Abh. Math. Sem. Univ. Hamburg **71** (2001), 161–180

Quasi-Isometries between Non-Locally-Finite Graphs and Structure Trees

By B. KRÖN

Abstract. We prove several criteria for quasi-isometry between non-locally-finite graphs and their structure trees. Results of MÖLLER in [11] for locally finite and transitive graphs are generalized. We also give a criterion in terms of correspondence between the ends of the graph and the ends of the structure tree.

1 Introduction

Quasi-isometry on graphs is a weakened form of isomorphism. Graphs which are quasi-isometric to each other have the same global structure but may have local differences. The main property of quasi-isometries is that a set has finite diameter if and only if its image has finite diameter, see Lemma 12 in Section 7.

In [5] and [6] GROMOV used the concept of quasi-isometry in the context of structural properties of infinite groups.

Usually a graph X is said to be almost transitive if its automorphism group $\mathrm{Aut}(X)$ has only finitely many orbits on the set of vertices VX. Let d_X be the natural metric on X and let

$$B(x,r) = \{y \in VX \mid d_X(x,y) \le r\}$$

be the *ball* (more precise d_X-ball or closed d_X-ball) with centre x and radius r. In Section 10 we call a graph *almost transitive* if there is a vertex x and a ball $B(x,r)$ such that

$$\bigcup_{g \in \mathrm{Aut}(X)} gB(x,r) = VX.$$

This ball is then called *covering ball.* In locally finite graphs these two definitions are equivalent but in non-locally-finite graphs the latter approach is more general.

A connected graph has infinite diameter if and only if it contains a ray of infinite diameter or a so-called *star ball,* confer Section 2 in [9]. The existence of structure trees, an axiomatic definition can be found in Section 2, was proved by DUNWOODY in [4] where he also generalized results of STALLINGS in [15]. Graphs which are not almost transitive or contain a star ball cannot be quasi-isometric to a structure

---

1991 *Mathematics Subject Classification.* 05C75, (05C25, 20B27).

The author is supported by the START-project Y96-MAT of the Austrian Science Fund.

tree. Let X be an almost transitive graph which does not contain a star ball. Then we characterize pairs of graphs and their structure trees which are quasi-isometric to each other in terms of correspondence between their spaces of ends, see Section 4 and Theorem 4 in Section 11.

Theorem 3 in Section 8 and Theorem 5 in Section 12 are generalizations of results of MÖLLER in [11].

2 Basic definitions and structure trees

Throughout this article let $X = (VX, EX)$ be a connected, undirected graph without loops or multiple edges. The set VX of all vertices consists of VX^L, the set of vertices with finite degree, and the set of vertices with infinite degree which we denote by VX^∞. Let e be a set of vertices in VX. We write e^* for the complement $VX\backslash e$ of e and $\operatorname{diam}_X e$ for the diameter of e with respect to the natural graph metric d_X on X. Let x and y be vertices in VX. A *path of length n* from x to y is a set of vertices

$$\{z_0 = x, z_1, \dots, z_n = y\}$$

such that z_i is adjacent to z_{i+1} for $0 \leq i \leq n-1$. The path is *geodesic* if its length n is $d_X(x, y)$.

The set e of vertices is called *connected* if any two vertices in e can be connected by a path that does not leave e.

The *vertex-boundary* θe of e is the set of vertices in e^* which are adjacent to a vertex in e. $I\theta e = \theta e^*$ is called *inner vertex-boundary* of e. The *edge-boundary* δe of e is defined as the set of edges connecting vertices in e with vertices in e^*. A non-empty set of vertices e is a *cut* (more precisely an *edge-cut*) if δe is finite. For $n = |\delta e|$ we also call e an *n-cut*. If both e and e^* are connected the cut e is said to be *tight*.

Definition 1. A set E of cuts in X is called a *tree set*, if it satisfies the following three axioms.

(S1) For all pairs of cuts e and f in E, one of the following inclusions holds:

$$e \subset f, \quad e \subset f^*, \quad e^* \subset f \quad \text{or} \quad e^* \subset f^*.$$

(S2) For any two cuts e and f in E there exist only finitely many cuts d in E such that $e \subset d \subset f$.

(S3) Neither $\emptyset$ nor VX are elements of E.

The tree set E is called *undirected* if

(S4) e is an element of E if and only if e^* is an element of E.

An undirected tree set that consists only of tight n-cuts for some fixed integer n is called a *tight* tree set.

We call an edge-cut e *non-trivial*, if both e and e^* are infinite. Non-trivial and tight edge-cuts e for which $\operatorname{Aut}(X)e \cup \operatorname{Aut}(X)e^*$ is a tree set are called *structure cuts*. Such a tree set is called a *structure tree set*.

Note that instead of $e \subset f^*$, $e^* \subset f$ and $e^* \subset f^*$ we can write $e \cap f = \emptyset$, $e \cup f = VX$ and $f \subset e$, respectively.

Theorem 1. *If a graph has a non-trivial cut then it also has a structure cut.*

This important theorem was originally stated by DUNWOODY in [4], Theorem 1.1. It is also an implication of a more general theorem in [2].

Definition 2. Let e and f be cuts in a tree set E. We say *e points to f* (notation $e \gg f$), if f is a subset of e and there is no third cut $d \in E$, such that $f \subset d \subset e$.

The ramifications that can be described by a tree set can always be represented by a tree which is called a *cut tree*. We want to give an axiomatic definition.

Definition 3. A *cut tree* of a tree set E is a connected directed tree $T = T(E)$, for which there exists a bijection $b : E \to ET$ with the following properties:

(T1) $b(e) = (u, v)$ is equivalent to $b(e^*) = (v, u)$ and
(T2) $e \gg f$ is equivalent to $t(b(e)) = o(b(f))$

where $o(p)$ is the origin and $t(p)$ is the terminus of a directed edge p. If E is a structure tree set then $T = T(E)$ is called a *structure tree*.

Note that (T1) an (T2) imply that $e \gg f$ is also equivalent to $t(b(f^*)) = o(b(e^*))$. To avoid complicated notation we will not distinguish between a cut e in E and the corresponding edge $b(e)$ in ET.

Theorem 2. *To every undirected tree set E there exists a cut tree $T = T(E)$ which is unique up to isomorphism.*

The existence of cut trees for a given tree set was proved in [3], Theorem 2.1. Various examples of structure trees can be found in [12], Section 2.3.

The following lemma is a generalization of a statement of DUNWOODY in [4], 2.3. THOMASSEN [13], Proposition 4.1 found a surprisingly simple proof by induction.

Lemma 1. *For every given natural number n and every edge p in a connected graph X there exist only finitely many tight n-cuts e such that p is element of the edge-boundary δe.*

Corollary 1. *Every strictly decreasing sequence of tight n-cuts whose intersection is non-empty must be finite.*

The edge-boundaries of the cuts in a tree set can be imagined as level lines of a topographic map or as pairwise disjoint Jordan curves in the plane, such that the two connected components in the complement of a curve correspond to cuts e and e^*. A cut tree T can be interpreted in the following way: VT is the the set of connected components in the complement of the union of all these Jordan curves. Two vertices in VT are adjacent if and only if their boundaries are not disjoint.

3 Edge-ends

A *ray* is a sequence $(x_n)_{n\in\mathbb{N}}$ of pairwise distinct vertices such that x_n is adjacent to x_{n+1} for all n. We write RX for the set of all rays in X. A ray *lies* in a set e of vertices or is *contained* in e, if e contains all but finitely many elements of the ray. Sometimes we will use the terms *contain* and *lie* at the same time in the sense above as well as in the sense of set theoretic inclusion. A set e of vertices *separates* two sets of vertices or rays, if one of them lies in e and the other lies in e^*.

Two rays are called *edge-equivalent* if they cannot be separated by an *edge-cut*. It is easy to see that this relation is an equivalence relation. Its equivalence classes are called *edge-ends of the first type*.

An end *lies* in a set of vertices e or is *contained* in e, if all of its rays lie in e. The set of edge-ends of the first type that lie in e is denoted by $\Omega_A e$. We write $\Omega_A X$ instead of $\Omega_A VX$. In fact, an edge-end of the first type ω lies in an edge-cut e if and only if one of its rays lies in e. So ω either lies in e or in e^*.

Lemma 2. *For a graph X, the set*

$$BX = \{e \cup \Omega_A e \mid e \subset VX \text{ and } |\delta e| < \infty\}$$

is closed under finite intersections.

For a proof see e.g. [9], Lemma 2. BX is a base of a topological space $(VX \cup \Omega_A X, \tau_A X)$ whose topology $\tau_A X$ is called *edge-topology of the first type*. By Theorem 2 and Example 3 in [9] we know that $\tau_A X$ is compact but in general not even T_0. For graphs with countably finite degree this compactness can easily be deduced from results of CARTWRIGHT, SOARDI and WOESS in [1].

To obtain better properties of separation we will now extend the edge-equivalence to $RX \cup VX^{\infty}$. This strategy was first adopted in the article [1] mentioned above.

Two elements of $RX \cup VX^{\infty}$ are called *edge-equivalent* if for every edge-cut e either both lie in e or both lie in e^*. Again it is easy to see that this relation is an equivalence-relation. We call its equivalence-classes *edge-ends* or *edge-ends of the second type*. The terms *to lie in* and *separate* are used in the same sense as before. The set of all ends lying in some set of vertices e is denoted by Ωe. We usually write ΩX instead of ΩVX.

In every edge-end of the second type containing some ray, there lies an edge-end of the first type. But note that there also may exist ends of the second type consisting only of vertices. In [1] these ends are called *improper ends*.

By the same construction as in the first case we now obtain the *edge-topology of the second type*. It is normal, Lindelöff and totally disconnected, see [9]. Compactness can be deduced from the compactness of the edge-topology of the first type.

4 Vertex and end structure mapping

To describe the connections between a graph X and its structure trees $T = T(E)$ we now want to define functions $\phi : VX \to VT$ and $\Phi : \Omega X \to \Omega T$. Another construction of ϕ can be found in [2]. In [11], Proposition 1 MÖLLER constructed a bijection between the end spaces of two arbitrary locally finite quasi-isometric graphs which is similar to our function Φ.

Definition 4. Let $T = T(E)$ be a cut tree of a graph X. A cut e in E *points at* some vertex x in VX (notation: $e \to x$), if x is an element of e and there is no other cut which contains x and is a subset of e.

Lemma 3. *Let x be a vertex in VX and E a tight tree set. Then there exists a cut $e \in E$ such that $e \to x$. The cuts that point at x, seen as edges in T, have all the same terminal point.*

Proof. Since a tight tree set is undirected there must exist a cut e_1 in E that contains x. By Corollary 1 every decreasing sequence $(e_n)_{n \geq 1}$ of cuts in E containing x must be finite. The last cut in such a sequence of maximal length must point at x.

Suppose that there are two cuts e_1 and e_2 in E, such that $e_1 \to x$, $e_2 \to x$ and $t(e_1) \neq t(e_2)$. By Axiom S1 of the definition of a tree set we distinguish between four cases. $e_1 \subset e_2$, $e_2 \subset e_1$ and $e_1 \cap e_2 = \emptyset$ would immediately imply a contradiction. If $e_1 \cup e_2 = VX$ then there must exist some $f \in E$ such that $e_1^* \subset f \subset e_2$ since $t(e_1) \neq t(e_2)$. If $x \in f$ then e_2 does not point at x, if x does not lie in f then e_1 does not point at x. □

Definition 5. Define ϕ as the function $VX \to VT$ such that for any $x \in VX$ $\phi(x) = t(e)$ for a cut $e \in E$ which points at x. We call ϕ the *vertex structure mapping* with respect to T.

By Lemma 3 the definition of the vertex structure mapping is well-defined since $\phi(x)$ is independent of the choice of the cut e which points at x.

Definition 6. In a similar way we now want to construct the *end structure mapping* $\Phi : \Omega X \to VT \cup \Omega T$. For each $\omega \in \Omega X$ there are two cases.

1. For every cut $e \in E$ containing ω there exists another cut $f \in E$ in which ω is contained and for which $e \gg f$. In this case any decreasing sequence of cuts in E containing ω defines a unique end $\varepsilon \in \Omega T$ and we set $\Phi(\omega) = \varepsilon$.
2. The end ω lies in some $e \in E$ but in no further cut in E which is contained in e. We set $\Phi(\omega) = t(e)$. The uniqueness of $t(e)$ can be seen by the same arguments that we used in the second part of the proof of Lemma 3.

Lemma 4. ([10], Lemma 2) *For a tight cut tree $T(E)$ the restriction of Φ to $\Phi^{-1}(\Omega T)$ is bijective.*

5 The action of Aut(X) on a structure tree

For a vertex x in VX and a structure tree set E we define

$$N(x) = \{e \in E \mid e \to x\}.$$

If g is an automorphism of X then we have

$$gN(x) = g\{e \in E \mid e \to x\} = \{ge \in E \mid e \to x\}.$$

Since the cut e points to x if and only if ge points to gx, this set is equal to

$$\{ge \in E \mid ge \to gx\} = \{f \in E \mid f \to gx\} = N(gx).$$

The images $g\phi^{-1}\phi(x)$ and $\phi^{-1}\phi g(x)$ are the sets of all vertices pointed at by cuts in $gN(x)$ or $N(gx)$, respectively. We now define a function

$$\bar{g}^T : \phi(VX) \to \phi(VX),\ v \mapsto \phi g \phi^{-1}(v).$$

By the above considerations we obtain

$$\bar{g}^T \phi(x) = \phi g \phi^{-1}\phi(x) = \phi\phi^{-1}\phi g(x) = \phi g(x).$$

For all $x \in VX$ and $v \in \phi(VX)$ we now have the following formulas

$$\begin{aligned} g\phi^{-1}\phi(x) &= \phi^{-1}\phi g(x), \\ \phi^{-1}\bar{g}^T(v) &= g\phi^{-1}(v), \\ \bar{g}^T\phi(x) &= \phi g(x). \end{aligned} \tag{5.1}$$

Thus $\bar{g}^T$ is a well defined function which is induced in a natural way by the automorphism g of X.

Every tree is bipartite. For connected trees the classes of the corresponding bipartition of the set of vertices are uniquely defined. We call them *classes of the bipartition of T*.

If we assume that $\phi(VX)$ does not cover the whole set of vertices VT then $\phi(VX)$ is one of the two classes of the bipartition of T. For any two ϕ-images $\phi(x)$ and $\phi(y)$ at distance 2 in T we can find cuts e and f in ET such that $e \to x$, $f \to y$, $e^* \gg f$ and $f^* \gg e$. There also exist cuts with this property for the vertices $g(x)$ and $g(y)$ in VX. By 5.1 we now obtain

$$d_T(\phi(x), \phi(y)) = d_T(\phi g(x), \phi g(y)) = d_T(\bar{g}^T\phi(x), \bar{g}^T\phi(y)) = 2.$$

Since T is a tree this implies the following

Lemma 5. *For all pairs of vertices x and y in VX*

$$d_T(\phi(x), \phi(y)) = d_T(\phi g(x), \phi g(y)) = d_T(\bar{g}^T\phi(x), \bar{g}^T\phi(y)).$$

The function $\bar{g}^T$ now can easily be extended to a bijective isometry on the whole set of vertices VT. This automorphism of T is denoted by g^T.

If $\phi(VX) = VT$ we can see by the same arguments that $\bar{g}^T$ itself is already an automorphism of T. In this case we define $g^T = \bar{g}^T$.

The set

$$\mathrm{Aut}^T(X) = \{g^T \mid g \in \mathrm{Aut}(X)\}$$

of these functions acts transitively at least on the classes of the bipartition of T. It acts transitively on the whole structure tree if and only if there exist cuts in $\mathrm{Aut}(X)e$ as well as in $\mathrm{Aut}(X)e^*$ that both point at some vertex x in VX.

To see that in the general case the function

$$L : \mathrm{Aut}(X) \to \mathrm{Aut}(T) : g \mapsto g^T$$

is neither surjective nor injective we give the following example.

Example 1. At each vertex of the cycle $C_4 = (v_1, v_2, v_3, v_4)$ of length 4 we fix a pair of hanging edges. The unions of the vertices in these pairs of hanging edges and their complements constitute a tree set E with eight elements. $T(E)$ is isomorphic to the star $K_{1,4}$. Its vertex of degree 4 is denoted by v. Instead of pairs of oppositely oriented edges in ET we draw undirected edges. See Figure 1.

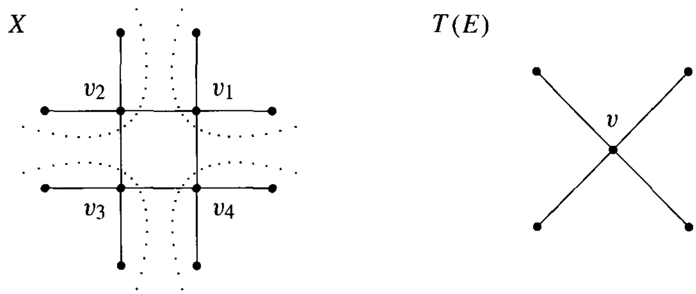

Figure 1

Each permutation of $VT\backslash\{v\}$ corresponds to an automorphism of the tree T, whereas automorphisms of X must respect the structure of the cycle C_4. Thus L is not surjective.

Automorphisms that have the same action on C_4, but different action on the vertices of degree one in VX are all mapped to the same automorphism of T. This means that the operator L is not injective.

Lemma 6. *For vertices v and w in $\phi(VX)$ and an automorphism g in* Aut(X) *we have*

$$g^T(v) = w \iff g\phi^{-1}(v) = \phi^{-1}(w).$$

Proof. By (5.1) $g\phi^{-1}(v) = \phi^{-1}(w)$ is equivalent to $\phi^{-1}g^T(v) = \phi^{-1}(w)$. □

Lemma 7. *For any two vertices v and w lying in the same class of the bipartition of T we have*

$$\mathrm{diam}_X\, \phi^{-1}(v) = \mathrm{diam}_X\, \phi^{-1}(w).$$

Proof. If these ϕ pre-images are non-empty there exists a $g^T \in \mathrm{Aut}^T(X)$ for which $g^T(v) = w$. The statement now is a consequence of $g\phi^{-1}(v) = \phi^{-1}(w)$. □

6 Regions

Definition 7. For a vertex v in a structure tree T we write $N(v)$ for the set of all cuts e in ET with $t(e) = v$. We write $N(v)^*$ for the set of all cuts f for which f^* is an element of $N(v)$. The set

$$\mathrm{Reg}(v) = \phi^{-1}(v) \cup \{x \mid x \in I\theta e, e \in N^*(v)\} = \phi^{-1}(v) \cup \{x \mid x \in \theta e, e \in N(v)\}$$

is called the *region of* v.

If $\phi^{-1}(v)$ is nonempty and $x \in \phi^{-1}(v)$ then we set $N(x) = N(v)$. Note that

$$N(v)^* = \{e \in E \mid o(e) = v\}.$$

Lemma 8.

$$\operatorname{Reg}(v) = \phi^{-1}(v) \cup \bigcup_{\substack{e \in N(v) \\ \{x,y\} \in \delta e}} \{x, y\} = \phi^{-1}(v) \cup \bigcup_{\substack{e \in N^*(v) \\ \{x,y\} \in \delta e}} \{x, y\}$$

Proof. It is clear that

$$\begin{aligned}\operatorname{Reg}(v) &= \phi^{-1}(v) \cup \{x \mid x \in I\theta e, e \in N^*(v)\} \\ &\subset \phi^{-1}(v) \cup \{x \mid x \in I\theta e, e \in N^*(v)\} \cup \{x \mid x \in \theta e, e \in N^*(v)\} \\ &= \phi^{-1}(v) \cup \bigcup_{\substack{e \in N(v) \\ \{x,y\} \in \delta e}} \{x, y\}.\end{aligned}$$

Let $x \in \theta e$ for some $e \in N^*(v)$. Either $\phi(x) = v$ or there is cut $f \in N^*(v)$ which points at $\phi(x)$. In the latter case x must lie in $I\theta f$ because x is adjacent to a vertex in e and, since e and f point away from v, $e \cap f = \emptyset$. □

Lemma 9. *For any two vertices v and w in the same class of the bipartition of T,*

$$\operatorname{diam}_X \operatorname{Reg}(v) = \operatorname{diam}_X \operatorname{Reg}(w).$$

Proof. We can find an automorphism $g \in \operatorname{Aut}(X)$ such that $g(v) = w$ and

$$\operatorname{Reg}(w) = \operatorname{Reg}(g^T(v)) = g \operatorname{Reg}(v).$$

□

Lemma 10. *Let v be a vertex in VT. Then*

$$\operatorname{diam}_T \phi \operatorname{Reg}(v) < \infty.$$

Proof. Let e be a cut in the structure tree set E of the structure tree T and $\{x, y\} \in EX$. By Lemma 1 there exist only finitely many cuts $e \in E$ which separate x from y. Since δe is finite and $\operatorname{Aut}(X)$ has not more than two orbits on E there exists a number k such that the vertices of any edge $\{x, y\}$ in the boundary of some cut $e \in E$ are separated by at most k cuts in E, which implies

$$d_T(\phi(x), \phi(y)) \leq k.$$

Let e be a cut which points at v. Seen as edge in T the cut e is an ordered pair (w, v). Let $\{x, y\} \in \delta e$ such that x lies in $I\theta e$ and y lies in θe. If we remove the edges $e = (w, v)$ and $e^* = (v, w)$ from the tree T then $\phi(y)$ lies in the same connected component as v, whereas $\phi(x)$ lies in another component.

Thus every geodesic path from $\phi(x)$ to $\phi(y)$ must contain v. Since

$$d_T(\phi(x), \phi(y)) \leq k$$

for any edge $\{x, y\}$ in any boundary of a cut which points at v we have

$$\phi \operatorname{Reg}(v) \subset B_T(v, k)$$

and

$$\operatorname{diam}_T \phi \operatorname{Reg}(v) \le 2k.$$

□

7 Quasi-isometries

Definition 8. Two metric spaces (M, d_M) and (N, d_N) are called *quasi-isometric with respect to the functions* $\phi : M \to N$ *and* $\psi : N \to M$ if there exist constants a_1, a_2, b_1, b_2, c and d such that for all x, x_1 and x_2 in M and y, y_1 and y_2 in N, the following conditions hold.

(Q1)' $d_N(\phi(x_1), \phi(x_2)) \le a_1 \cdot d_M(x_1, x_2) + a_2$ (boundedness of ϕ)
(Q2)' $d_M(\psi(y_1), \psi(y_2)) \le b_1 \cdot d_N(y_1, y_2) + b_2$ (boundedness of ψ)
(Q3) $d_M(\psi\phi(x), x) \le c$ (quasi-injectivity of ϕ)
(Q4) $d_N(\phi\psi(y), y) \le d$ (quasi-surjectivity of ϕ)

We call ϕ and ψ *quasi-isometries.* They are said to be *quasi-inverse* to each other.

Lemma 11. *Let (M, d_M) and (N, d_N) be metric spaces such that all positive values of d_M and d_N are greater than a positive constant α. Then the functions $\phi : M \to N$ and $\psi : N \to M$ satisfy Axioms (Q1)' and (Q2)', respectively, if and only if they are Lipschitz continuous.*

Proof. Let (Q1)' be satisfied and let $x_1 \neq x_2$. Then

$$d_N(\phi(x_1), \phi(x_2)) \le a_1 \cdot d_M(x_1, x_2) + a_2 \le (a_1 + \tfrac{a_2}{\alpha}) \cdot d_M(x_1, x_2).$$

Analogously (Q2)' implies a Lipschitz inequality with constant $b_1 + \frac{b_2}{\alpha}$. □

We say that two graphs X and Y are *quasi-isometric* if the metric spaces (VX, d_X) and (VY, d_Y) are quasi-isometric.

Corollary 2. *Two graphs X and Y are quasi-isometric if there are functions $\phi : VX \to VY$ and $\psi : VY \to VX$ and constants a and b such that*

(Q1) $d_N(\phi(x_1), \phi(x_2)) \le a \cdot d_M(x_1, x_2)$ *and*
(Q2) $d_M(\psi(y_1), \psi(y_2)) \le b \cdot d_N(y_1, y_2)$

for all x_1 and x_2 in VX and all y_1 and y_2 in VY, and the Axioms (Q3) and (Q4) are satisfied.

Quasi-isometry is an equivalence relation on the family of all graphs. Various examples of quasi-isometric graphs can be found in [9], Example 6.

Bounded geometry is an invariant under quasi-isometry. For graphs with bounded geometry transience and recurrence of the simple random walk, amenability, growth and number of ends are properties which are invariant under quasi-isometry.

Quasi-isometries between non-locally-finite graphs can be extended continuously to the so-called *metric end compactification* such that its restriction on the space of ends is a homeomorphism, see [9], Section 7.

More informal one could say that quasi-isometries may change local structures as long as the differences between the graph and its image can be bounded uniformly. They preserve the global structure of graphs.

The following lemma describes an important property of quasi-isometries.

Lemma 12. *Let $\phi : VX \to VY$ be a quasi-isometry and A a subset of VX. Then*

$$\operatorname{diam}_X A < \infty \Leftrightarrow \operatorname{diam}_Y \phi(A) < \infty.$$

8 A general criterion for ϕ being a quasi-isometry

The following extends a result of MÖLLER, Lemma 1 in [11], from locally finite graphs to arbitrary graphs.

Theorem 3. *A connected graph is quasi-isometric to a structure tree by the vertex structure mapping if and only if the regions of origin and terminus of an edge (equivalently: all edges) in the tree are bounded.*

Proof. By Lemma 10 $\operatorname{diam}_T \phi \operatorname{Reg}(v)$ is finite for all $v \in VT$. If ϕ is a quasi-isometry then Lemma 12 implies that $\operatorname{diam}_X \operatorname{Reg}(v)$ is finite.

To prove the theorem we have to show that ϕ is a quasi-isometry if $\operatorname{diam}_X \operatorname{Reg}(v)$ is finite. First we construct a function $\psi : VT \to VX$ which will turn out to be a quasi-inverse of ϕ.

If a vertex v in VT is not contained in $\phi(VX)$ then let $\operatorname{Reg}(v)$ be $\{w\}$ for any vertex w in VT which is adjacent to v. Otherwise we set $\operatorname{Reg}(v) = \{v\}$. Let $\psi(v)$ be an arbitrary element of $\phi^{-1}(\operatorname{Reg}(v))$. The function ψ is defined on the whole set of vertices VT. We have $\phi\psi(v) = \operatorname{Reg}(v)$ and $d_T(\phi\psi(v), v) \leq 1$ for all $v \in VT$ which implies Axiom (Q4). To prove that ϕ and ψ are quasi-isometries which are quasi-inverse to each other we check the remaining Axioms (Q1) to (Q3).

(Q1): If there is no cut e in E which separates two adjacent vertices x and y then $\phi(x) = \phi(y)$. Otherwise for any $e \in E$ there must be an automorphism $g \in \operatorname{Aut}(X)$ such that $g(\{x, y\})$ is in the edge-boundary δe. By Lemma 5 and since δe is finite the distance $d_T(\phi(x), \phi(y))$ can only have finitely many values for adjacent vertices x and y. Thus the set

$$\{d_T(\phi(x), \phi(y)) \mid d_X(x, y) = 1\}$$

has a maximal element a. For any two vertices x and y in VX there is a path $\{x = z_0, z_1, \ldots, z_n = y\}$ in X of length $d_X(x, y)$ and we obtain

$$\begin{aligned} d_T(\phi(x), \phi(y)) &\leq d_T(\phi(x = z_0), \phi(z_1)) + \cdots + d_T(\phi(z_{n-1}), \phi(z_n = y)) \\ &\leq a \cdot d_X(x, y). \end{aligned}$$

(Q2): <u>Case 1.</u>: $\phi(VX) \neq VT$

Let v_1 and v_2 be two vertices in $\phi(VX)$ with $d_T(v_1, v_2) = 2$ and let w be the vertex in $VT \backslash \phi(VX)$ which is adjacent to v_1 and v_2. Then

$$\begin{aligned} &\operatorname{diam}_X \phi^{-1}(v_1) + \operatorname{diam}_X \phi^{-1}(v_2) + d_X(\phi^{-1}(v_1), \phi^{-1}(v_2)) \leq \\ &\qquad \operatorname{diam}_X \phi^{-1}(v_1) + \operatorname{diam}_X \phi^{-1}(v_2) + \operatorname{diam}_X \operatorname{Reg}(w). \end{aligned}$$

By Lemma 7 and Lemma 9 the latter sum does not depend on the choice of the vertices v_1 and v_2. Thus it is a constant which we denote by $2\bar{b}$. For two vertices x and y in VX and a path

$$(v_0 = \phi(x), v_1, v_2, \ldots, v_{2k} = \phi(y))$$

of length $d_T(\phi(x), \phi(y)) = 2k$ such that $v_{2i} \in \phi(VX)$ for $0 \leq i \leq k$ we have

$$\begin{aligned} d_X(x, y) &\leq \sum_{i=0}^{k-1} \Big(\operatorname{diam}_X \phi^{-1}(v_{2i}) + \operatorname{diam}_X \phi^{-1}(v_{2(i+1)}) + \\ &\qquad d_X\big(\phi^{-1}(v_{2i}), \phi^{-1}(v_{2(i+1)})\big)\Big) \\ &\leq k \cdot 2\bar{b} = \bar{b} \cdot d_T(\phi(x), \phi(y)). \end{aligned}$$

Let $\psi(v)$ and $\psi(w)$ be any vertices in $\psi(VT)$. Then

$$d_X(\psi(v), \psi(w)) \leq \bar{b} \cdot d_T(\phi\psi(v), \phi\psi(w)).$$

Since $d_T(\phi\psi(v), v) \leq 1$ for all $v \in VT$

$$\bar{b} \cdot d_T(\phi\psi(v), \phi\psi(w)) \leq \bar{b} \cdot (d_T(v, w) + 2).$$

If $\psi(v) = \psi(w)$ then (Q2) is satisfied trivially. Otherwise v is not equal w and

$$\bar{b} \cdot (d_T(v, w) + 2) \leq 3\bar{b} \cdot d_T(v, w)$$

and $b = 3\bar{b}$ satisfies (Q2).
Case 2.: $\phi(VX) = VT$

In this case the proof of Case 1 works analogously by using the inequality

$$\begin{aligned} &\operatorname{diam}_X \phi^{-1}(v) + \operatorname{diam}_X \phi^{-1}(w) + d_X(\phi^{-1}(v), \phi^{-1}(w)) \\ &\qquad \leq \operatorname{diam}_X \operatorname{Reg}(v) + \operatorname{diam}_X \operatorname{Reg}(w) \end{aligned}$$

for any adjacent vertices v and w in VT.
(Q3): A vertex x in VX and the vertex $\psi\phi(x)$ always lie in the same ϕ-pre-image of some vertex in VT. By defining

$$c = \max\{\operatorname{diam}_X \phi^{-1}(t(e)), \operatorname{diam}_X \phi^{-1}(t(e^*))\}$$

for any $e \in E$ we obtain $d_X(\psi\phi(x), x) \leq c$ for all vertices x in VX. □

Since we did not use the assumption $\operatorname{diam}_X \operatorname{Reg}(v) < \infty$ in verifying Axiom (Q1) we have proved the following lemma.

Lemma 13. *For a structure tree T there exists a constant a such that*

$$d_T(\phi(x), \phi(y)) \leq a \cdot d_X(x, y)$$

for all vertices x and y in VX.

Lemma 14. *If $\phi^{-1}(v)$ is non-empty for some vertex v of a structure tree $T = T(E)$ of a graph X, then* Reg(v) *has finite diameter if and only if $\phi^{-1}(v)$ has finite diameter.*

Proof. Since $\phi^{-1}(v)$ is a subset of $\mathrm{Reg}(v)$ we need only prove that $\mathrm{diam}_X \mathrm{Reg}(v)$ is finite if $\phi^{-1}(v)$ has finite diameter. By the definition of a structure tree set the stabilizer $\mathrm{Aut}_{\phi^{-1}(v)}(X)$ of the set $\phi^{-1}(v)$ has at most two orbits O_1 and O_2 on $N(v)^*$. For every cut e in O_1 the set

$$\{d_X(x, \phi^{-1}(v)) \mid x \in I\theta e\}$$

has a maximal element. The same holds for the orbit O_2. The larger of these two maxima is the maximal distance between a vertex in $\mathrm{Reg}(v)$ and a vertex in $\phi^{-1}(v)$. Since $\phi^{-1}(v)$ has finite diameter the same must hold for $\mathrm{Reg}(v)$. □

The following example shows that for a connected graph and its structure tree to be quasi-isometric, it is not enough to have finite diameters of the ϕ-pre-images.

Example 2. For the two-sided infinite line L with $VL = \{x_k \mid k \in \mathbb{Z}\}$ the set

$$E_L = \left\{\{x_k\}, \{x_k\}^* \mid k \in \mathbb{Z}\right\}$$

is a structure set. The corresponding structure tree T looks like a star with one vertex v of infinite degree and infinitely many vertices of degree one. The ϕ-pre-images of the vertices with degree one consist of one vertex whereas $\phi^{-1}(v)$ is empty. All these pre-images have finite diameter but $\mathrm{Reg}(v)$ equals VL, and therefore it has infinite diameter. Thus L and T are not quasi-isometric to each other.

9 Uniform ramification

For locally finite graphs X the following properties are equivalent:
(a) VX is infinite, (b) X has infinite diameter and (c) X contains a ray of infinite diameter.

In the non-locally-finite case we have

$$\text{(c)} \Rightarrow \text{(b)} \Rightarrow \text{(a)}$$

but none of inverse implications is true in general. See Example 3.

Definition 9. For a set of vertices B in VX we write $\mathcal{C}(B)$ for the set of all connected components of B^* and $\mathcal{C}_0(B)$ for the set of all components in B^* with finite diameter. A *star ball* in a graph X is a ball S such that

$$\sup\{\mathrm{diam}_X C \mid C \in \mathcal{C}_0(S)\} = \infty.$$

A graph has *uniform ramification* if it is connected, has infinite diameter and does not contain a star ball. A ray that does not contain infinite subsets of finite diameter is called *metric ray.*

Example 3. The star T in Example 2 is an infinite graph with finite diameter.

Let $\{P_n \mid n \in \mathbb{N}, \mathrm{diam}\, P_n = n\}$ be a set of disjoint paths. We join together the initial vertices of these paths and obtain a graph X with a vertex x of infinite degree. The diameter of X is infinite. There does not exist any ray but any ball with centre x is a star ball. See Figure 2. This example can also be found in [9], Example 1.

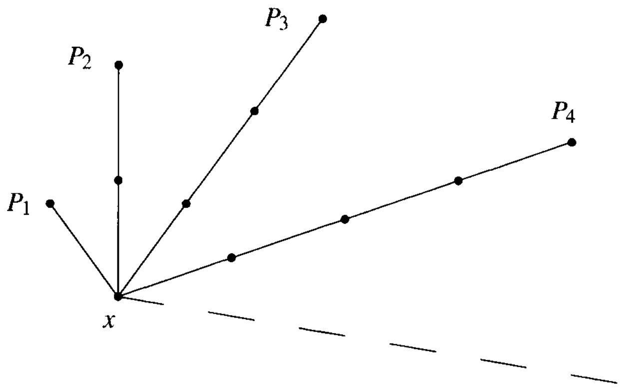

Figure 2

The following lemma, which is a slightly modified version of Lemma 1 in [9], characterizes graphs with infinite diameter. A stronger version of his Lemma will be Lemma 18. In [7] HALIN characterizes rayless graphs in a similar way but without taking their metric into consideration.

Lemma 15. *The diameter of a non-locally-finite graph is infinite if and only if it contains a metric ray or a star ball.*

Lemma 16. *Every ball B which contains some star ball S is a star ball.*

Proof. Let S be the ball $B_X(z, r)$. In $\bigcup \mathcal{C}_0(S)$ there is a sequence $(x_n)_{n>r}$ of vertices such that $d_X(x_n, z) = n$. Since only finitely many vertices of the sequence lie in B and for all other elements of the sequence there exist components in $\mathcal{C}_0(B)$ which contain them, B must be a star ball, too. □

Lemma 17. *The complement B_1^* of every ball B_1 in a graph with uniform ramification contains a connected component C with infinite diameter. For every ball B_2 containing B_1 there is also a connected component in $C \setminus B_2$ which has an infinite diameter.*

Proof. A graph with uniform ramification has infinite diameter. If B_1^* consisted only of components with finite diameter then B_1 would be a star ball.

If there were only connected components with finite diameter in $C \setminus B_2$ then B_2 would be a star ball since $C \setminus B_2$ has infinite diameter. □

Lemma 18. *Let X be a graph with uniform ramification. Then every component of infinite diameter in the complement of a ball contains a metric ray.*

Proof. For some x_0 in VX let C_1 be a component of infinite diameter in $\mathcal{C}(\{x_0\})$. By induction we now choose a sequence $(C_n)_{n\in\mathbb{N}}$ of components having infinite diameter such that

$$C_n \in \mathcal{C}(B_X(x_0, n)) \text{ and } C_{n+1} \subset C_n.$$

The existence of such a sequence is a consequence of Lemma 17. Let x_1 be an arbitrary vertex in the inner vertex-boundary $I\theta C_1$. Since C_1 is connected we can find a path from x_1 to some vertex x_2 in $I\theta C_2$ that does not leave $C_1 \backslash C_2$. By induction we obtain a sequence of paths whose union is a metric ray. □

10 Almost transitive graphs

Usually almost transitive graphs are defined as graphs with only finitely many orbits of vertices under the action of the automorphism group. Our definition is based on the natural metric on a graph. It is equivalent to the definition above in the locally finite case but includes a bigger class of graphs in the non-locally-finite case.

Definition 10. The automorphism group Aut(X) of a graph X acts *almost transitively* on VX if there exists a vertex $x_0 \in VX$ and a constant $r \in \mathbb{N}$ such that $d_X(\mathrm{Aut}(X)x_0, x) \leq r$ for all vertices x in VX. The graph X is called *almost transitive* if Aut(X) acts almost transitively on VX.

Remark 1. A graph is almost transitive if and only if it contains a covering ball.

Lemma 19. *A graph X which is quasi-isometric to a structure tree by the vertex structure mapping ϕ is almost transitive.*

Proof. The set of automorphisms $\mathrm{Aut}^T(X)$ acts transitively on both classes of the bipartition of T. Every ball B in T with radius at least 2 is a covering ball of T. Since ϕ is a quasi-isometry the pre-image $\phi^{-1}(B)$, by Lemma 12, has finite diameter. Every ball in X containing $\phi^{-1}(B)$ is a covering ball. □

Lemma 20. *Every connected almost transitive graph X is ramifying uniformly.*

Proof. We assume that there is a star ball $B_X(z,n)$. Let r be a number such that $B_X(z,r)$ is a covering ball. In $\mathcal{C}_0(B_X(z,n))$ there is a component C containing a vertex $\bar{y}$ whose distance to $B_X(z,n)$ is greater then $2n+r$. Let y denote a vertex of the z-orbit such that $\bar{y}$ is an element of the covering ball $B_X(y,r)$. $B_X(y,n)$ is again a star ball which is now contained in C. Since C^* is connected it is completely contained in one of the components of $B_X(y,n)^*$. All other components of $B_X(y,n)^*$ are contained in C. Thus $B_X(y,n)$ cannot be a star ball. □

11 The general Φ-criterion

The arguments in the proof of the following lemma are similar to those of Theorem 6 in [9].

Lemma 21. *Let X be a graph which is quasi-isometric to a structure tree T by the vertex structure mapping ϕ and let ω be an end in ΩX.*

1. *If ω contains a ray with infinite diameter then $\Phi(\omega) \in \Omega T$.*
2. *If ω contains a vertex then $\Phi(\omega) \in VT$.*

Proof. If x and y are adjacent vertices in X then $\phi(x)$ and $\phi(y)$ have at most distance a in T, where a is the constant in Axiom (Q1) of the definition of quasi-isometry. By connecting the ϕ-images of adjacent vertices of a ray L_1 of infinite diameter in ω by geodesic paths whose lengths are at most the constant a we obtain a path P in T. By Lemma 12 its diameter is infinite. Again by Lemma 12 all d_T-balls in P, as a subgraph of T, contain at most finitely many ϕ-images of vertices in L_1. Since we have constructed P only with paths of length at most a, the subgraph P of T must be locally finite. By Lemma 15, P must contain some ray L_2. The end of L_2 has to be the Φ-image of ω.

By Corollary 1 there is no infinite decreasing sequence of cuts in a structure cut set with nonempty intersection. Thus an end containing a vertex in X cannot be mapped by Φ to an end of T. □

Definition 11. An end is called *thick* if it contains infinitely many disjoint rays. An end which is not thick is called *thin*. Let ΘX denote the set of thick ends in a graph X and ΔX the set of thin ends. An end that does only contain vertices and rays of finite diameter is called a *point end*. A *mixed end* contains a ray with infinite diameter and a vertex of infinite degree. All other ends are called *proper ends*. The set of point ends is denoted by $\Omega_0 X$, the set of mixed ends by $\Omega_1 X$ and the set of proper ends by $\Omega_2 X$. Furthermore we define

$$\Delta_2 X = \Delta X \cap \Omega_2 X,$$
$$\Theta_2 X = \Theta X \cap \Omega_2 X.$$

For the following observations it will not be necessary to distinguish between thick and thin ends in $\Omega_0 X$ and $\Omega_1 X$.

Example 4. A graph with one thin mixed end.

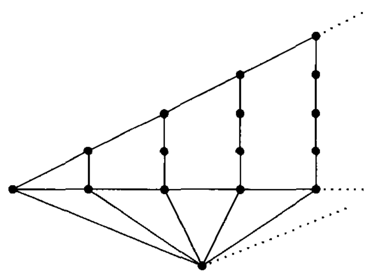

Figure 3

Lemma 22. *Proper ends only consist of rays with infinite diameter.*

Proof. We have to prove that a ray R which is not equivalent to any vertex has infinite diameter. Let C_0 be a cut containing R. Every vertex x in $I\theta C_0$ can be separated from R by a cut D_x. The set

$$C_1 = C_0 \setminus \bigcup \{D_x \mid x \in I\theta C_0\}$$

is again an edge-cut containing R. By induction we obtain a strictly decreasing sequence $(C_n)_{n\in\mathbb{N}}$ with empty intersection such that R lies in all the cuts C_n and the inner vertex-boundaries of these cuts are pairwise disjoint. The distance of a vertex in θC_0 to any vertex in C_n is greater than n. Since R lies in all cuts C_n it must have infinite diameter. □

For the end structure mapping Φ we define two properties:

(P1) $\Phi^{-1}(VT) = \Omega_0 X$

(P2) $\Phi^{-1}(VT) = \Omega_0 X, \quad \Omega_1 X = \emptyset, \quad \Theta_2 X = \emptyset, \quad$ and $\quad \Phi(\Delta_2 X) = \Omega T$

Theorem 4. *For a graph X with a structure tree $T(E)$ the following statements are equivalent.*

1. *X is quasi-isometric to T by the vertex structure mapping ϕ.*
2. *X is uniformly ramifying and has property* (P1).
3. *X is uniformly ramifying and has property* (P2).
4. *X is almost transitive and has property* (P1).
5. *X is almost transitive and has property* (P2).

Proof. The implications (3) $\Rightarrow$ (2) and (5) $\Rightarrow$ (4) are trivial. Lemma 20 implies (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3). We will now prove (1) $\Rightarrow$ (5) and (2) $\Rightarrow$ (1).
(1) $\Rightarrow$ (5): If X is quasi-isometric to T then, by Lemma 21, there cannot exist a mixed end in ΩX. Thick proper ends cannot be mapped to ΩT under Φ since thick ends cannot be described by a sequence of n-cuts. By Lemma 21 they also cannot be mapped to VT and therefore $\Theta_2 X = \emptyset$.

Another consequence of Lemma 21 is $\Phi(\Omega_0) \subset VT$. The only remaining ends with rays of infinite diameter are the thin proper ends. By Lemma 4 Φ is bijective on $\Phi^{-1}(\Omega T)$. Thus we have $\Phi(\Delta_2 X) = \Omega T$. Now it is also clear that $\Phi^{-1}(VT)$ equals $\Omega_0 X$.

By Lemma 19 X is almost transitive.
(2) $\Rightarrow$ (1): Assuming that $\Phi^{-1}(VT) = \Omega_0 X$ we want to prove that a graph with uniform ramification is quasi-isometric under the vertex structure mapping ϕ to a structure tree $T = T(E)$.

We suppose that there exists a vertex v in VT having a region $\text{Reg}(v)$ of infinite diameter. The stabilizer $\text{Aut}_v^T(X)$ of v lets the set of neighbours of v invariant. The group $L^{-1}(\text{Aut}_v^T(X))$ of the corresponding automorphisms in $\text{Aut}(X)$ acts on

$$N^*(v) = \{e \in E \mid o(e) = v\}$$

with at most two orbits O_1 and O_2, where L is the function defined in Section 5. For some cut $e \in O_1$ we now choose a finite and connected subgraph W_1 of X with $VW_1 \subset e$ so that W_1 connects all pairs of vertices in $I\theta e$ by paths of minimal length that do not leave e. Note that these paths are not necessarily geodesic in X. We define

$$\bar{W_1} = \bigcup_{g \in \text{Aut}_e(X)} gW_1$$

where $\text{Aut}_e(X)$ is the stabilizer of e in the group $\text{Aut}(X)$. $\bar{W_1}$ has finite diameter, because the distance of every vertex in W_1 to θe is at most $\text{diam}_X W_1$. For every $f \in O_1$ we now replace the restriction of X to f by an automorphic image of $\bar{W_1}$.

With the possibly existing orbit O_2 we proceed analogously. Thereby we obtain a connected subgraph $\bar{X}$ of X. We set

$$\bar{O}_i = \{Vg\bar{W}_i \mid g \in L^{-1}(\mathrm{Aut}_v^T(X))\} \quad i = 1, 2$$
$$\text{and} \quad \bar{O} = \bar{O}_1 \cup \bar{O}_2.$$

Let x and y be two vertices in $\mathrm{Reg}(v)$ and $P(x, y)$ a d_X-geodesic path connecting them. All parts of maximal length in $P(x, y)$ that are completely contained in one of the orbits O_1 and O_2 can be replaced by a path in an element of $\bar{O}$ of the same length. Thus we have

$$d_{\bar{X}}(x, y) = d_X(x, y)$$

for all pairs of vertices x and y in $\mathrm{Reg}(v)$.

The automorphism group $\mathrm{Aut}(\bar{X})$ generates not more than two orbits on $\bar{O}$. In order to prove the existence of a ray with infinite diameter, we now proceed analogously to the proof of Lemma 20. Assuming that there is no ray with infinite diameter in $\bar{X}$, by Lemma 15, there must exist some star ball $S = B_{\bar{X}}(z, r)$. By Lemma 16, S can be chosen so that it contains an element of both $\bar{O}_1$ and $\bar{O}_2$. There is no radius r_1 such that all sets in $\bar{O}$ which are contained in components of $\mathcal{C}_0(S)$ are subsets of $B_{\bar{X}}(z, r_1)$, because then, by Lemma 16, $B_X(z, r_1)$ would be a star ball in X, which would be a contradiction to the uniform ramification of X.

Let $\bar{w} \in \bar{O}$ be a set of vertices which is contained in a component C in $\mathcal{C}_0(S)$ such that its distance to S is at least $2r$. Since S contains an element of both orbits $\bar{O}_1$ and $\bar{O}_2$ there must exist an automorphism $g \in \mathrm{Aut}(\bar{X})$ such that $g(\bar{w})$ is contained in S. Now $g^{-1}(S)$ is completely contained in C. $V\bar{X}\backslash C$ is connected and therefore part of a component in $\mathcal{C}(g^{-1}(S))$. But this is impossible, because then $V\bar{X}\backslash g^{-1}(S)$ would have one component of infinite diameter, all other components would be contained in C and $g^{-1}(S)$ would not be a star ball in $\bar{X}$.

Thus, by Lemma 20, there exists a ray L with infinite diameter in $\bar{X}$. Since $d_{\bar{X}}(x, y) = d_X(x, y)$ for all vertices x and y in $\mathrm{Reg}(v)$, the ray L also has infinite diameter in X. The ray L has finite intersection with every cut in $N(v)^*$, because the intersection of these cuts with $V\bar{X}$ is finite. Hence the Φ-image of the end of L must be v. This is a contradiction to the condition $\Phi^{-1}(VT) = \Omega_0 X$.

We have now proved that, assuming $\Phi^{-1}(VT) = \Omega_0 X$, there is no vertex in VT whose region has infinite diameter. So, by Theorem 3, the graph X is quasi-isometric to its structure tree T. □

12 Another criterion for quasi-isometry

The stabilizer $\mathrm{Aut}_\omega(X)$ of an end ω in ΩX is the group of automorphisms in $\mathrm{Aut}(X)$ that map rays and vertices in ω onto rays and vertices in ω.

The following theorem was proved by Möller for locally finite graphs with infinitely many ends where the stabilizer $\mathrm{Aut}_\omega(X)$ acts transitively, see [11], Theorem 1.

Theorem 5. *Let X be a connected graph with a structure tree $T = T(E)$. If there is an end $\omega \in \Omega X$ such that the stabilizer $\mathrm{Aut}_\omega(X)$ acts almost transitively on X, then X is quasi-isometric to T by the vertex structure mapping ϕ.*

Proof. If $\mathrm{Aut}_\omega(X)$ acts almost transitively on X, then also $\mathrm{Aut}_\omega^T(X)$ must act almost transitively on T. Thus $\Phi(\omega)$ must be an end in ΩT.

First we prove that all ϕ-pre-images of vertices in VT have finite diameter. If there is a constant n_0 such that any two vertices in VX with d_X-distance at least n_0 have a different ϕ-image, then $\mathrm{diam}_X\, \phi^{-1}(v) \leq n_0$ for all vertices $v \in VT$. This is equivalent to the condition that for all pairs of vertices in VX with d_X-distance at least n_0 there exists a cut in E which separates them. Let $f \in ET$ be a cut containing ω and let $B_X(x_0, r_0)$ be a covering ball of X with respect to $\mathrm{Aut}_\omega(X)$. We define

$$M_f = \{x \in f^* \mid d_X(f, x) \leq 4r_0\}.$$

M_f has finite diameter. Let y be a vertex in M_f with $d_X(f, y) = 2r_0$. The ball $B_X(y, r_0)$ contains a vertex y_0 of the x_0-orbit with respect to $\mathrm{Aut}_\omega(X)$. As $B_X(y_0, r_0)$ is also a subset of M_f, we have

$$B_X(y_0, r_0) \subset B_X(y, 2r_0) \subset M_f.$$

We define $n_0 = 2\,\mathrm{diam}_X\, M_f$ and choose two arbitrary vertices x_1 and x_2 with distance larger then n_0. Since $\bigcup \mathrm{Aut}_\omega(X) M_f$ is the whole set of vertices VX, there is a cut e_1 containing ω for which $x_1 \in M_{e_1}$. If x_2 is an element of e_1 there is nothing more to prove because then e_1 is the desired cut which separates x_1 and x_2. So we suppose that x_2 is an element of e_1^*. Let e_2 be a cut that contains ω such that $x_2 \in M_{e_2}$. The edge-boundaries δe_1 and δe_2 are disjoint and M_{e_2} is a subset of e_1^*. Thus $e_1 \cup M_{e_1}$ must be a subset the component of e_2 which contains ω. Thus $x_1 \in e_2$ and $x_2 \in e_2^*$.

To prove the theorem we have to show that the region of any vertex $v \in VT$ has finite diameter. By Lemma 14 we just have to deal with the case $\phi^{-1}(v) = \emptyset$. Let

$$L = \{w_0 = v, w_1, w_2, \dots\}$$

be the ray which starts at v and lies in $\Phi(\omega)$. We furthermore define

$$U(v) = \{u \in VT \mid u \text{ adjacent to } v \text{ and } u \neq w_1\}.$$

The ϕ-pre-image of any vertex v_0 in $U(v)$ is non-empty and has finite diameter. Thus it is contained in some covering ball B with respect to $\mathrm{Aut}_\omega(X)$. Let M_0 be the set of all ϕ-pre-images of vertices in VT which have a non-empty intersection with B. The diameter of M_0 is finite. By Lemma 13 this also holds for $\phi(M_0)$. Let n be the smallest index such that no vertex in

$$\{w_n, w_{n+1}, w_{n+2}, \dots\}$$

lies in $\phi(M_0)$. We define

$$d = \mathrm{diam}_T\, \phi(M_0) \quad \text{and}$$
$$A = \{w_n, w_{n+1}, \dots, w_{n+2d}\}.$$

For every $u \in U(v)$ there exists an automorphism $g_u \in \mathrm{Aut}_\omega^T(X)$ with $g_u(u) \in \phi(M_0)$. Since such an automorphism g_u must fix $\Phi(\omega)$ and maps u to a vertex in VT which has a distance to u that is at most d, it causes a translation on L of maximal length d and therefore it must map w_{n+d} onto a vertex in A. Hence

$$\max\{d_X(\phi^{-1}(u), \phi^{-1}(w_{n+d})) \mid u \in U(v)\} \leq \max\{d_X(\phi^{-1}(v_0), \phi^{-1}(w)) \mid w \in A\} < \infty.$$

This implies

$$\operatorname{diam}_X \phi^{-1}(w_1) \cup \bigcup_{u \in U(v)} \phi^{-1}(u)) < \infty.$$

Since

$$\operatorname{Reg}(v) \subset \phi^{-1}(w_1) \cup \bigcup_{u \in U(v)} \phi^{-1}(u) = \phi^{-1}(B_T(v, 1))$$

we finally have

$$\operatorname{diam}_X \operatorname{Reg}(v) < \infty.$$

□

Example 5.

1. For a semi-regular tree T there is a unique structure tree set. The corresponding structure tree is isomorphic to T.
2. Let T again be be a semi-regular tree. By adding a graph Y of finite diameter to each vertex in one of the classes of the bipartition of T, such that this vertex is identified with any vertex in VY, we obtain a graph X which is quasi-isometric to T.
3. Let X be the Cayley graph of the free product

$$\mathbb{Z} * \mathbb{Z}^2 = \langle a, b, c \mid bc = cb \rangle$$

with generating set $\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$. This graph is 6-regular and transitive. By removing edges that correspond to the generating elements $a^{\pm 1}$ we obtain pairs of structure cuts. The structure tree T is regular of countably infinite degree. Φ maps thick ends to vertices and thin ends to thin ends. T and X are not quasi-isometric.

Taking $\{a^{\pm 1}\} \cup \mathbb{Z}^2$ as a generating system we obtain the Cayley graph $\bar{X}$. The ends in the copies of $\mathbb{Z}^2$ now are thick point ends. Again removing the edges that correspond to the generating elements $a^{\pm 1}$ we obtain a structure tree which is isomorphic to T. $\bar{X}$ and T are quasi-isometric, because Φ maps only point ends onto vertices in VT.

In [14] TROFIMOV gave an example of a graph whose automorphism group fixes an end and acts transitively on the set of vertices.

This article is based on Chapter 3 of the author's masters thesis [8] at the University of Salzburg under supervision of Prof. W. WOESS, and the author wants to thank him for many useful suggestions. The main part of this thesis was written during a stay at Milan supported by the Italian Ministry of Foreign Affairs.

References

[1] DONALD I. CARTWRIGHT, PAOLO M. SOARDI, and WOLFGANG WOESS, Martin and end compactifications for non-locally finite graphs, *Trans. Amer. Math. Soc.* **338** (2) (1993), 679–693.

[2] WARREN DICKS and MARTIN J. DUNWOODY, *Groups acting on graphs*. Cambridge University Press, Cambridge, 1989.

[3] MARTIN J. DUNWOODY, Accessibility and groups of cohomological dimension one, *Proc. London Math. Soc. (3)* **38** (2) (1979), 193–215.

[4] ———, Cutting up graphs, *Combinatorica* **2** (1) (1982), 15–23.

[5] MIKHAEL GROMOV, Infinite groups as geometric objects. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 385–392. Warsaw, 1984. PWN.

[6] ———, Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[7] RUDOLF HALIN, Graphen ohne unendliche Wege, *Math. Nachr.* **31** (1966), 111–123.

[8] BERNHARD KRÖN, Topologische Strukturtheorie nicht lokalendlicher Graphen, Master's thesis, Universität Salzburg, 1998.

[9] ———, End compactifications in non-locally-finite graphs. *Math. Proc. Cambridge Philos. Soc.* to appear.

[10] RÖGNVALDUR G. MÖLLER, Ends of graphs, *Math. Proc. Cambridge Philos. Soc.* **111** (2) (1992), 255–266.

[11] ———, Ends of graphs. II. *Math. Proc. Cambridge Philos. Soc.* **111** (3) (1992), 455–460.

[12] ———, Groups acting on locally finite graphs—a survey of the infinitely ended case. In *Groups '93 Galway/St. Andrews, Vol. 2*, pages 426–456. Cambridge Univ. Press, Cambridge, 1995.

[13] CARSTEN THOMASSEN and WOLFGANG WOESS, Vertex-transitive graphs and accessibility, *J. in. Theory Ser. B* **58** (2) (1993), 248–268.

[14] VLADIMIR I. TROFIMOV, Groups of automorphisms of graphs as topological groups, *Mat. Zametki* **38** (3) (1985), 378–385, 476. English translation: *Math. Notes* **38** (1985), no. 3-4, 717–720.

[15] JOHN R. STALLINGS, *Group theory and three-dimensional manifolds*. Yale University Press, 1971.

Eingegangen am: 15. Juli 2000

Author's address: B. Krön, Institut für Mathematik C, Technische Universität Graz, Steyrergasse 30, 8010 - Graz

E-Mail: `kroen@finanz.math.tu-graz.ac.at`.