$. Thus $x$ is an accumulation point of $\xi$.

LEMMA 5. $(VX \cup \Omega_{\mathrm{V}}X, \mathcal{T}_{\mathrm{V}}X)$ *is sequentially compact.*

*Proof.* Let $\xi$ be a sequence of pairwise different elements of $VX \cup \Omega_{\mathrm{V}}X$. If $VX$ is the only vertex-cut, then every element of $VX \cup \Omega_{\mathrm{E}}X$ is an accumulation point of the sequence. Otherwise let $z$ be a vertex in the complement of a vertex-cut $e$, in which lie infinitely many elements of $\xi$. If the sequence has no accumulation point in $\theta e$, then by Lemma 4 there exists a connected component $e_0$ of $e$, such that infinitely many elements of $\xi$ lie in $e_0$.

In the case that $\xi$ has no accumulation point in $\theta e_0$, we construct a connected vertex-cut $e_1$, that is a subset of $e_0 \backslash I\theta e_0$ which contains infinitely many elements of $\xi$:

For a vertex $y$ in $\theta e_0$ of infinite degree, which is no accumulation point of $\xi$, there exists a vertex-cut $f(y)$, that contains $y$ and for which $f(y) \cup \Omega_{\mathrm{V}} f(y)$ contains only finitely many elements of $\xi$. All but finitely many vertices in $I\theta e_0$, that are neighbours of $y$, lie in $f(y)$. Thus $e_0 \backslash f(y)$ is a cut, such that infinitely many elements of the sequence $\xi$ lie in it. This and the finiteness of $\theta e_0$ imply that

$$e_1^{\mathrm{A}} := e_0 \backslash \bigcup \{ f(y) \mid y \in \theta e_0 \}$$

is a vertex-cut, which is a subset of $e_0$ and contains all but finitely many elements of $I\theta e_0$. Thus

$$e_1^{\mathrm{B}} := e_1^{\mathrm{A}} \# \backslash I\theta e_0$$

is also a vertex-cut. It is a subset of $e_0 \backslash I\theta e_0$. If $\xi$ has no accumulation point in $\theta e_1^{\mathrm{B}}$, once more by Lemma 4, there exists a connected component $e_1$ of $e_1^{\mathrm{B}}$ that has again the properties requested before.

By induction we obtain a strictly decreasing sequence $(e_n)_{n\in\mathbb{N}}$ of connected vertex-cuts, in all of which lie infinitely many elements of the sequence $\xi$. Since $e_{n+1}$ is a subset of $e_n \backslash I\theta e_n$ and the vertex $z$ lies in the complement of $e_0$ we have

$$B(z, n) \cap e_n = \varnothing$$

and thus

$$\bigcap_{n\in\mathbb{N}} e_n = \varnothing.$$

Following Lemma 3 we obtain a ray $L$, whose end $\omega$ lies in every cut $e_n$. Every neighbourhood $U$ of $\omega$ contains a base element $f \cup \Omega_{\mathrm{V}} f$ with $\omega \in \Omega_{\mathrm{V}} f$, for which $\theta f$ is a subset of one of the balls $B(z, n)$. This implies, that all but finitely many cuts $e_n$ and thus infinitely many elements of the sequence $\xi$ lie in any neighbourhood of $\omega$.

LEMMA 6. $(VX \cup \Omega_{\mathrm{V}} X, \mathcal{T}_{\mathrm{V}} X)$ *is a Lindelöf space.*

*Proof.* We may assume that an open cover $\mathscr{U}$ consists of base elements. Given a set $e \cup \Omega_{\mathrm{V}} e$ of the cover, we choose a vertex $z$ in $e$. Since $\theta e$ is finite there exists a finite subcover $\mathscr{U}_1$ of $e \cup \theta e$ in $\mathscr{U}$. The set $e_1$ of vertices in $\bigcup \mathscr{U}_1$ is a vertex-cut. Every ray in $e_1$ must lie in one of those vertex-cuts $f$ for which $f \cup \Omega_{\mathrm{V}} f$ is an element of the cover $\mathscr{U}_1$. The same holds for any end containing this ray and therefore

$$e_1 \cup \Omega_{\mathrm{V}} e_1 = \bigcup \mathscr{U}_1.$$

By induction we obtain a sequence of vertex-cuts $(e_n)_{n\in\mathbb{N}}$ and corresponding finite subcovers $\mathscr{U}_n$ of $\mathscr{U}$, such that $B(z, n)$ is a subset of $e_n$. In other words:

$$e_n \cup \Omega_{\mathrm{V}} e_n = \bigcup \mathscr{U}_n \quad \text{and} \quad \bigcup \{ e_n \mid n \in \mathbb{N} \} = VX.$$

For an end $\omega$ and a vertex $z$ we define $\mathrm{d}_{\mathscr{U}}(\omega, z)$, the *distance of $\omega$ to $z$ with respect to the cover $\mathscr{U}$*, as the minimal radius $r$ for which there exists a base element $f \cup \Omega_{\mathrm{V}} f$ in $\mathscr{U}$ such that $\theta f$ is a subset of $B(z, r)$ and $\omega$ is an element of $\Omega_{\mathrm{V}} f$. Note that this radius exists for all vertex-ends. Let $f \cup \Omega_{\mathrm{V}} f$ be a base element in $\mathscr{U}$. If the vertex boundary of $f$ is contained in $e_n$ then for any $x$ in $\theta e_n \cap f$ all connected components of $(e_n \cup \theta e_n)^*$ which are adjacent to $x$ are completely contained in $f$. Let $\mathscr{W}_n$ denote the set of elements of $\mathscr{U}$ whose vertex-boundaries are subsets of $e_n$. We now choose a finite subcover $\mathscr{V}_n$ of $\mathscr{W}_n$ which covers $\theta e_n \cap \bigcup \mathscr{W}_n$. Now $\mathscr{V}_n$ covers all ends $\omega$ with $\mathrm{d}_{\mathscr{U}}(\omega, z) \leqslant n$ and the union

$$\bigcup_{n \in \mathbb{N}} \mathscr{U}_n \cup \mathscr{V}_n$$

is a countable covering of both $VX$ and $\Omega_{\mathrm{V}}X$ which meets the statement of the lemma.

THEOREM 1. *$(VX \cup \Omega_{\mathrm{V}}X, \tau_{\mathrm{V}}X)$ is a compact $T_0$-space.*

*Proof.* A sequentially compact Lindelöf space is compact. Thus we can conclude from Lemma 5 and Lemma 6, that $(VX \cup \Omega_{\mathrm{V}}X, \tau_{\mathrm{V}}X)$ is compact.

It remains to show that at least one of two elements $x$ and $y$ of $VX \cup \Omega_{\mathrm{V}}X$ has an open environment, in which the other element is not contained. We distinguish:

(i) One of the two elements is a vertex. If $x$ is a vertex, then $(VX \setminus \{x\} \cup \Omega_{\mathrm{V}}X)$ is the requested neighbourhood of $y$.

(ii) $x$ and $y$ are ends. In the complement of a finite set of vertices that separates $x$ and $y$ there exist two disjoint base elements, of which one contains the end $x$ and the other $y$.

LEMMA 7. *There exists a continuous surjection $f$ from $(VX \cup \Omega_{\mathrm{V}}X, \tau_{\mathrm{V}}X)$ onto $(VX \cup \Omega_{\mathrm{E}}X, \tau_{\mathrm{E}}X)$, whose restriction on $VX$ is the identity.*

*Proof.* We define the $f$-image of an end in $\Omega_{\mathrm{V}}X$ as the end in $\Omega_{\mathrm{E}}X$ that contains it. The preimage of every base element of the edge-topology is open in the vertex-topology.

THEOREM 2. *$(VX \cup \Omega_{\mathrm{E}}X, \tau_{\mathrm{E}}X)$ is compact.*

*Proof.* The statement is a consequence of Theorem 1 and Lemma 7.

*Example* 3. We connect two adjacent vertices $x_1$ and $x_2$ with each vertex of two disjoint rays $L_1$ and $L_2$, respectively. The resulting graph $X_1$ has two vertex- and two edge-ends, see Fig. 4($a$). When we take away the edge $\{x_1, x_2\}$ and identify the vertices $x_1$ and $x_2$ we obtain a graph $X_2$ in which $L_1$ and $L_2$ are edge-equivalent but not vertex-equivalent. In other words, there exist two vertex-ends but only one edge-end, see Fig. 4($b$). $L_1$ and $L_2$, and therefore their corresponding vertex-ends, lie in every neighbourhood of $x$ in the vertex-topology. Thus $(VX_2 \cup \Omega_{\mathrm{V}}X_2, \tau_{\mathrm{V}}X_2)$ is not a $T_1$-space. In the edge-topology the edge-end of $X_2$ and the vertex $x$ cannot even be separated in the sense of the $T_0$-axiom. In the graph $X_1$ the same holds for the edge-end of $L_1$ and $L_2$ and the vertices $x_1$ and $x_2$, respectively.

As shown in Example 3 there may exist vertices of infinite degree, whose neighbourhoods in the edge-topology all contain a fixed edge-end. To avoid such complications and to obtain better properties of separation the edge equivalence can be extended to the set of rays and vertices with infinite degree. In other words, vertices with infinite degree are considered as degenerated rays. They also can define new ends, so-called *improper vertices*. We then obtain *edge topologies $\tau_{\mathrm{E}_2}$ of the second type* which are compact, totally disconnected and normal, either by defining $\{x\}$ as open

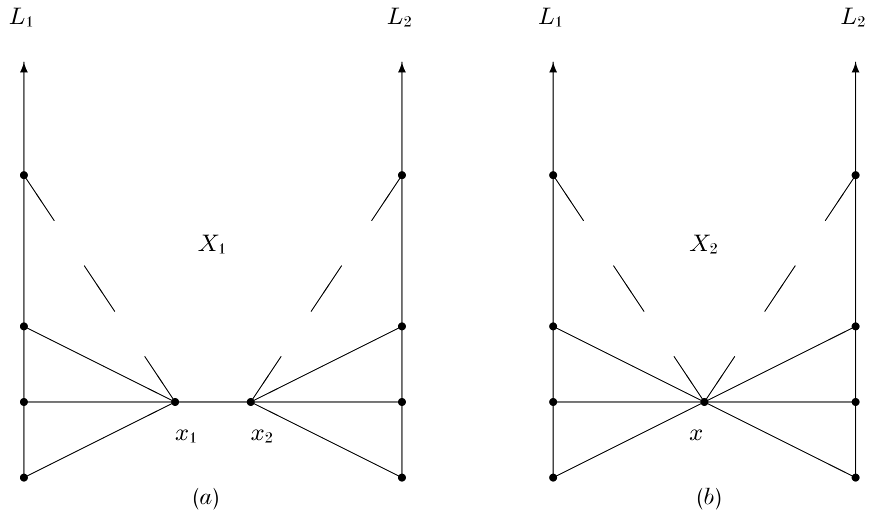


Fig. 4.

sets a priori for all vertices $x$, see [**1**] or by considering vertices $x$ with infinite degree as degenerated rays only and not as vertices (cf. [**9**]). In the first case the edge-topology is a compactification of the discrete topology on $VX$. The compactness of these modified topologies can be deduced easily from Theorem 2.

Polat [**14–18**] used uniformities to define and study the topology which is generated by $\mathscr{B}_V X \cup \mathscr{P}(VX)$. In general it is not compact.

Theorem 3. *Let* $c_{\mathrm{E}} = e_{\mathrm{E}} \cup \varepsilon_{\mathrm{E}}$ *and* $c_{\mathrm{V}} = e_{\mathrm{V}} \cup \varepsilon_{\mathrm{V}}$ *be such that* $e_{\mathrm{E}} \subset VX$, $e_{\mathrm{V}} \subset VX$, $\varepsilon_{\mathrm{E}} \subset \mathbf{\Omega}_{\mathrm{E}}X$ *and* $\varepsilon_{\mathrm{V}} \subset \mathbf{\Omega}_{\mathrm{V}}X$. $c_{\mathrm{E}}$ *is open and closed in* $\tau_{\mathrm{E}}$ *if and only if* $e_{\mathrm{E}}$ *is an edge-cut and* $\varepsilon_{\mathrm{E}} = \mathbf{\Omega}_{\mathrm{E}}e_{\mathrm{E}}$. *The set* $c_{\mathrm{V}}$ *is open and closed in* $\tau_{\mathrm{V}}$ *if and only if* $e_{\mathrm{V}}$ *is an edge-cut and* $\varepsilon_{\mathrm{V}} = \mathbf{\Omega}_{\mathrm{V}}e_{\mathrm{V}}$.

*Proof.* Let $c_{\mathrm{E}}$ and $c_{\mathrm{V}}$ be open and closed in their corresponding topologies. They can be represented as a finite union of elements of the base, because they are open and compact. Hence $e_{\mathrm{E}}$ is an edge-cut and $e_{\mathrm{V}}$ is a vertex-cut. The same argument holds for $(VX \cup \Omega_{\mathrm{V}}X)\backslash c_{\mathrm{V}}$ and therefore $e_{\mathrm{V}}^*$ is a vertex-cut, too. Thus $e_{\mathrm{V}}$ is an edge-cut.

An end in $\varepsilon_{\mathrm{E}}$ or $\varepsilon_{\mathrm{V}}$ must lie in $e_{\mathrm{E}}$ or $e_{\mathrm{V}}$, respectively, because $c_{\mathrm{E}}$ and $c_{\mathrm{V}}$ are open. On the other hand every end lying in $e_{\mathrm{E}}$ or $e_{\mathrm{V}}$ must be an element of $\varepsilon_{\mathrm{E}}$ or $\varepsilon_{\mathrm{V}}$, respectively. Otherwise the complements of $c_{\mathrm{E}}$ and $c_{\mathrm{V}}$ would not be open.

## 5. *Metric ends*

Sometimes theorems for locally finite graphs can be generalized for non-locally-finite graphs or their proofs can be simplified by replacing arguments that use the finiteness of sets of vertices by arguments of finite diameters (cf. [**9, 10**]). At the other hand there even exist rather simple structures whose ramifications can neither be captured by the vertex- nor by the edge-topology (cf. Example 4 and Example 5). This motivates a new definition of ends in non-locally-finite graphs, which is only based on the natural graph metric.

A *metric cut* is a set of vertices with a vertex-boundary of finite diameter. A ray whose infinite subsequences have all infinite diameters is called *metric ray*. Two metric rays are *metrically equivalent* if they cannot be separated by metric cuts. Metrical equivalence is an equivalence relation on the set of metric rays of a graph. We call its equivalence classes *proper metric ends*. A proper metric end *lies in a set of vertices* $e$ if all of its rays lie in $e$.

We denote the set of proper metric ends that lie in $e$ by $\Omega_{\mathrm{P}}e$ and write $\Omega_{\mathrm{P}}X$ instead of $\Omega_{\mathrm{P}}VX$.

Lemma 8. *For any graph $X$, the set $\mathscr{B}_{\mathrm{P}}X := \{e \cup \Omega_{\mathrm{P}}e \mid e \subset VX$ and $\operatorname{diam} \theta e < \infty\}$ is closed under finite intersection.*

The proof can be copied word by word from Lemma 2.

We now can define the *proper metric end topology* $\tau_{\mathrm{P}}X$ as the topology on $VX \cup \Omega_{\mathrm{P}}X$ which is generated by $\mathscr{B}_{\mathrm{P}}X$.

Theorem 4. *The proper metric end topology is a totally disconnected Tychonoff topology. In the case that $X$ is locally countably infinite it is Lindelöf and normal. If there exists a vertex with infinite degree it is not paracompact.*

*Proof.* For any closed set $F$ not containing some point $x$ we can find a neighbourhood $A$ of $x$ which is element of the base and contained in the complement of $F$. As the base elements in $\mathscr{B}_{\mathrm{P}}X$ are open and closed the indicator function on $A$ is continuous. Thus the proper metric end topology is $T_{3\frac{1}{2}}$ or completely regular. Zero-dimensionality is also an immediate consequence of the fact that elements of the base are open and closed. It is easy to see that this topology is Hausdorff which now implies that it is Tychnoff and totally disconnected.

Let $X$ be a locally countable graph. For some given vertex $x$ let $\mathscr{V}$ denote the set of all base elements $e \cup \Omega_{\mathrm{P}}e$ such that $e$ is a connected component of $B(x,r)^*$ for some natural $r$. $\mathscr{V}$ is countable. Any element of an open cover $\mathscr{U}$ containing a proper metric end must contain an element of $\mathscr{V}$. Thus we can find a countable subcover of $\mathscr{U}$ of the set of proper metric ends. The set of vertices is countable anyway and hence the proper metric end topology is Lindelöf. Every regular Lindelöf space is normal.

For any vertex $x$ of infinite degree

$$\{\{x,y\} \mid x \sim y\} \cup \{B(x,1)^* \cup \Omega_{\mathrm{P}}X\}$$

is an open cover which is not locally finite in $x$ and therefore the proper metric end topology is not paracompact.

We do not know whether the proper metric end topology is normal in the general case.

*Example* 4. Let $X$ be the graph which arises from a tree $T$ with only vertices of infinite degree to which we add edges connecting all pairs of vertices with distance two. The proper metric topologies on $X$ and $T$ are homeomorphic, in contrast to the vertex- and the edge-topology on $X$, which are indiscrete and therefore describe no structure at all.

To motivate the definitions of the following section we remark that every sequence of vertices that has neither an accumulation point in $VX$ (equivalently: has infinitely many identical elements) nor an accumulation point in $\Omega_{\mathrm{P}}X$ can be divided up into two types of subsequences:

(i) bounded sequences;

(ii) sequences with no bounded subsequences for which there exists a metric cut, whose complement consists only of components that contain at most finitely many elements of the sequence.

## 6. *A compactification of the proper metric end topology*

We will now add two additional points to the set of metric ends and modify the proper metric topology correspondingly.

A *star-cut* is a metric cut $e$ such that every union $r$ of all but finitely many components of its complement has infinite diameter. For a star-cut $e$ we call such a union $r$ *star-boundary of* $e$. A set $s$ of vertices is called a *global star-set* if to every star-cut $e$ there exists a star-boundary $r$, such that $r\backslash s$ has finite diameter.

An infinite set of vertices $p$ is called *locally complete*, if it contains all but finitely many elements of every ball.

If there exists a star-cut in $VX$ we add an element $\sigma_X$ called the *star-end* to the set of proper metric ends. Furthermore we add an element $\lambda_X$ to the set of ends which we call the *local end*. The set $\mathbf{\Omega}_{\mathrm{M}}X$ so obtained is called *set of metric ends of* $X$, $\lambda_X$ and $\sigma_X$ are called *improper metric ends*. We say that the local end *lies in a set of vertices* $e$, if it is a locally complete set. If $e$ is a global star-set we say that the star-end *lies in* $e$. Let $\mathbf{\Omega}_{\mathrm{M}}e$ denote the set of all metric ends, that lie in $e$.

Although the local end will be useless in locally finite graphs (equivalently: it is an isolated point, $\{\lambda_X\}$ is open), we add it to the set of ends in any way. This will help use to find homeomorphisms between end-spaces in locally and non-locally-finite graphs.

Lemma 9. *The set*

$$\mathscr{B}_{\mathrm{M}}X := \{(e \cap s \cap p) \cup \mathbf{\Omega}_{\mathrm{M}}(e \cap s \cap p) \mid e \textit{ is a metric cut, } s \textit{ is a global star-set and } p \textit{ is a locally complete set}\}$$

*is closed under finite intersection.*

We call the topology $\tau_{\mathrm{M}}X$ on $VX \cup \mathbf{\Omega}_{\mathrm{M}}X$ which is generated by $\mathscr{B}_{\mathrm{M}}X$ *metric end topology*.

Lemma 10. $(VX \cup \mathbf{\Omega}_{\mathrm{M}}X, \tau_{\mathrm{M}}X)$ *is a Lindelöf space.*

*Proof.* Let $\mathscr{U}$ be an open cover of base elements. It must contain sets $p \cup \mathbf{\Omega}_{\mathrm{M}}p$ and $s \cup \mathbf{\Omega}_{\mathrm{M}}s$ such that $p$ is a locally complete set and $s$ is a global star-set. After removing these base elements from $VX \cup \mathbf{\Omega}_{\mathrm{M}}X$ we obtain a set $A \cup \mathbf{\Omega}A$ such that $A \subset p^* \cap s^*$ and $\mathbf{\Omega}A \subset \mathbf{\Omega}_{\mathrm{P}}X \backslash \mathbf{\Omega}_{\mathrm{P}}s$. $A$ is a set of vertices, that contains only finitely many elements of any ball in $X$. Thus $A$ is countable and we can find a countable subcover $\mathscr{V}$ of $VX$ in $\mathscr{U}$. We now copy the ideas in the second part of the proof of Lemma 6 to construct a countable subset of $\mathscr{U}$ which covers $\mathbf{\Omega}A$.

Let $\mathrm{d}_{\mathscr{U}}(\omega, z)$ again denote the distance of the end $\omega$ to the vertex $z$ with respect to the cover $\mathscr{U}$. As $A$ contains no star-boundary, it has the property that for every metric cut $e$ we can find a union $u$ of finitely many components of $e^*$, such that $A\backslash u$ has a finite diameter. Thus there are only finitely many connected components in $(s \cup B(z,n))^*$ for any positive integer $n$. We can find a finite subcover $\mathscr{V}_n$ of $\mathscr{U}$

covering all ends with $\mathrm{d}_{\mathscr{U}}(\omega, z) \leqslant n$, that lie in $\Omega A$ and

$$\{p \cup \Omega_{\mathrm{M}} p\} \cup \{s \cup \Omega_{\mathrm{M}} s\} \cup \mathscr{V} \cup \bigcup_{n=1}^{\infty} \mathscr{V}_n$$

is a countable subcover of $\mathscr{U}$ covering $VX \cup \Omega_{\mathrm{M}} X$.

Theorem 5. *$(VX \cup \Omega_{\mathrm{M}} X, \tau_{\mathrm{M}} X)$ is a compact topological space. All limits of convergent sequences are unique. For a convergent sequence $\xi$ with $\alpha = \lim \xi$ exactly one of the following cases must hold.*

(i) *The limit $\alpha$ is a vertex. All elements of the sequence are equal to $\alpha$ from an index on.*

(ii) *From some index onwards the elements of the sequence $\xi$ are the local end or they are vertices lying in some ball such that at the most finitely many vertices are identical. In this case $\xi$ converges to the local end $\lambda_X$.*

(iii) *In the complement of any ball exactly one component contains all but finitely many elements of the sequence $\xi$. There exists exactly one proper metric end lying in all these components, which is the limit $\alpha$.*

(iv) *There exists a vertex $x$ and a radius $r$ such that for any natural n the ball $B(z, r+n)$ is a star-cut, whose star-boundaries contain all but finitely many of those elements of the sequence, that do not equal the star-end, but contain only finitely many of them in each of their connected components. The sequence converges to the star-end $\sigma_X$.*

*Proof.* To show that every sequence in $VX \cup \Omega_{\mathrm{M}} X$ has an accumulation point we can assume that it contains an infinite partial sequence $\xi$ consisting of pairwise different elements. Otherwise the existence of an accumulation point is immediate. If $\xi$ contains a bounded subsequence of vertices, the local end is an accumulation point. In the other case we choose a vertex $x$ and distinguish between two cases.

(i) There exists a sequence $(e_n)_{n \in \mathbb{N}}$ of components $e_n$ of $K(x,n)^*$ such that $e_n$ contains infinitely many elements of the sequence $\xi$ and is a superset of $e_{n+1}$. Following the idea of Lemma 3 we now can construct a metric ray, that lies in all cuts $e_n$. Its end is an accumulation point of $\xi$.

(ii) There exists a ball $B(x,n)$ such that only finitely many elements of $\xi$ lie in every component of $B(x,n)^*$. Since $\xi$ does not contain a bounded subsequence $B(x,n)$ is a star-ball. Let $s \cup \Omega_{\mathrm{M}} s$ be a base element that contains the star-end $\sigma_X$, which means that $s$ must be a global star-set. By the definition of a global star-set there exists a star-boundary $r$ of $B(x,n)$ such that $r \backslash s$ has a finite diameter. The set $r \cup \Omega_{\mathrm{M}} r$ must contain all but finitely many elements of $\xi$, but as $\xi$ contains no bounded subsequence, only finitely many elements of the sequence $\xi$ lie in $r \backslash s$. Now $s \cup \Omega_{\mathrm{M}} s$ must contain all but finitely many elements of $\xi$, which means that $\sigma_X$ in an accumulation point of $\xi$.

We have shown that $(VX \cup \Omega_{\mathrm{M}} X, \tau_{\mathrm{M}} X)$ is sequentially compact. By Lemma 10 it is also a Lindelöf space and hence it is compact. The other statements of Theorem 5 now follow easily by the above considerations.

*Example* 5. Let $K_{\mathbb{N}}$ denote the complete graph with vertex-set $\mathbb{N}$, see Fig. 5($a$). We take two graphs $K_{\mathbb{N}}^{(1)}$ and $K_{\mathbb{N}}^{(2)}$ that are isomorphic to $K_{\mathbb{N}}$ and connect each vertex with adjacent vertices $x_1$ and $x_2$, respectively (Fig. 5($b$)). As in Example 3 we take

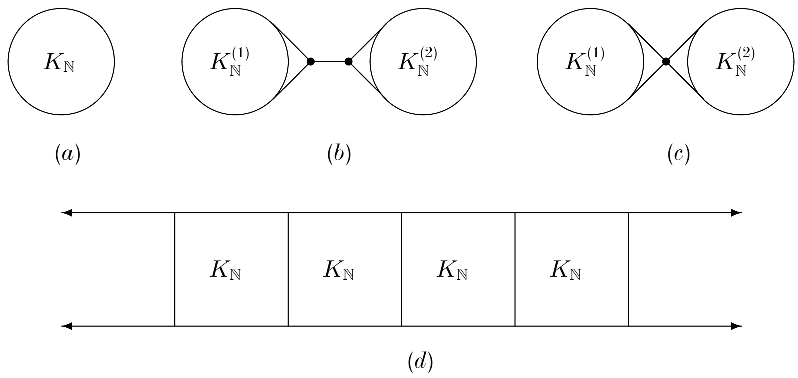


Fig. 5.

away the edge $(x_1, x_2)$ from the graph in Fig. 5($b$) and identify $x_1$ and $x_2$ to a vertex $x$, see Fig. 5($c$). In Fig. 5($d$) we take a sequence $(K_{\mathbb{N}}^{(n)})_{n\in\mathbb{Z}}$ of copies of the graph $K_{\mathbb{N}}$ and connect each vertex in $K_{\mathbb{N}}^{(n)}$ with each of the vertices in $K_{\mathbb{N}}^{(n+1)}$.

Every sequence of pairwise distinct vertices in the metric end topology of the graphs in Fig. 5($a$), 5($b$) and 5($c$) converges to the pointend. For the edge-end and vertex-end compactification in Fig. 5($b$) and 5($c$) we have the same situation as in Fig. 4. In Fig. 5($d$) they contain only one end whereas the metric topology contains two proper metric ends, compare with Example 2.

## 7. *Metric ends and quasi-isometries*

*Definition* 1. Two graphs $X$ and $Y$ are called *quasi-isometric with respect to the functions* $\phi\colon VX \to VY$ *and* $\psi\colon VY \to VX$ if there exist constants $a$, $b$, $c$ and $d$ such that for all vertices $x$, $x_1$ and $x_2$ in $VX$ and vertices $y$, $y_1$ and $y_2$ in $VY$, the following conditions hold

| | | |
|---|---|---|
| $(Q1)$ | $\mathrm{d}_Y(\phi(x_1), \phi(x_2)) \leqslant a \cdot \mathrm{d}_X(x_1, x_2)$ | (boundedness of $\phi$) |
| $(Q2)$ | $\mathrm{d}_X(\psi(y_1), \psi(y_2)) \leqslant b \cdot \mathrm{d}_Y(y_1, y_2)$ | (boundedness of $\psi$) |
| $(Q3)$ | $\mathrm{d}_X(\psi\phi(x), x) \leqslant c$ | (quasiinjectivity of $\phi$) |
| $(Q4)$ | $\mathrm{d}_Y(\phi\psi(y), y) \leqslant d$ | (quasisurjectivity of $\phi$). |

We call $\phi$ and $\psi$ *quasi-inverse* to each other.

For general metric spaces the definition of quasi-isometries allows further additive constants in the Axioms (Q1) and (Q2). In case the positive values of the metric are greater than some positive real number these additive constants are useless.

Quasi-isometries may change structures as long as the differences can be bounded uniformly. In other words we could say that they preserve the global structure of graphs when we consider graphs as discrete metric spaces only.

Without proof we remark that quasi-isometry is an equivalence relation on the family of all graphs.

*Example* 6. The graphs in Example 3, the ladder graph (Example 2) and the graph in Example 5, Fig. 4($d$), as well as the tree $T$ and the graph $X$ in Example 4 are quasi-isometric.

For all constants $r$ the Cayley graphs $X$ of the free group with respect to the generating system $A^r$ in Section 8 are quasi-isometric to each other.

Replacing every edge in a graph $X$ by a path with a length smaller or equal some constant we obtain a graph which is quasi-isometric to $X$.

LEMMA 11. *Let $\phi : VX \to VY$ be a quasi-isometry and $A$ a subset of $VX$, then*

$$\operatorname{diam}_X A < \infty \Leftrightarrow \operatorname{diam}_Y \phi(A) < \infty.$$

*Proof.* $\operatorname{diam}_X A < \infty$ implies

$$\begin{aligned}\operatorname{diam}_Y \phi(A) &= \sup\{\mathrm{d}_Y(\phi(x_1), \phi(x_2)) \mid x_1, x_2 \in A\}\\ &\leqslant \sup\{a \cdot \mathrm{d}_X(x_1, x_2) \mid x_1, x_2 \in A\} = a \cdot \operatorname{diam}_X A < \infty.\end{aligned}$$

Let $\psi$ be a quasi-inverse to $\phi$. If $\operatorname{diam}_X A$ is infinite, then, by (Q3), this also holds for $\operatorname{diam}_X \psi\phi(A)$. Now

$$\mathrm{d}_X(\psi\phi(x_1), \psi\phi(x_2)) \leqslant b \cdot \mathrm{d}_Y(\phi(x_1), \phi(x_2))$$

implies that

$$\{\mathrm{d}_Y(\phi(x_1), \phi(x_2)) \mid x_1, x_2 \in A\}$$

has no upper bound.

COROLLARY 1. *The pre-image of a metric cut or a global star-set set under a quasi-isometry is a metric cut or a global star-set, respectively.*

We now want to extend the concept of quasi-isometry to the set of proper metric ends.

THEOREM 6. *To every quasi-isometry $\phi\colon VX \to VY$ there exists a unique extension*

$$\Phi : VX \cup \Omega_{\mathrm{P}}X \to VY \cup \Omega_{\mathrm{P}}Y,$$

*such that*

(i) $\Phi|_{VX} = \phi$
(ii) $\Phi$ *is continuous and*
(iii) $\Phi|_{\Omega_{\mathrm{P}}X}$ *is a homeomorphism of $\Omega_{\mathrm{P}}X$ and $\Omega_{\mathrm{P}}Y$ with respect to the corresponding relative topologies.*

*Proof.* By connecting the $\phi$-images of adjacent vertices of a metric ray $L$ in $X$ with geodesic paths with lengths, that are smaller or equal $a$, we obtain a path $P$ in $Y$. Its diameter is infinite by Lemma 11. If it had an infinite subset $M$ with finite diameter, we also could find an infinite subset of $\phi(L)$ whose diameter is finite, contradicting Lemma 11 and the assumption that $L$ is a metric ray. As a graph is locally finite if and only if every bounded set of vertices is finite, $P$ is a locally finite subgraph of $Y$. Lemma 1 implies that it must contain a ray which we denote with $\tilde{\phi}(L)$, it is a metric ray in $Y$. Thus $\tilde{\phi}$ maps metric rays in $X$ to metric rays in $Y$.

Let $L_1$ and $L_2$ be metric equivalent rays in $X$. If $\tilde{\phi}(L_1)$ and $\tilde{\phi}(L_2)$ were not metrically equivalent, we could find disjoint metric cuts $f_1$ and $f_2$, such that $\tilde{\phi}(L_1)$ lies in $f_1$ and $\tilde{\phi}(L_2)$ lies in $f_2$; $\phi^{-1}(f_1) \cap \phi^{-1}(f_2)$ is empty, $L_1$ lies in $\phi^{-1}(fn)$ and $L_2$ lies in $\phi^{-1}(f_2)$ in contradiction to the assumption that they are metrically equivalent. Thus we can say that metric equivalence of metric rays is an invariance under $\tilde{\phi}$ on sets of metric rays.

Now for every end $\omega$ in $\Omega_{\mathrm{P}}X$ we define $\Phi(\omega)$ as the unique end in $\Omega_{\mathrm{P}}Y$ which contains the $\tilde{\phi}$-images of the elements of $\omega$ and set $\Phi(x) = \phi(x)$ for every vertex $x$.

By the above invariance of the metric equivalence and the construction of $\tilde{\phi}$ we obtain

$$\Phi(\Omega_{\mathrm{P}} e) \subset \Omega_{\mathrm{P}}\Phi(e)$$

for every metric cut $e$ in $X$, and therefore by Corollary 1

$$\Phi(\Omega_{\mathrm{P}}\Phi^{-1}(f)) \subset \Omega_{\mathrm{P}}\Phi(\Phi^{-1}(f)) = \Omega_{\mathrm{P}} f$$

and

$$\Omega_{\mathrm{P}}\Phi^{-1}(f) \subset \Phi^{-1}(\Omega_{\mathrm{P}} f)$$

for every metric cut $f$ in $Y$.

Now let $\omega$ be a proper metric end in $\Phi^{-1}(\Omega_{\mathrm{P}} f)$. Then of course $\Phi(\omega) \in \Omega_{\mathrm{P}} f$. If $\omega$ did not lie in $\Phi^{-1}(f)$ then there would exist infinitely many vertices $x$ in a ray $L$ of $\omega$ which are not elements of $\Phi^{-1}(f)$. Their $\phi$-images would not be elements of $f$, $\tilde{\phi}(L)$ would not lie in $f$ and $\Phi(\omega)$ would not be an element of $\Omega_{\mathrm{P}} f$. Hence

$$\Phi^{-1}(\Omega_{\mathrm{P}} f) \subset \Omega_{\mathrm{P}}\Phi^{-1}(f)$$

and

$$\Phi^{-1}(\Omega_{\mathrm{P}} f) = \Omega_{\mathrm{P}}\Phi^{-1}(f). \tag{7·1}$$

For every base element $f \cup \Omega_{\mathrm{P}} f$ in $\mathscr{B}_{\mathrm{P}} Y$ we obtain

$$\Phi^{-1}(f \cup \Omega_{\mathrm{P}} f) = \Phi^{-1}(f) \cup \Omega_{\mathrm{P}}\Phi^{-1}(f)$$

and $\Phi$ is continuous.

To prove the third property of $\Phi$, it now suffices to show that the restriction $\Phi|_{\Omega_{\mathrm{P}} X}$ is a bijection of the sets of ends in $X$ and $Y$. To a given end $\varepsilon$ in $\Omega_{\mathrm{P}} Y$ we can find a decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of connected metric cuts in $Y$ such that $\bigcap_{n\in\mathbb{N}} f_n = \varnothing$ and $\bigcap_{n\in\mathbb{N}} \Omega_{\mathrm{P}} f_n = \varepsilon$. Now we choose an increasing sequence $(B(z, r_n))_{n\in\mathbb{N}}$ of concentric balls in $X$ such that $\theta\Phi^{-1}(f_n)$ is contained in $B(z, r_n)$. To every integer $n$ there exists a connected component $e_n$ of $B(z, r_n)^*$ which contains all but finitely many vertices of the $\Phi$-preimage of a ray $L$ of $\varepsilon$. Copying the idea of Lemma 3 we can find a metric ray $R$ that lies in all cuts $e_n$. The end $\omega$ of $R$ is mapped onto $\varepsilon$ under $\Phi$. Thus $\Phi$ is surjective.

Let $\varepsilon_1$ and $\varepsilon_2$ be two different ends of the graph $Y$ and $f_1$ and $f_2$ two disjoint metric cuts such that $\varepsilon_1$ lies in $f_1$ and $\varepsilon_2$ lies in $f_2$. Now $\Phi^{-1}(\Omega_{\mathrm{P}} f_1)$ and $\Phi^{-1}(\Omega_{\mathrm{P}} f_2)$ must be disjoint too, and therefore $\Phi$ is injective.

To complete the proof of the theorem, it remains to show that $\Phi$ is the unique extension of $\phi$ with the requested properties. We assume that there exists another extension $\hat{\Phi}$ which has these three properties but does not equal $\Phi$. Let $\omega$ be an end in $X$ and $\Phi(\omega) = \varepsilon_1$ and $\hat{\Phi}(\omega) = \varepsilon_2$ for two different ends $\varepsilon_1$ and $\varepsilon_2$ in $Y$. Now we choose again two disjoint metric cuts $f_1$ and $f_2$ such that $\varepsilon_1$ lies in $f_1$ and $\varepsilon_2$ lies in $f_2$. As $\Phi^{-1}(f_1) = \phi^{-1}(f_1)$ and $\hat{\Phi}^{-1}(f_2) = \phi^{-1}(f_2)$, the preimages $\Phi^{-1}(f_1)$ and $\hat{\Phi}^{-1}(f_2)$ are disjoint metric cuts in $X$. We know that $\omega$ cannot lie in both, $\Phi^{-1}(f_1)$ and $\Phi^{-1}(f_2)$. Suppose $\omega$ lies in $\hat{\Phi}^{-1}(f_2)^*$. This implies that every open neighbourhood of $\omega$ has a nonempty intersection with $\hat{\Phi}^{-1}(f_2)^*$. In other words, no open neighbourhood of $\omega$ is completely contained in $\hat{\Phi}^{-1}(f_2 \cup \Omega_{\mathrm{P}} f_2)$ and $\hat{\Phi}$ is not continuous.

For a subset $A$ of $VX \cup \Omega_{\mathrm{P}}X$ and an integer $r$ we define

$$A + r := \{x \in VX \mid \mathrm{d}_X(x, A \backslash \Omega_{\mathrm{P}}A) \leqslant r\} \cup \Omega_{\mathrm{P}}A.$$

In the following sense we could call $\Phi$ a *quasi-open* function.

THEOREM 7. *For any base element $e \cup \Omega_{\mathrm{P}}e$ in $\mathscr{B}_{\mathrm{P}}X$, the set $\Phi(e \cup \Omega_{\mathrm{P}}e) + d + 1$ is open in $\tau_{\mathrm{P}}Y$.*

Although the proof seems a little technical, its idea is simple. Quasi-isometry is a weakened form of isomorphy, it does not ensure that the image $\phi(VX)$ covers $VY$ completely but says that it does not have 'holes' that are bigger than a circle with radius $d$.

*Proof.* Let $y$ be an element of $\theta(\phi(e) + d + 1)$. As $\mathrm{d}_Y(\phi\psi(y), y) \leqslant d$ the vertex $\phi\psi(y)$ must not be contained in $\phi(e)$, so $\phi\psi(y) \in \phi(e^*)$ and $\psi(y) \in e^*$. Hence

$$\psi(\theta(\phi(e) + d + 1)) \subset e^*.$$

Now let $y_1$ be an element of $\phi(e) + d + 1$ which is adjacent to $y$ and choose an element $y_2$ of $\phi(e)$ such that $\mathrm{d}_Y(y_1, y_2) \leqslant d + 1$. For an $x$ in $e$ with $\phi(x) = y_2$ we now have

$$\mathrm{d}_Y(y, \phi(x)) \leqslant d + 2.$$

By the boundedness of $\psi$ we have $\mathrm{d}_X(\psi(y), \psi\phi(x)) \leqslant b \cdot (d + 2)$ and by the quasi-injectivity of $\phi$ we get $\mathrm{d}_X(x, \psi\phi(x)) \leqslant c$ which implies

$$\mathrm{d}_X(\psi(y), x) \leqslant b \cdot (d + 2) + c.$$

In other words

$$\max\{\mathrm{d}_X(\psi(y), e) \mid y \in \theta(\phi(e) + d + 1)\} \leqslant b \cdot (d + 2) + c.$$

As $\psi(\theta(\phi(e) + d + 1))$ is a subset of $e^*$ we now obtain

$$\psi(\theta(\phi(e) + d + 1)) \subset \theta e + b \cdot (d + 2) + c - 1.$$

Consequently $\psi(\theta(\phi(e) + d + 1))$ has a finite diameter and by Lemma 11 this also holds for $\theta(\phi(e) + d + 1)$ and $\phi(e) + d + 1$ is a metric cut in $Y$.

By the definition of $\Phi$

$$\Phi(e \cup \Omega_{\mathrm{P}}e) + d + 1 = \Phi(\Omega_{\mathrm{P}}e) \cup (\phi(e) + d + 1)$$

and to prove this set is open in $\tau_{\mathrm{P}}Y$ we show that $\Phi(\Omega_{\mathrm{P}}e)$ is contained in $\Omega_{\mathrm{P}}(\phi(e)+d+1)$. For every metric cut $f$ in $Y$ the equation (7·1) implies

$$\Omega_{\mathrm{P}}f = \Phi(\Omega_{\mathrm{P}}\Phi^{-1}(f)).$$

As $\Omega_{\mathrm{P}}e$ is a subset of $\Omega_{\mathrm{P}}\Phi^{-1}(\phi(e) + d + 1)$ and $\phi(e) + d + 1$ is a metric cut we now obtain

$$\Phi(\Omega_{\mathrm{P}}e) \subset \Phi(\Omega_{\mathrm{P}}\Phi^{-1}(\phi(e) + d + 1)) = \Omega_{\mathrm{P}}(\phi(e) + d + 1)$$

by replacing $f$ with $\phi(e) + d + 1$.

For applications of quasi-isometries in the study of ends of graphs confer [**9**], [**10**] and [**12**].

## 8. *Bounded random walk on the free group*

We now want to give a further example for an application of the metric end compactification concerning the random walk on the free group with countably infinitely many generators. Random walks on free groups were first introduced and studied by Kesten [**8**]. Let $\Gamma$ be a free group with a symmetric and countable set of generators $A$ containing at least four elements. Every element $x$ of $\Gamma$ can be represented in a unique way by the shortest product of elements of $A$ that equals $x$. The length of this product is called *length* of $x$, the length of the neutral element $o$ is set zero. Let $A^r$ denote the set of elements with positive lengths that are less or equal some integer $r$. The Cayley graph $X$ of $\Gamma$ with respect to $A^r$ has vertex set $VX = \Gamma$. Two vertices $x$ and $y$ are adjacent if and only if $x^{-1}y$ is an element of $A^r$. $X$ is isomorphic to a graph that arises from a regular tree $T$ with degree $|A|$ by connecting pairs of distinct vertices $x$ and $y$ with distance $\mathrm{d}_T(x,y) \leqslant r$. Compare with Example 4.

Let $\mu$ be a probability measure on $A^r$ whose support generates the whole group $\Gamma$. Now $\mu$ defines a random walk $(Z_n)_{n\in\mathbb{N}}$ on $\Gamma$ with respect to the probability measure Pr on $\Gamma^{\mathbb{N}}$ which is generated by the cylindric sets in $(A^r)^{\mathbb{N}}$ together with the corresponding powers of $\mu$.

THEOREM 8. *The random walk $(Z_n)_{n\in\mathbb{N}}$ on the countably infinite free group $\Gamma$ converges almost surely to some proper metric end in the Cayley graph $X$.*

*Proof.* As the support of $\mu$ generates $\Gamma$ we can choose four distinct elements $a_1$, $a_2$, $a_1^{-1}$ and $a_2^{-1}$ in $A^r$ such that $\mu(a_1)$ and $\mu(a_2)$ are positive. We define a random walk $(\tilde{Z}_n)_{n\in\mathbb{N}}$ on the free group $\tilde{\Gamma}$ with generating system $\tilde{A} = \{a_1, a_2, a_1^{-1}, a_2^{-1}\}$ by choosing $\mu(a_1)$, $\mu(a_2)$, $\mu(a_1^{-1})$ and $\mu(a_2^{-1})$ as the probabilities for the right multiplication with the generating elements, respectively. The probability of not making a move is set $1-(\mu(a_1)+\mu(a_2)+\mu(a_1^{-1})+\mu(a_2^{-1}))$. We know that $(\tilde{Z}_n)_{n\in\mathbb{N}}$ is transient (cf. for example [**21**]). The corresponding probability measure on the set of trajectories is denoted by $\tilde{\Pr}$. As the corresponding Cayley graph $\tilde{X}$ is locally finite, the random walk $(\tilde{Z}_n)_{n\in\mathbb{N}}$ does not enter any ball from an index on with probability one. Thus

$$\tilde{\Pr}[\lim_{n\to\infty} \mathrm{d}_{\tilde{X}}(o, \tilde{Z}_n) = \infty] = 1.$$

Now $\mathrm{d}_{\tilde{X}}(x,y) \leqslant r \cdot \mathrm{d}_X(x,y)$ for all elements $x$ and $y$ of $\Gamma$ and therefore

$$\Pr[\lim_{n\to\infty} \mathrm{d}_X(o, Z_n) = \infty] = 1.$$

This is equivalent to the almost sure convergence of $Z_n$ to a proper metric end in the metric end topology of $X$.

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