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Infinite faces and ends of almost transitive plane graphs

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INFINITE FACES AND ENDS OF ALMOST TRANSITIVE PLANE GRAPHS

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Abstract. We prove a conjecture of Bonnington, Richter and Watkins [2] that in almost transitive, locally finite, plane graphs, disjoint tails of a facial double-ray lie in different ends. For the infinitely ended case we use the theory of structure trees and accessibility of almost transitive plane graphs.

As application we classify infinite faces of a locally finite, 2-connected, almost transitive, plane graphs with a topological embedding.

1. Introduction

1.1. **The main result.** Rays in graphs are 1-sided infinite paths (consisting of distinct vertices). Ends of graphs are equivalence classes of rays. Two rays are equivalent if there is a third ray who has infinitely many vertices in common with each of them. A plane automorphism of a plane graph is a graph automorphism which extends to a homeomorphism of the sphere. A plane graph is called *almost transitive in the sphere* if the group of plane automorphisms acts almost transitively (i.e., with finitely many orbits) on the vertices of the graph. A *facial double-ray* in clockwise (counter clockwise) direction is a 2-sided infinite path with vertices $\ldots, x_{-1}, x_0, x_1, \ldots$ such that x_{i+1} is the next neighbour of x_i that one encounters after x_{i-1} in counter clockwise (clockwise) direction about x_i. Bonnington, Richter and Watkins [2, Conjecture 2] conjectured that disjoint tails (i.e., subrays) of any facial double-ray in a locally finite almost transitive plane graph belong to different ends. In the present paper we prove this conjecture.

1.2. **Related literature.** Freudenthal defined ends of locally finite graphs in [16]. The definition goes back to his more general approach for connected, locally connected, locally compact Hausdorff space, see [14, 15]. A graph theoretic definition of ends as equivalence classes of rays was given by Halin in [17].

An infinite, connected, almost transitive, locally finite graph has either one or two ends or the end boundary is a Cantor set. This fact was first observed by Hopf in [19] in the more general context of Freudenthal ends. In [18] it is shown that in the one-ended case, the end of the graph is thick (contains infinitely many disjoint rays), whereas in the two-ended case the ends are thin.

The number of ends of a finitely generated group is defined as the number of ends of its finitely generated Cayley graphs. This number does not depend on the choice of the finite generating set. Stallings' Structure theorem says that a finitely generated infinite group G has more than one end if and only if it splits over a finite subgroup C, see [31]. That is, G is an HNN-extension or a nontrivial free product with amalgamation over C. If A or B has more than one end then this splitting process can be iterated. The group G is called *accessible* if this process stops after finitely many steps. For more details we refer to [10, 12, 26, 32]. In

[12], Dunwoody proved that finitely presented groups are accessible. Thomassen and Woess [32] have called a graph *accessible* if there is a number n such that every pair of ends can be separated by removing n edges. They showed that a finitely generated group G is accessible if and only if some (equivalently: any) finitely generated Cayley graph is accessible. Dunwoody has recently proved in [13] that all almost transitive plane graphs are accessible.

Let X be a graph with more than one end. The group of automorphisms of X can be lifted to an action on a so-called *structure tree* with the help of certain automorphism invariant partitions (*tree-sets*) of the graph. To prove the existence of these partitions is the difficult part of this theory. This was first done by Dunwoody in [11] and later developed to a more powerful theorem in [7, Theorem II.2.20]. The construction of structure trees was first treated in [10, Theorem 2.1], for further references see [7, 21, 24, 25, 32]. Structure tree theory will be one of the ingredients that we need to reduce the infinitely ended case to the one and two ended cases.

The paper [30] of Richter and Thomassen is important concerning topological properties of almost transitive plane graphs as subsets of the sphere. They showed that 3-connected planar graphs have a unique embedding. Moreover, let X be a 2-connected graph where the vertices of each ray in an end converge to the same point of the sphere. Then the faces of X are bounded by a closed curve. For details and precise definitions see below. The results of Richter and Thomassen will allow us to use specific topological arguments rather than just combinatorial graph theoretic arguments. These topological arguments will not only play an important role in the proof of the main result, they will also enable us to simplify proofs of results in [2].

Bonnington, Imrich and Watkins have studied locally finite, almost transitive, plane graphs in [4] by means of certain path separation properties. They showed that locally finite, 2-connected, almost transitive, plane graphs have no infinite faces.

Bonnington, Richter and Watkins [2] studied bundles of infinite graphs. Bundles are a refinement of ends. They could show that disjoint tails of facial double-rays lie in distinct bundles. And they conjectured that this is also true for ends. The proof of this conjecture is our main result.

Cayley graphs (group diagrams) are a rich source for interesting examples of infinite transitive plane graphs, see Bonnington and Watkins [3], Droms, Servatius and Servatius [8], Droms [9], Mohar [27, 28] and Renault [29].

1.3. **Contents and methodology.** In Section 2 we give basic definitions for graphs as 1-complexes and we define the end compactification $|X|$ of locally finite graphs.

A plane graph is a graph which is embedded in the sphere S^2. We will see in Section 3 that for infinite plane graphs, this embedding can be of different quality in a topological sense. A plane graph X is *accumulation free* if no point x of X is accumulation point of edges which do not contain x. A plane embedding of a graph is called *pointed* if the vertices of each ray in an end converge to the same point of the sphere. That is, every end corresponds to exactly one point in the sphere. Such a point will be called *projected end*. A *face* of a plane graph is a component of the complement of the closure of the graph as subset of the sphere. For us important is the result from [30] that the faces of 2-connected graphs with a pointed embedding are bounded by a closed curve, see Theorem 1. Finally, a plane graph X is *topological* if the identity on X extends to a homeomorphism between the closure $\overline{X}$ of X in S^2 and the end compactification $|X|$. In [30] it is proved that every plane, 2-connected, locally finite graph has a topological embedding, see Theorem 2.

In locally finite plane graphs, the edges which are incident with some vertex can be ordered in clockwise direction. With this order we define facial walks as mentioned above. In infinite plane graphs, there is not necessarily a one-to-one correspondence between facial walks and faces. It can happen that facial walks are not contained in the boundary of any face or that there are faces whose boundaries do not intersect the graph at all, see Example 1. In Section 4 we discuss the connection between facial paths and faces for 2-connected accumulation free plane graphs. In Theorem 4 we show that no two facial rays in an end have the same orientation. As corollary, this yields a theorem of Bonnington, Richter and Watkins (Theorem 5) that the number of facial double-rays in a locally finite 2-connected plane graph is less or equal the number of ends. If in addition the graph is accumulation free then the number of infinite faces is also less or equal the number of ends, see Theorem 6. The previously mentioned result from [30], that faces of locally finite 2-connected pointed plane graphs are bounded by a closed curve, enables us to derive a relatively simple topological proof of Theorem 5.

A graph is called *almost transitive* if the group of automorphisms acts almost transitively (i.e., with finitely many orbits) on the set of vertices. In Section 5 we discuss some elementary properties of almost transitive (not necessarily plane) graphs concerning ends and translations of double-rays.

The conjecture of Bonnington, Richter and Watkins [2, Conjecture 2] (see above) is formulated in Theorem 7. In Section 6 we prove this conjecture for the case of almost transitive graphs with one or two ends and give a detailed discussion of this fact in Theorem 8. On the one hand, we use specific arguments concerning almost transitive graphs with one or two ends where the graph metric plays an important role. Similar techniques are used in the theory of metric ends, see [20, 22, 23]. On the other hand, Theorem 5 (the number of infinite faces is less or equal the number of ends) plays an important role in the proof.

In Section 7 we state basic definitions and results on structure trees and accessibility. For us important will be the main theorem in structure tree theory from Dicks and Dunwoody [7] about the existence of certain partitions of the graph which are invariant under the action of all graph automorphisms, see Theorem 9. Another crucial tool will be is a recent result of Dunwoody: In [13] he proved that almost transitive locally finite plane graphs are accessible, see Theorem 10.

In Section 8 we prove the conjecture of Bonnington, Richter and Watkins for almost transitive, locally finite 2-connected graphs with infinitely many ends, and we give a more detailed discussion of this case in Theorem 11. The ends of the graph X are mapped to vertices or to ends of the structure tree by a function Φ in a natural way. Using [7, Theorem II.2.20], Thomassen and Woess showed in [32] that if X is accessible then there exists a structure tree such that Φ is injective. They constructed certain subgraphs X_v of X which correspond to vertices v of the structure tree. These subgraphs have three nice properties. First, if X is almost transitive then so is X_v. Second, every ray of an end in $\Phi^{-1}(v)$ has infinite intersection with X_v. Third, there is a bijection between $\Phi^{-1}(v)$ and the end space of X_v. In particular, if Φ is injective and $\Phi^{-1}(v)$ is not empty then it follows that X_v has one thick end. Hence by combining this construction with the accessibility of almost transitive plane graphs, we are able to trace back the infinitely ended case to the one-ended and two-ended case that we have treated before in Section 6. The proof for thin ends in infinite graphs uses the fact that facial double-rays in locally finite almost transitive graphs are translatable together with a well known result of Halin from [18]: Disjoint tails of such translatable double-rays lie in different ends if and only if these ends are thin.

What remains is the 1-connected case. In Section 9 we give a simple construction with the help of which we can turn a 1-connected almost transitive plane graph into a 2-connected almost transitive plane graph, see Theorem 12. We state some results on facial walks in 1-connected graphs which are analogues to previous results on 2-connected graphs, see Lemma 12 and Theorem 13, followed by a discussion of our main result for 1-connected graphs in Theorem 14.

Finally, in Section 10, we apply our results in order to classify faces of almost transitive locally finite plane graphs which have a topological embedding. The interesting case is the case of infinite faces in infinitely ended graphs. The boundary of such a face either consists of two double rays and two projected ends, or it is the union of countably many double-rays and a Cantor set consisting of projected ends.

2. Preliminaries

A *graph* X is a union $VX \cup EX$ of a *set of vertices* VX and a *set of edges* EX with the following properties.

(i) Edges are arcs of simple curves between distinct vertices whose parameter run through the unit interval.
(ii) Different edges have different sets of end-vertices.
(iii) The interior of an edge contains no vertex and no point of any other edge.

The two vertices which are contained in an edge are called *end-vertices* of the edge. The edge with end vertices x and y is denoted by $[x, y]$ or $[y, x]$. Vertices x and y are *adjacent* (or *neighbours*) if there is an edge $[x, y]$. A graph is *locally finite* if every vertex is only adjacent to finitely many other vertices. A *subgraph* of X is a graph Y such that $VY \subset VX$ and $EY \subset EX$.

A *walk* π *of length* n from a vertex x to a vertex y is a subgraph of the form

$$\{x = x_0\} \cup [x_0, x_1] \cup \{x_1\} \cup [x_1, x_2] \cup \ldots \cup [x_{n-1}, x_n] \cup \{x_n = y\} \tag{2.1}$$

where $x_i \in VX$ and $0 \le i \le n$. If $x_0 = x_n$ then π is called a *closed walk.* A *subwalk* of π is a walk $\{x_i\} \cup [x_i, x_{i+1}] \cup \{x_{i+1}\} \cup \ldots \cup [x_{j-1}, x_j] \cup \{x_j\}$. A *1-way* or *2-way infinite walk* is a walk with vertices x_i and edges $[x_i, x_{i+1}]$, for $i \in \mathbb{N}$ or $i \in \mathbb{Z}$, respectively. A 1-way infinite subwalk of an infinite walk π is called a *tail* of π.

Walks with distinct vertices are called *paths. Rays* and *double-rays* are 1-way infinite paths and 2-way infinite paths, respectively. Edges are arcs of simple curves whose parametrization induces a metric on the edges such that adjacent vertices have distance 1. If X is connected and the interiors of the edges are disjoint then the metric on the edges extends to a metric d on X, such that the *distance* $d(x, y)$ between two vertices x and y is the length of a shortest path from x to y. We call d the *graph metric* of X. When we talk about subgraphs Y of X then with d we always mean the distance with respect to underlying graph X and not with respect to Y. For a point x in X we define

$$d(x, Y) = \min\{d(x, y) \mid y \in Y\}.$$

Let $B(Y, r)$ denote the set $\{x \in X \mid d(x, Y) \leq r\}$. Note that if r is an integer and Y is a subgraph of X then $B(Y, r)$ is also a subgraph of X. If x is a vertex then $B(x, r) = B(\{x\}, r)$ is the *closed ball* with center x and radius r.

A *finite graph* is a graph with finitely many vertices. Let A, B and F be subgraphs of X. We say that F *separates* A from B if every path from a vertex in A to a vertex in B contains an element of F. *Ends* are equivalence classes of rays, where two rays are equivalent whenever there is a third ray who has infinitely many vertices in common with each of them. This is equivalent to saying that rays are equivalent if they cannot be separated by a finite subgraph.

Let C be a subset of X. The *vertex boundary* NC is the set of vertices in $X \setminus C$ which are adjacent to a vertex in C. A subgraph Y of X is said to *live in* a subgraph C of X if $VY \setminus C$ is finite. Let ω be an end and let NC be finite. Then either all rays in ω live in $X \setminus C$, or they all live in C. In the latter case, we say that ω *lives in* C. We write ΩC to denote the set of ends which live in C. A neighbourood of an end ω is a subset of $X \cup \Omega X$ that contains a set $C \cup \Omega C$ where C is a subgraph, NC is finite and ω lives in C. This extends the topology of X to $X \cup \Omega X$. If X is locally finite then the resulting topological space $|X|$ is metrizable and compact and ΩX is totally disconnected and compact. For more details we refer to [6].

In non-locally finite graphs there are various ways to topologize $X \cup \Omega X$, see [5, 20]. For locally finite graphs, all these concepts coincide with the Freudenthal compactification of connected, locally connected, locally compact Hausdorff space, see [14, 15, 16].

3. Plane graphs

A graph is *plane* if it is a subset of the sphere $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Let $\overline{X}$ denote the closure of X with respect to the topology of S^2. The connected components of $S^2 \setminus \overline{X}$ are called *faces* of X. Let z be an element of a plane graph X and let $E(z)$ be the union of edges which contain z. We call X *accumulation free* if no element z of X is in the closure of $X \setminus E(z)$. Equivalently, if no z is an accumulation point of $X \setminus E(z)$. Note that if z is such an accumulation point then z must be an accumulation point of infinitely many distinct edges, because edges are closed and the interiors of distinct edges are disjoint, see Example 1.

The following can be found in [30, Lemma 4]. For the convenience of the reader we give a simple direct proof.

Lemma 1 (Lemma 4 [30])**.** *The boundary of a face of a connected plane graph X is connected.*

Proof. Suppose there is a face f of a connected graph X whose boundary is disconnected. Any boundary is closed and S^2 is compact. Hence ∂f is compact. There is a simple closed curve c in $S^2 \setminus \partial f$ which separates components of ∂f. (To see this, consider an open cover of a component C of ∂f consisting of open balls whose closures are disjoint with $\partial f \setminus C$. The union of a finite subcover is bounded by a simple closed curve.) The complement $S^2 \setminus c$ consists of two open components A and B which both contain elements of ∂f. Hence A and B both contain elements of X as well as of f. The graph X is pathwise connected by its construction and f

is pathwise connected, because it is an open connected subset of S^2. It follows that the curve c must intersect X and f. Hence c intersects ∂f, a contradiction. □

Let $[x, y]$ be an edge. We set $(x, y) = [x, y] \setminus \{x, y\}$, and we use the same notation for subarcs of $[x, y]$.

Lemma 2. *Let f be a face of an accumulation free plane graph and let $[x, y]$ be an edge. If $\partial f \cap (x, y) \neq \emptyset$ then $[x, y] \subset \partial f$.*

Proof. If $(x, y) \subset \partial f$ then $[x, y] \subset \partial f$, because ∂f is closed. Otherwise, there is an open subarc (a, b) of $[x, y]$ whose end point z is in $\partial f \setminus \{x, y\}$ and $(a, b) \cap \partial f = \emptyset$. Suppose $z = a$.

Case 1. If a were an accumulation point of $\partial f \setminus [x, y]$ then a would also be an accumulation point of $X \setminus [x, y] = X \setminus E(a)$, and X would not be accumulation free.

Case 2. There is an open neighbourhood O of a (with respect to the topology of S^2) such that $O \cap \partial f \subset [x, y]$. This implies that a part of $O \cap \partial f$ is a subarc of $[x, y]$ with a as end-point, which is also impossible. □

A plane graph X is called *pointed* if every ray has exactly one accumulation point in S^2. In pointed plane graphs, rays in the same end have the same accumulation point. In this sense, every end of a pointed plane graph corresponds to exactly one point on the sphere. We call this point the *projected end* or *projection of the end.* An element of S^2 may be the projection of more than one end. Note that these projections may be elements of the graph, see Example 1.

Theorem 1. [30, Theorem 7, Proposition 3] *Let X be a locally finite 2-connected pointed plane graph. Then $\overline{X}$ is locally connected in the sphere. The faces of X are bounded by a simple closed curve.*

A plane graph is said to embed *topologically* in S^2 if the identity on X extends to a homeomorphism $f : |X| \to \overline{X}$ where $f(\Omega X) = \overline{X} \setminus X$. That is, $|X|$ embeds in the sphere. Let there be a sequence of vertices which converges to an end ω in the end topology. Then this sequence converges to the projection $f(\omega)$ of ω in the topology of the sphere, and vice versa. Note that the topology of $|X|$ is by definition independent of the embedding. In topological plane embeddings, rays which lie in different ends have to converge to different points in the sphere.

An *isomorphism* between plane graphs X and Y is a homeomorphism $g : X \to Y$ such that $g(VX) = VY$. An isomorphism $X \to X$ is called an automorphism of X. Two graphs Y and Y' are called *homeomorphic in the sphere* if there is an isomorphism $g : Y \to Y'$ that can be extended to an orientation preserving homeomorphism of S^2.

It is clear that plane graphs which are embedded topologically are pointed and accumulation free. The following example shows that there are pointed plane graphs which are not accumulation free and there accumulation free plane graphs which are not pointed.

A face f is called *proper* if $X \cap \partial f \neq \emptyset$.

Example 1. The graph in Figure 1a is accumulation free but not pointed, whereas the graph in Figure 1b is pointed but not accumulation free. The graph in Figure 1c is embedded topologically.

There are also graphs which are pointed and accumulation free, but which are not topological. Simply consider a double-ray whose projected ends coincide, such that $\overline{X}$ is a circle.

The following graph X is accumulation free and pointed, but not topological and it has a face which is not proper. Let S_n, $n \in \mathbb{N} \setminus \{1\}$, be circles in S^2 with center o and radius $1 - 1/n$. Let S be the circle with the same center but with radius 1. We can choose 2^n vertices on each circle S_n such that every element of S is an accumulation point of vertices. Now we choose edges such that the resulting graph X is a binary tree which is rooted in o and such that no interior of any edge intersects one of the circles S_n. The resulting graph X is pointed and accumulation free, but it is not embedded topologically, because $\overline{X} \setminus X$ is the circle S, but $|X| \setminus X$ is ΩX which is a Cantor set (with respect to the topology of $|X|$). Note that S is the boundary of a face f of X. This means that $\partial f \cap X = \emptyset$ and f is not a proper face.

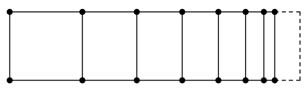

Figure 1a

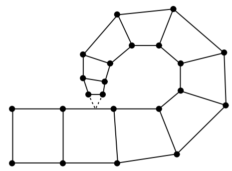

Figure 1b

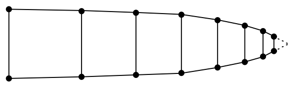

Figure 1c

Let X and Y be a locally finite 2-connected plane graphs such that there is a isomorphism $g : X \to Y$ with the property that for all finite subgraphs Z of X, the graphs Z and $g(Z)$ are homeomorphic in the sphere. Then X is called *locally homeomorphic* to Y.

Lemma 12 of Richter and Thomassen in [30] says that every locally finite 2-connected plane graph has a topological embedding in the sphere. But in fact, they have proved the following.

Theorem 2. *Every 2-connected locally finite plane graph has a topological embedding in the sphere which is locally homeomorphic.*

4. Facial walks

Let X be a locally finite plane graph. Then for every vertex y and every edge $[x, y]$ there is a next edge $[y, z]$ in the clockwise (or counter clockwise) direction with respect to y. A *facial walk* with *clockwise (counter clockwise) orientation* is a walk $(x = x_0, [x_0, x_1], x_1, \ldots, [x_{n-1}, x_n], x_n = y)$ where $[x_i, x_{i+1}]$ is the next edge to $[x_{i-1}, x_i]$ in counter clockwise (clockwise) direction with respect to x_i and such that if $i \neq j$ then $x_{i-1} \neq x_{j-1}$ or $x_i \neq x_j$. The last condition means that a facial walk does not go through one edge two times in the same direction. Maximal facial walks are either finite and closed or 2-way infinite. In 2-connected graphs, every facial walk is a path.

Let R be a facial ray in a locally finite 2-connected pointed plane graph. Then the orientation of R as facial ray coincides with the orientation of R as part of the closed curve from Theorem 1. But note that facial paths and their directions are defined by neighbours in clockwise or counter clockwise direction, and not by curves in boundaries of faces and their orientation.

Lemma 3. *Let X be a 2-connected accumulation free plane graph. A path is a maximal facial path if and only if it is a maximal path in the boundary of a face. Such paths are either finite closed paths or double-rays.*

Proof. Let π be a maximal facial path. Then π is a finite closed path or a double-ray, because X is 2-connected. Let the vertices of π be x_i, either $0 \leq i \leq n$ and $x_0 = x_n$ or $i \in \mathbb{Z}$, and such that x_i is adjacent to x_{i+1}. Since X is accumulation free, π has to intersect the boundary of some face f. If π is completely contained in ∂f then π is of course maximal in ∂f. Suppose not. If ∂f contains an inner point of some edge then it contains the whole edge, see Lemma 2. Hence there is a subpath of π with vertices x_i, $j \leq i \leq k$, which is contained in ∂f, but either x_{j-1} is not in ∂f or x_{k+1} is not in ∂f. Suppose the latter is the case. Then the interior of $[x_i, x_{i+1}]$ is disjoint with ∂f. Either there is an open neighbourhood of x_i (with respect to the topology of S^2) whose intersection with X is contained in $E(x_i)$. Then $[x_i, x_{i+1}]$ cannot be the next edge after $[x_{i-1}, x_i]$ in the sense of orientation corresponding to the facial walk. Or there is no such neighbourhood. Then x_i is an accumulation point of infinitely many distinct edges, and X would not be accumulation free. Hence π is contained in ∂f.

Let π be a path with vertices x_i which is maximal in the boundary of a face f. Following an edge $[x_i, x_{i+1}]$ from x_i to x_{i+1}, the face f is on the left or on the right side of $[x_i, x_{i+1}]$, or on both sides. Suppose π is not a facial path.

Case 1. The face f is on the left side of all edges of π. Then there must be a vertex x_i such that $[x_i, y]$ is the next edge after $[x_{i-1}, x_i]$ in clockwise direction (seen from x_i) and $y \neq x_{i+1}$. Then $[x_i, y]$ is also contained in ∂f and $f \cup x_i$, which is a connected set, separates y from x_{i-1} and x_{i+1} and X is not 2-connected.

Case 2. There are edges $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ such that f is on the left side of $[x_{i-1}, x_i]$ and on the right side of $[x_i, x_{i+1}]$. Then $f \cup x_i$ separates x_{i-1} from x_{i+1} and X is not 2-connected.

Hence π is a facial path. If π is finite and closed or 2-way infinite then π is also maximal as facial path. Otherwise, we can add edges to π such that we obtain a maximal facial path π'. We have seen before that π' is contained ∂f. This would then contradict the assumption that π is maximal in ∂f. □

As corollary we obtain the following.

Theorem 3. *Let f be a proper face of a 2-connected accumulation free plane graph X. Then either ∂f is a closed finite path of X, or $X \cap \partial f$ is the union of disjoint facial double-rays.*

Proof. Let f be a proper face. Suppose $X \cap \partial f$ contains a vertex x. Since X is accumulation free, there is a neighborhood U of x (with respect to the topology of S^2) such that $U \cap \partial f \subset E(x)$. Hence ∂f contains an inner point of an edge e which is incident with x. One of the maximal facial paths (the one in clockwise or the one in counter clockwise direction) has to be contained ∂f, see Lemma 3. If this path π is finite, then it is closed and $\pi = \partial f$. Otherwise π is a double-ray. □

Theorem 4. *No end of a locally finite 2-connected plane graph contains two facial rays with the same orientation.*

Proof. Suppose there is an end of X which contains two facial rays R and S with the same orientation. Let us consider a topological locally homeomorphic embedding Y of X. Such an embedding exists by Theorem 2. We write P and Q for the rays in Y which correspond to R and S in X. Since X and Y are locally homeomorphic, R and S have the the same orientation as P and Q, respectively. Moreover, the rays P and Q also have an orientation as part of the closed curve which is the boundary of their faces, see Theorem 1.

Let π be a path from P to Q. Let T_P and T_Q be tales of P and Q with vertices $(p_0, p_1, \ldots)$ and $(q_0, q_1, \ldots)$, respectively, such that p_0 and q_0 are the only vertices in these tails which are in π.

If P and Q were in the boundary of the same face then π would separate P from Q. This would contradict the assumption that P and Q are in the same end.

Suppose P and Q are in boundaries of different faces f_P and f_Q. The path π is contained in an open set U (with respect to S^2) such that $U' = U \setminus (\pi \cup f_P \cup f_Q)$ has two components. Because T_P and T_Q have the same orientation, there is a subarc (p_0, a) of (p_0, p_1) which is in one component of U' and a subarc (q_0, b) of (q_0, q_1) which is in one component of U'. In this sense we can say that $T_P \setminus \{p_0\}$ begins on one side of π and $T_Q \setminus \{q_0\}$ begins on the other side of π. But $T_P \setminus \{p_0\}$ and $T_Q \setminus \{q_0\}$ are contained in the same component C of $Y \setminus \pi$, because they are in the same end. Hence there is a finite walk π_C from p_0 to q_0, such that π_C is completely contained in C. Then the arc $\pi_C \setminus \{p_0, q_0\}$ starts in one component of U' and ends in the other component of U'. It follows that $\pi \cup \pi_C$ separates f_P from f_Q. This contradicts the assumption that P and Q are in the same end. □

Bonnington, Richter and Watkins showed in [2, Theorem 7.1] that if a 2-connected locally finite plane graph has exactly m ends then it has at most m facial double-rays. Theorem 4 yields this result as corollary.

Theorem 5. *An end of a locally finite 2-connected plane graph X contains at most two facial rays. The number of facial double-rays is less or equal the number of ends.*

For the next corollary we also use Theorem 3.

Theorem 6. *The number of infinite proper faces of a locally finite accumulation free 2-connected plane graph is less or equal the number of ends.*

5. Almost transitive plane graphs

A *plane isomorphism* of plane graphs X and Y is a map $X \to Y$ which can be extended to an orientation preserving homeomorphism of the sphere. A *plane automorphism* of X is a plane isomorphism $X \to X$. A group G *acts almost transitively* on a set A if it has only finitely many orbits. That is, if there is a finite set $F \subset A$ such that $\bigcup_{g \in G} g(F) = A$. A graph X is *almost transitive* if the set of graph automorphisms acts almost transitively on VX. A plane graph X is called *almost transitive in the sphere* if the plane automorphisms act almost transitively on VX. Let g be an automorphism of a graph X. Let D be a double-ray with vertices x_i, $i \in \mathbb{Z}$, such that x_i is adjacent to x_{i+1}. We call g a *translation of D* if there is a positive integer k such that $g(x_i) = x_{i+k}$, for all $i \in \mathbb{Z}$.

Ends are called *thick* if they contain infinitely many disjoint rays, and they are called *thin* otherwise. The next two lemmas are well known and have been proved for different definitions of thin and thick ends for locally finite and non-locally finite graphs.

Lemma 4 (Theorem 9 in [18])**.** *Let g be a translation of a double-ray D. Let ω_1 and ω_2 be the ends of disjoint tails of D. Then either $\omega_1 = \omega_2$ and this end is thick. Or $\omega_1 \neq \omega_2$ and these ends are both thin. The ends ω_1 and ω_2 are fixed by g, and they are the only ends which are fixed by g.*

Lemma 5 ([17, 19])**.** *Let X be an infinite, almost transitive, locally finite graph. Then either*

(1) *X has one end and this end is thick,*
(2) *X has two ends and these ends are thin, or*
(3) *X has infinitely many ends and ΩX is a Cantor set (with respect to the topology of $|X|$).*

A similar theorem was proved by Hopf for Freudenthal ends of locally compact connected spaces which have a compact set whose translates under a group G of homeomorphisms cover the whole space. This property is the topological analogue to almost transitivity in locally finite graphs. In [1], Abels observed that this can be weakened to the property that every end is an accumulation point of elements of an orbit $\{g(x) \mid g \in G\}$, for some point x. Abels' theorem can be applied to locally finite graphs.

Lemma 6. *Let X be a locally finite almost transitive graph with two ends. Let D be a double-ray whose tails lie in different ends. Then there is a number r such that $B(D, r) = X$.*

Proof. Since X is almost transitive and has two ends, there is an integer c such that every ball with radius c separates the two ends. If $X \setminus B(D, r)$ were non-empty for all numbers r then there would be a vertex y such that $B(y, c)$ is disjoint with D, and this ball would then not separate the two ends. □

Lemma 7 (Lemma 7.5 in [2])**.** *Facial double-rays of almost transitive, locally finite graphs are translatable by plane automorphisms.*

Proof. Let G be the group of plane automorphisms. Then there is an infinite orbit of G in VD. Let x be an element of such an orbit. Then there are elements f and h of G such that $f(x)$ and $h(x)$ are distinct vertices of D and such that f and h map the same pair of neighbours of x to D. Here we use the fact that X is locally finite. It follows that $fh^{-1}(D) = D$. The automorphism $g = fh^{-1}$ does not fix the vertex $h(x)$. Hence g translates D. □

6. The conjecture of Bonnington, Richter and Watkins for graphs with one or two ends

We will prove the following theorem which was conjectured by Bonnington, Richter and Watkins.

Theorem 7 (Conjecture 2 in [2])**.** *Disjoint tails of a facial double-ray in a locally finite almost transitive plane graph belong to different ends.*

By Lemma 5, an infinite connected almost transitive graph has either one, two or infinitely many ends. First we give a classification of all possible situations for the cases of one and two ends for 2-connected graphs.

Theorem 8. *Let X be a locally finite almost transitive 2-connected plane graph.*

(1) *If X has one end then there is no infinite facial walk.*
(2) *If X has two ends then there is either no infinite facial walk or there are exactly two facial double-rays. Disjoint tails of these double-rays lie in different ends.*

The one-ended case of Theorem 8 was already proved in [4, Theorem 2.3]. The present proof works for the one- and the two-ended case at the same time. Before giving this proof we want to have a look at some simple examples.

Example 2. All four graphs are 2-connected, locally finite, plane and transitive in the graph theoretic sense. The graphs in Figure 2a, b and c are also transitive as plane graphs. That is, the plane automorphisms act transitively on the set of vertices. The graph in Figure 2d is not almost transitive as plane graph. It is the only one out of these four graphs for which Theorem 8 does not apply. The graphs in Figure 2b, c and d are isomorphic in the graph theoretic sense, but not in the sense of plane isomorphisms. Each of these three graphs has two thin ends.

The first graph (Figure 2a) has one thick end and no infinite facial path. The graph in Figure 2b has two infinite facial double-rays whose tails lie in distinct ends. The third graph has no infinite facial paths. The graph in Figure 2d has one facial double-ray whose tails all lie in the same end.

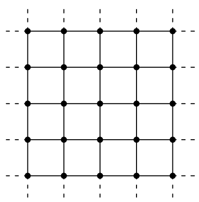

Figure 2a

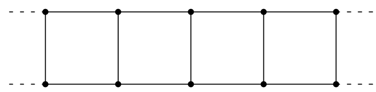

Figure 2b

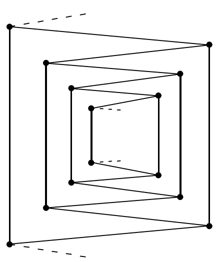

Figure 2c

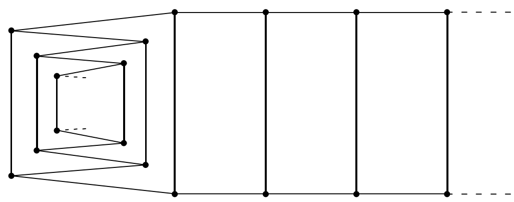

Figure 2d

Proof of Theorem 8. Let X be connected, locally finite, almost transitive, 2-connected, plane and with one or two ends. If there is no infinite facial walk then there is nothing to prove. Suppose there is an infinite facial walk. Since X is 2-connected, this walk is a double-ray D.

<u>Case 1.</u> There is an integer c such that $B(D, c) = X$.

This implies that an end which contains a tail of D has to be thin. By Lemma 5, X has exactly two ends which are both thin. The double ray D is translatable by an automorphism, see Lemma 7. Lemma 4 implies that disjoint tails of D are in different ends ω_1 and ω_2. Let F be a finite subgraph which separates these ends and let C_1 and C_2 be the subgraphs which are spanned by components of $X \setminus F$ that ω_1 and ω_2 live in, respectively. Note that these graphs C_i are not necessarily 2-connected. Let D_i be the 2-way infinite facial walk in C_i which contains the tail $D \cap C_i$. Let R_i be a tail of D_i which does not intersect F. Then R_i has to be a ray, because of the 2-connectedness of X. Hence D_i contains two tails which are rays and which do no not intersect F. These tails are disjoint, otherwise X would not be 2-connected. And these rays are also facial in X. Hence there are at least four disjoint facial rays in X. By Theorem 5, a graph with two ends has at most two facial double-rays. Hence X has exactly two facial double-rays, one of them is D and the other we denote by D'. Because the tails of D lie in different ends and because each end can at most contain two facial rays, it follows that ω_1 and ω_2 each contain one tail of D and one tail of D'. This completes the proof of the theorem for case 1.

Case 2. The complement $X \setminus B(D, c)$ is not empty, for all integers c.

Since the plane automorphisms of X act almost transitively on VX, there is an integer r such that $\bigcup_{g\in G} g(B) = X$, for any closed ball B with radius r. There is a vertex x such that $d(x, D) = r + 1$. Let y be any vertex in D. Then there is a $g \in G$ such that $g(y) \in B(x, r)$. Hence $g(y) \notin D$, which implies $g(D) \neq D$. The image $g(D)$ is a facial double-ray, because plane automorphisms map facial double-rays to facial double-rays. Theorem 5 implies that X has two ends. These ends are thin, see Lemma 5. If the tails of D were in different ends then Lemma 6 would imply that there is an integer such that $B(D, c) = X$. The same holds for $g(D)$. By Theorem 5, it follows that the tails of D are in one end, and the tails of $g(D)$ are in the other end of X. There is a finite connected subgraph F which contains elements of D and of $g(D)$, and which has the property that the complement $X \setminus F$ has exactly two infinite components C_1 and C_2, one of which is disjoint with D the other disjoint with $g(D)$. We can find a plane automorphism h such that $h(F) \subset C_1$. The graph F intersects two facial double-rays. The same holds for $h(F)$. Hence C_1 intersects two facial double-rays. This means there is a third facial double-ray. By Theorem 5, there is a third end and thus case 2 is impossible. □

7. Structure trees and accessibility

The *coboundary* δe of a subgraph e of X is the subgraph consisting of all edges which connect a vertex of e with a vertex of $VX \setminus e$. Let e^* denote the subgraph spanned by $VX \setminus e$. Note that $e \cup e^* \cup \delta e = X$. A *cut* is a subgraph of X whose coboundary is finite. If $|\delta e| = n$ then e is called an *n-cut*. Define $\mathcal{B}_n X$ to be the Boolean ring generated by all n-cuts. Let $\mathcal{B}X$ denote the Boolean ring generated by all cuts. All the elements in $\mathcal{B}X$ are cuts. A set E of cuts is said to be a *nested* if for each choice of e and f in E, one of the intersections

$$e \cap f, \quad e \cap f^*, \quad e^* \cap f, \quad \text{or} \quad e^* \cap f^*$$

is empty. A nested set E is called a *tree set* if for all elements e and f in E such that $e \subset f$, there are only finitely many elements $g \in E$ such that $e \subset g \subset f$. A tree set E is called *undirected* if whenever $e \in E$ then $e^* \in E$. Let G be a group of automorphisms of X. A tree set E is called *G-invariant* if

$$G(E) = \{g(e) \mid g \in G\} = E.$$

A cut e is said to be *tight* if both e and e^* are connected.

Lemma 8 (Proposition 4.1 in [32])**.** *Let n be an integer and let q be an edge of a connected graph. Then there are only finitely many tight n-cuts whose coboundary contains q.*

Corollary 1 (Corollary 4.3 in [32])**.** *Let G be a group of automorphisms acting almost transitively on the vertices of a locally finite connected graph. Then, up to G-equivalence, there are only finitely many tight n-cuts. In other words, there is a finite set M of tight n-cuts with the property that for every tight n-cut e there is a g in G such that $g(e)$ is in M.*

Dicks and Dunwoody have proved the following remarkable theorem in their book [7].

Theorem 9 (Theorem II.2.20 in [7])**.** *Let X be a connected graph and let G be a subgroup of the group of automorphisms of X. Then there is a chain of G-invariant undirected tree sets $E_1 \subset E_2 \subset \ldots$ in $\mathcal{B}X$ such that the elements of E_n are tight k-cuts, where $k \leq n$, and E_n generates $\mathcal{B}_n X$.*

Corollary 2 (Proposition 7.1 in [32])**.** *Let X be a locally finite connected graph and G be a subgroup of the group of automorphisms of X. Let ω_1 and ω_2 be ends which can be separated by removing n edges. Let E_n be a tree set according to Theorem 9. Then there is a cut e in E_n such that ω_1 lives in e and ω_2 lives in e^*.*

From a tight undirected G-invariant tree set E in $\mathcal{B}X$, we can build a directed tree $T(E)$. Such a construction was first described in [10, Theorem 2.1]. It is also treated in [7, Section II.1], [21], [25] and [32]. In the following, we will only give a brief description of the structure tree $T(E)$. The reader is referred to the above references for more details and proofs.

Cuts e and f in an undirected tree set E are defined as equivalent if $e \supset f^*$ and there is no third cut g in E such that $e \supset g \supset f^*$. This is an equivalence relation and the corresponding equivalence classes are the vertices of $T(E)$. Vertices x and y of $T(E)$ are adjacent if and only if there is a cut e in E such that e is in x and e^* is in y. In this way, each element e of E corresponds to exactly one edge of $T(E)$ which is a directed tree. If E is a G-invariant tree set then $T(E)$ is called a *structure tree* of E (with respect to G).

For every vertex x in VX there is exactly one vertex v of $T(E)$ such that x is element of all cuts in v. We set $\phi(x) = v$. This defines a map $\phi : VX \to VT(E)$ such that $\phi^{-1}(v) = \bigcap_{e \in v} e$. Let ω be an end of X. Then either there is a vertex v of $T(E)$ such that every ray in ω lives in each cut of v. Then we set $\Phi(\omega) = v$. Note that this does not necessarily mean that the rays of ω live in $\phi^{-1}(v)$. Or there is a sequence of cuts $e_0 \supset e_1 \supset \ldots$ in E such that every ray of ω lives in all cuts e_i. The edges of T which correspond to these cuts constitute a ray in $T(E)$. We write $\Phi(\omega)$ for the end in $T(E)$ of this ray. This yields a map $\Phi : \Omega X \to VT(E) \cup \Omega T(E)$.

Let E_n be a tree set as in Theorem 9 and let ω be an end of X. If $\Phi(\omega) \in \Omega T(E_n)$ then ω is thin. Or equivalently, if ω is thick then $\Phi(\omega) \in VT(E_n)$.

The following recent result of Dunwoody will be crucial for the proof of the infinitely ended case of our main result.

Theorem 10 ([13]). *Almost transitive locally finite plane graphs are accessible.*

8. The infinitely ended case

A *geodesic path* π from a vertex x to a vertex y is a path of minimal length. That is, the length of π is $d(x, y)$. A *geodesic ray* is a ray whose finite subpaths are all geodesic.

Lemma 9 ([33]). *Every end of a locally finite graph contains a geodesic ray.*

Proof. Let R be a ray of an end ω with vertices x_i, $i \geq 0$, such that x_i is adjacent to x_{i+1}. Let y be any vertex and let π_i be a geodesic path from y to x_i. Since X is locally finite, there is a sequence π_{i_j}, $j \geq 0$, which converges to a ray S. Then S is a geodesic ray in ω. □

The following construction is similar to a construction by Thomassen and Woess [32, Section 7]. The authors treated the case where the tree set generates the Boolean ring of all cuts. They showed that this is equivalent to X being accessible. Let G be a group of automorphisms of a connected, locally finite graph X which acts almost transitively on VX. Let E be a G-invariant tree set and let y be a vertex of the corresponding structure tree $T(E)$. Recall that the vertex v is an equivalence class of cuts and $\phi^{-1}(v)$ is the intersection of all cuts in v. Note that this intersection might be empty. Let G_v denote the stabilizer of v. This the set of elements of G which fix v. Let e be a cut in v. There is a union $\Pi(e)$ of finitely many finite paths in $X \setminus e$ with distinct end vertices in Ne such that

(G) $\Pi(e)$ contains all geodesics (with respect to the metric of X) in e^* between end vertices of Ne which are contained in e^*. And

(P) if $\pi_1, \pi_2, \ldots, \pi_r$ are pairwise disjoint paths in e^* connecting distinct vertices of Ne then there are pairwise disjoint paths $\pi'_1, \pi'_2, \ldots, \pi'_r$ in $\Pi(e)$ with the same end vertices.

For all cuts $g(e)$ in $G_v(e) = \{g(e) \mid g \in G_v\}$, we set $\Pi(g(e)) = g(\Pi(e))$. We do the same for all other orbits $G_v(f)$ of cuts f in v. By Corollary 1, there are only finitely such orbits $G_v(f)$ in v. We define X_v as the graph spanned by the union of $\phi^{-1}(v)$ with all subgraphs $\Pi(e)$, for all e in v.

Lemma 10.

(i) *If X is 2-connected that X_v is also 2-connected.*
(ii) *All geodesic rays of ends in $\Phi^{-1}(v)$ live in X_v.*
(iii) *Let R be any ray of an end in $\Phi^{-1}(v)$. Then infinitely many vertices of R are in X_v.*
(iv) *The number of ends of X_v is the same as the number of ends in $\Phi^{-1}(v)$.*
(v) *The stabilizer G_v acts almost transitively on VX_v.*

Proof.

(i) Suppose X is 2-connected. Let x be any vertex of X_v. Suppose x is in $\phi^{-1}(v)$. If $X_v \setminus \{x\}$ would be disconnected then x would either separate the paths of a set $\Pi(e)$ from the rest of X_v, or there would be a component of $X_v \setminus \{x\}$ whose vertices are all in $\phi^{-1}(v)$. In either case, $X \setminus \{x\}$ would also be disconnected. If $x \in X_v \setminus \phi^{-1}(v)$ then x is in some path π in $\Pi(e)$ which connects distinct vertices in Ne. The cut e is connected, because it is tight. Property (P) ensures that $X_v \setminus \Pi(e)$ is connected. It follows that $X_v \setminus \{x\}$ is connected. Hence X_v is 2-connected.

(ii) This is a consequence property (G).

(iii) This follows from the fact that R lives in all cuts e in v.

(iv) Let ω be an end in $\Phi^{-1}(v)$. We choose a geodesic ray R in ω, see Lemma 9. By (G), the ray R lives in X_v. Define $f(\omega)$ as the end of X_v which contains a tail of R. We will show that the resulting map $f : \Phi^{-1}(v) \to \Omega X_v$ is a bijection.

Let ω_1 and ω_2 be distinct ends in $\Phi^{-1}(v)$. Their rays can be separated by a finite set of vertices $F \subset VX$. Then $F \cap X_v$ separates $f(\omega_1)$ from $f(\omega_2)$ in X_v. Hence f is injective.

Let η be an end of X_v. A ray S in η is also a ray in X. Let ω be the end of X which contains S and let R be the geodesic ray in ω which defines the end $f(\omega)$. Suppose $f(\omega)$ is not the same end as η. Then there is a finite subgraph F in X_v, and there are distinct components C and D of $X_v \setminus F$ such that R is contained in $F \cup C$ and S is contained in $F \cup D$. Since the graphs $\Pi(e)$ are finite we may assume that the following is satisfied:

(R) For all sets $\Pi(e)$, $e \in v$, either $\Pi(e) \subset F$ or $\Pi(e) \cap F = \emptyset$.

Suppose there is a path in X from C to D which is disjoint with F. Then this path has to contain a subpath π with vertices $x = z_0, z_1, \ldots, z_n = y$, such that $x \in C$ and $y \in D$ (or vice versa), z_i is adjacent to z_{i+1} and the vertices $z_1, \ldots, z_{n-1}$ are not in X_v. Suppose $n \geq 2$. Then x and y have to be contained in the sets $\Pi(e)$, $e \in v$. The vertices x and y cannot lie in distinct sets $\Pi(e)$ and $\Pi(e')$, because the tree set is nested. Suppose x and y are in the same set $\Pi(e)$. Then $\Pi(e) \cap F = \emptyset$, because $\Pi(e) \subset F$ is impossible, see property (R). Then property (P) implies that x and y can be connected by a path in $X_v \setminus F$, a contradiction. If $n = 1$ then $[x, y]$ is an edge of X_v and we have a contradiction to the fact that F separates C from D in X_v. Hence $\eta = f(\omega)$ and f is surjective.

(v) The group G acts almost transitively on $\phi^{-1}(v)$. Let g be an element of G which maps some vertex of $\phi^{-1}(v)$ to another vertex of $\phi^{-1}(v)$. The sets $\phi^{-1}(v)$, $v \in VT(E)$, form a G-invariant partition of VX. Hence $g \in G_v$ and G_v acts almost transitively on $\phi^{-1}(v)$. Since G_v has only finitely many orbits on the cuts in v and the graphs $\Pi(e)$ are finite, it follows that G_v acts almost transitively on VX_v. □

We are now ready to prove the conjecture of Bonnington, Richter and Watkins for the infinitely ended case.

Theorem 11. *Let X be a 2-connected, locally finite, almost transitive plane graph with infinitely many ends.*

Then no thick end contains a facial ray.

A thin end contains at most two facial rays. Disjoint tails of a facial double-ray lie in different ends.

Proof. Suppose X has a thick end which contains a facial ray. Let Y be a locally homeomorphic, topological embedding of X, see Theorem 2. Then Y has again a thick end which contains a facial ray. We denote this end by ω and the facial ray by R.

By Theorem 10, Y is accessible. That is, there is an integer n such that each pair of ends can be separated by removing n edges. Let G be the group of plane automorphisms of Y and let E_n be a tree set according to Theorem 9. Let ω_1 and ω_2 be any pair of ends. By Corollary 2, there is a cut e in E_n such that ω_1 lives in E and ω_2 lives in e^*. This implies that the map $\Phi : \Omega X \to VT(E_n) \cup \Omega T(E_n)$ is injective.

Since ω is thick, Φ maps ω to a vertex v of $T(E_n)$. Let Y_v be the graph according to Lemma 10. Then infinitely many vertices of R are in Y_v. The ray R is contained in a facial double-ray, because Y is 2-connected, and this double-ray is in the boundary of a face f of Y, see Lemma 3. The face f is a component of $S^2 \setminus \overline{Y}$. And this component must be contained in a component f' of $S^2 \setminus \overline{Y_v}$, because Y_v is a subgraph of Y. We have $R \cap Y_v \cap \partial f \subset R \cap Y_v \cap \partial f'$. Hence $\partial f'$ contains infinitely many vertices of R, and Y_v has an infinite face. Since Y_v has only one end, this is a contradiction to Theorem 8 (1).

Let ω be a thin end which contains a facial ray. Then ω contains a facial double-ray D which is translated by some plane automorphism g of X, see Lemma 7. By Lemma 4, the tails of D lie in distinct thin ends. That ω contains no more than two facial rays is what we have proved in Theorem 5. □

9. 1-connected plane graphs

To complete the proof of Theorem 7, the conjecture of Bonnington, Richter and Watkins we still have to consider the 1-connected case.

From a locally finite almost transitive plane graph X we will construct a 2-connected graph X^d, called a *doubling* of X, which is again locally finite and almost transitive. For all vertices x, we choose a counter clockwise ordering of its neighbours $x_0, x_1, \ldots, x_{\deg(x)-1}$. The vertices of VX^d are elements $x(x_i, j)$, for $i \in \{0, \ldots, \deg(x)-1\}$, $j \in \{0,1\}$ and $x \in VX$. The edges of X^d are (i) $[x(x_i,0), x(x_i,1)]$, (ii) $[x(x_i,1), x(x_{i+1},0)]$ (we consider the index $i+1$ modulo $\deg(x)$), and (iii) for all edges $[x,y]$ of X we have two edges $[x(y,0), y(x,1)]$ and $[x(y,1), y(x,0)]$ in EX^d. This means that every vertex of x of X is replaced by a circle of length $2\deg(x)$ and every edge $[x,y]$ of X is replaced by two edges. For each edge $[x,y]$ of X there is a closed path of length 4 in X^d with vertices

$$x(y,0),\ x(y,1),\ y(x,0),\ y(x,1). \tag{9.1}$$

And the circles of length $2\deg(x)$ have vertices

$$x(x_0,0),\ x(x_0,1),\ x(x_1,0),\ x(x_1,1), \ldots, x(x_{\deg(x)},0), x(x_{\deg(x)},1). \tag{9.2}$$

Note that all doublings of a locally finite, plane graph X are isomorphic in the graph theoretic sense. The above construction implies the following.

Lemma 11. *If X is connected then X^d is 2-connected.*

Let g be a plane automorphism of X. We define $g^d(x(y,i)) = g(x)(g(y),i)$.

Theorem 12. *Let X be locally finite, plane and accumulation free. Then there is a plane and accumulation free doubling X^d of X such that if g is a plane automorphism of X then g^d is a plane automorphism of X^d. Let G be a group of automorphisms of X which acts almost transitively on VX. Then $G^d = \{g^d \mid g \in G\}$ acts almost transitively on VX^d.*

Proof. Let Y be a topological embedding of X, see Theorem 2. Edges $[x,y]$ are closed arcs from x to y and (x,y) has no accumulation point on any other edge except for x and y. There is a family $\mathcal{U}$ of pairwise disjoint connected open neighbourhoods U_x of x, for all $x \in VX$, such that ∂U_x is a simple closed curve which intersects each of the edges $[x,x_0], \ldots, [x,x_{\deg x-1}]$ in one single point and these points are ordered in counter clockwise direction around x along this curve. There is also a family of pairwise disjoint, connected neighbourhoods $V_{[x,y]}$ of $[x,y] \setminus (U_x \cup U_y)$, for all edges $[x,y]$, such that the only neighbourhoods in $\mathcal{U}$ which intersect $V_{[x,y]}$ are U_x and U_y. We can construct these neighbourhoods by induction, because Y is locally finite. There is a plane embedding of the circle in (9.2) in U_x such that the edges $[x(x_i,0), x(x_i,1)]$ are in $U_x \cap V_{[x,x_i]}$ and the edges $[x(x_i,1), x(x_{i+1},0)]$ are contained in

$$U_x \setminus \bigcup_{j\in\{0,1,\ldots,\deg(x)\}\setminus\{i,i+1\}} V_{[x,x_j]}.$$

The edges of type $[x(y,0), y(x,1)]$ and $[x(y,1), y(x,0)]$ can now be embedded in $V_{[x,y]}$ such that the resulting graph is a plane, accumulation free doubling of X.

Let g be an automorphism of X. Since g is a bijection $VX \to VX$, the map g^d is a bijection of the closed paths of type (9.2). The vertices of each of these circles are mapped bijectively to the vertices of another closed path of this type. Hence g^d is a bijection $VX^d \to VX^d$. Let u and v be vertices of X^d. Suppose u and v are adjacent in X^d. If (i) $u = x(x_i,0)$ and $v = x(x_i,1)$ then $g(u) = g(x)(g(x_i),0)$ and $g(v) = g(x)(g(x_i),1)$ are again adjacent in X^d, in the sense of (i), because $g(x)$ is adjacent to $g(x_i)$ in X. Suppose (ii) $u = x(x_i,1)$ and $v = x(x_{i+1},0)$. Then $g(x_{i+1})$ is the the next neighbour of $g(x)$ after $g(x_i)$ in clockwise direction, because g is orientation preserving. Hence $g(x)(g(x_i),1)$ is adjacent to $g(x)(g(x_{i+1}),0)$. If (iii) $u = x(y,0)$ and $v = y(x,1)$ then x is adjacent to y in X, $g(x)$ is adjacent to $g(y)$ in X, and hence $g(u) = g(x)(g(y),0)$ is adjacent to $g(v) = g(y)(g(x),1)$ in X^d. Thus g^d preserves adjacency. The same holds for $(g^d)^{-1}$. Hence g^d is an automorphism of X^d. Let X be plane and let g be a plane automorphism of X. Then by the above construction, every closed path of type (9.1) or (9.2) is the boundary of a face. All other faces of X^D correspond to faces of X in an obvious way and g^D acts on these faces in the same way as g acts on these faces of X, and g^d is a plane automorphism of X^D.

Suppose G has p orbits on VX. Then there are at most p different degrees for the vertices of X. Since X is locally finite, the maximal degree q is finite. The group G^d acts on the set of closed paths of type (9.2) with at most p orbits. The maximal length of these paths is $2q$. Hence G^d has at most $2pq$ orbits on VX^d. □

A doubling X^d according to the construction in the proof of Theorem 12 is called a *plane doubling* of X. A *quasi ray* is a 1-sided infinite walk with vertices x_i, $i \in \mathbb{Z}$, and edges $[x_i, x_{i+1}]$, such that for every vertex y there are only finitely many $i \in \mathbb{Z}$ with $x_i = y$. A *quasi double-ray* is a 2-sided infinite walk which has disjoint tails that are quasi rays. Results on plane 2-connected graphs can sometimes easily be reformulated for connected graphs by replacing the terms "maximal facial path/ray/double-ray" by "maximal facial walk/quasi-ray/quasi double-ray", respectively. For example, the following is the analogue to Lemma 3.

Lemma 12. *Maximal facial walks of connected accumulation free plane graphs X are contained in the boundary of a face.*

An infinite maximal facial walk in a locally finite graph is either a quasi double-ray or any pair of its tails have infinitely many vertices in common.

Proof. Let X be locally finite and plane. An infinite maximal facial walk D is 2-way infinite. Let D have vertices x_i, $i \in \mathbb{Z}$, and edges $[x_i, x_{i+1}]$.

By definition, a facial walk does not go through the same edge twice in the same direction. Hence there is no vertex y such that $y = x_i$ for infinitely many i, because

X is locally finite. This implies that if D is not a quasi double-ray then any pair of tails of D has infinitely many vertices in common.

Let X be any connected accumulation free plane graph. Then there is a face f such that ∂f intersects the interior of an edge of D. To prove that D is contained in ∂f, we can now repeat the arguments used in the proof of Lemma 3. □

Let D be a two-sided infinite facial walk with vertices x_i and edges $[x_i, x_{i+1}]$, $i \in \mathbb{Z}$. Suppose D has counter clockwise orientation. Recall that this means that among the edges which are incident with x_i, $[x_i, x_{i+1}]$ is the next edge after $[x_i, x_{i-1}]$ in clockwise direction. Let X^d be a plane doubling of a plane graph X. We define D^d as the walk in X^d with vertices

$$\dots x_{-1}(x_0, 1),\ x_0(x_{-1}, 0),\ x_0(y_1, 1),\ x_1(x_0, 0),\ x_1(x_2, 1),\ x_2(x_1, 0),\ x_2(x_3, 1), \dots$$

If D has clockwise orientation then define D^d as the walk in X^d with vertices

$$\dots x_{-1}(x_0, 0),\ x_0(x_{-1}, 1),\ x_0(x_1, 0),\ x_1(x_0, 1),\ x_1(x_2, 0),\ x_2(x_1, 1),\ x_2(x_3, 0), \dots$$

From the construction of X^d, see the proof of Theorem 12, it follows that D^d is a facial double-ray with the same orientation as the facial walk D.

Theorem 13. *Every infinite maximal facial walk of a connected, locally finite, plane graph which is almost transitive in the sphere is a quasi double-ray. Each pair of disjoint tails of such a walk are in different ends.*

Proof. Let X be a connected, locally finite, plane graph which is almost transitive in the sphere. Let Y be a locally homeomorphic, topological embedding of X. Let D be an infinite maximal facial walk in Y with vertices x_i, $i \in \mathbb{Z}$, and edges $[x_i, x_{i+1}]$. Suppose D is not a quasi double-ray. Then by Lemma 12, any pair of tails has infinitely many vertices in common. This implies that disjoint tails of the facial double-ray D^d lie in the same end of X^d. If X^d has one or two ends then this contradicts Theorem 8. If X^d has infinitely many ends then this contradicts Theorem 11. □

Theorem 13 completes the proof of the conjecture of Bonnington, Richter and Watkins, see Theorem 7. With the construction of plane doublings we can derive Theorems 8 and 11 also for 1-connected graphs.

Theorem 14. *Let X be a connected, locally finite, almost transitive, plane graph.*

(1) *If X has one end then there is no infinite facial walk.*
(2) *If X has two ends then there is either no infinite facial walk, or there are exactly two facial quasi double-rays. Disjoint tails of these quasi double-rays are in different ends.*
(3) *If X has infinitely many ends then no thick end contains a facial quasi-ray. Thin ends contain at most two facial quasi-rays. Disjoint tails of quasi double-rays lie in different ends.*

10. An application

Recall that in graphs with a topological embedding, projections of ends ω are elements η of $\overline{X} \setminus X$ such that every ray in ω converges to η in the topology of S^2. As a consequence of our main result we obtain the following classification.

Theorem 15. *Let f be an infinite face of an infinitely ended, 2-connected, almost transitive, plane graph with a topological embedding. Then either ∂f is the disjoint union of two double-rays and two projected ends. Or ∂f is the disjoint union of countably many double-rays and a Cantor set consisting of projected ends.*

Proof. Theorem 3 says that ∂f contains a double-ray D. Suppose $\partial f \setminus D$ consists of a single point. Then any disjoint tails of D belong to the same end ω, because X is embedded topologically. This is a contradiction to Theorem 7. If $\partial f \setminus X$ consists of exactly two points then ∂f is the union of two facial double-rays and the projections of the ends which contain tails of these double rays.

Suppose $\partial f \setminus X$ has more than two elements. The end boundary ΩX is compact and totally disconnected in $|X|$. Since X is embedded topologically, the corresponding set in S^2, which is $\overline{X} \setminus X$, is compact and totally disconnected in S^2. The set $\partial f \setminus X$ is compact and totally disconnected in S^2 because it is the intersection of $\overline{X} \setminus X$ and ∂f. Suppose $\partial f \setminus X$ contains an isolated point η which is the projection of an end ω. Then η has to be in the closure of two double-rays D_1 and D_2, because ∂f is a closed curve, see Theorem 1. There is a translation g of D_1 which fixes f, see Lemma 7. But D_2 is also invariant under g. This can be seen by the fact that ω does not contain a tail of any facial double ray other that D_1 and D_2, see Theorem 5, or by the fact that g has to fix the closed curve which is bounding f. The boundaries ∂D_1 and ∂D_2 each consist of two elements. Say $\partial D_1 = \{\eta, \eta_1\}$ and $\partial D_2 = \{\eta, \eta_2\}$. Then $\eta_1 \neq \eta_2$, because $\partial f \setminus X$ has more than two elements. Hence g fixes three distinct points η, η_1 and η_2. And g fixes the corresponding three ends of X. But this is a contradiction to Lemma 4. Hence ∂f has no isolated points. Thus $\partial f \setminus X$ is a Cantor set. And $\partial f \cap X$ has to be the union of infinitely many disjoint double-rays. That there are only countably many such double-rays can be seen by the fact that ∂f is bounded by a closed curve. □

Both cases of Theorem 15 may occur.

Example 3. The doubling of a regular tree can be embedded topologically such that there is only one infinite face whose boundary is the union of countably many double-rays and a Cantor set of projected ends.

The following infinitely ended, plane Cayley graph X, see Figure 3, has finite faces (bounded by closed paths of length four) and infinite faces whose boundaries consist of two facial double-rays and, if X is embedded topologically, of two projected ends.

$$G = (\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z} =$$
$$\left\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ad)^2 = (bd)^2 = (cd)^2 = 1 \right\rangle,$$

and $X = \mathrm{Cay}(G, S)$ for $S = \{a, b, c, d\}$. That is, $VX = G$ and x is adjacent to y if and only if xy^{-1} is in S. The graph consists of an "upper" and a "lower" 3-regular tree, each of which correspond to one of the left cosets of $\{1, a\} * \{1, b\} * \{1, c\} \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Pairs of vertices of the upper and the lower tree are connected by edges which correspond to the left cosets of $\{1, d\} \simeq \mathbb{Z}/2\mathbb{Z}$. To see that the plane automorphisms act transitively on the vertices we suggest to imagine the graph with a topological embedding in the sphere, so that the faces have no "inside" and no "outside" as in the plane.

The same graph can be found [28], but there it is not embedded almost transitively in the sphere.

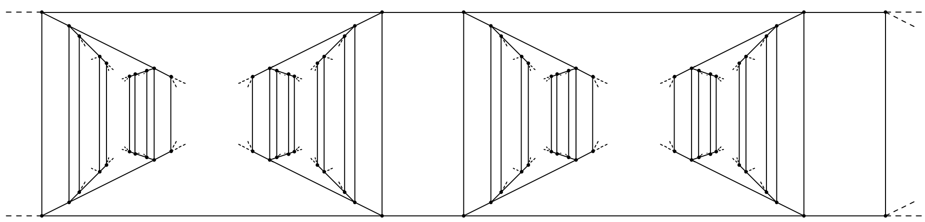

Figure 3

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