Introduction to Ends of Graphs

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Abstract. We give an elementary introduction to the theory of ends of locally finite graphs for students and mathematicians from other fields. The focus is on basic topological properties of the end space. We will also discuss Cayley graphs and some connections between ends and group actions on graphs.

[1]This is a preliminary version. Please help me to improve the paper by telling me any mistakes, typos and bad formulations, or make any kind of comments.

1. Introduction

The basic idea behind the concept of an end is to distinguish between different ways of going to infinity. Ends carry a natural topology which is often not mentioned explicitly. Ends of graphs were introduced by Freudenthal in 1945 (see [**8**]). His "abzählbare, diskrete Räume" (countable, discrete spaces) with "Nachbarschaft zweier Punkte" (adjacency of two points) are what we nowadays call "graphs". Freudenthal only considered locally finite graphs. These are graphs where a vertex has only finitely many neighbours. His work goes back to his thesis [**5**] which was also published in [**6**]. In [**7**] he generalized his previous work and weakened the assumptions on the topological space as far as possible. Halin was the first to consider ends of non-locally finite graphs in [**10**]. He defined ends as equivalence classes of rays where two rays are equivalent if they have infinitely many vertices in common with a third ray. We will refer to this type of ends as vertex ends.

If we identify the edges of a graph with unit intervals then we obtain a so-called 1-complex equipped with the topology induced by the natural metric of the unit intervals. The 1-complex of a locally finite connected graph is a connected locally compact space. Freudenthal's end theories for topological spaces in [**6**] and [**7**] apply to such spaces and yield the same definition of ends as the graph theoretic definition in [**8**], and in locally finite graphs these ends are the same as defined by Halin in [**10**]. The connection between Halin's and Freudenthal's ends in non-locally finite graphs was discussed in [**3**]. There are a couple of other types of ends which do not coincide in non-locally finite graphs, see [**3**] and [**14**]. But for locally finite graphs, they all yield the same ends. Thus for locally finite graphs, there is one standard theory of ends.

There is a well written survey paper on group actions on graphs and their end space by Möller [**16**]. In [**2**], Diestel gives a survey on the existence of end faithful spanning trees and ends of graphs. There is also a diploma thesis [**12**] on ends of graphs by Hien.

In the present paper we do not want to present a survey of the state of the art. Our aim is to give an easy introduction to the theory of ends of locally finite graphs which is accessible for a broad audience. We address mathematicians which are not familiar with the subject and students who have a basic knowledge in group theory and topology. It suffices if the reader knows what groups, subgroups and free groups are, and we will use the concept of compact sets and convergence of sequences in topological spaces. The necessary background from topology and group theory can be found in the appendix.

There is a series of exercises. Most of them are not difficult. They should help the reader to make sure that he has understood the definitions and the basic concepts.

We start with some basic definitions in Section 2 and discuss simple properties of graphs as metric spaces.

We discuss some basic concepts of the theory of ends of graphs for the example of trees in Section 3. This serves as a motivation for the following sections. Results or notation will not be used later. The advanced reader may skip this section.

Section 4 is devoted to Cayley graphs. These are graphs which describe the structure of a group G, depending on a generating set S. The vertices of the graph are the group elements. The edges correspond to multiplications with elements of S. If there are vertices x and y and an element s of S such that $xs = y$ then x and y are connected by an edge which is coloured (or labelled) with the element s. Given a graph where each edge is coloured with an element of a set S, we classify those colourings which stem from a Cayley graph. Moreover, we prove a theorem of Sabidussi, which says that a graph is a Cayley graph if and only if it permits a transitive and free group action. Exercises deal with certain finite Cayley graphs which are related to Platonic bodies. The Petersen graph, which is defined in Section 2, is an example of a graph which permits a transitive group action but which is no Cayley graph.

Ends are defined as equivalence classes of rays in Section 5. To construct a topology on the set of ends together with the set of vertices of the graph, we define an open base which is shown to be closed under finite intersection. In the exercises of this section the reader should determine the number of ends of some given graphs and determine cluster points of certain sequences of vertices.

In Section 6 we discuss the most important properties of the end topology, such as compactness and separation properties. In other papers, the proof of the compactness is often left to the reader. We give all the necessary details.

The topological properties discussed in Section 6 allow us to apply a metrisation theorem from general topology. Nevertheless, in Section 7 we give an explicit metric which induces the end topology.

In [**13**] Hopf used the end concept of Freudenthal in [**5**] and [**6**] and observed that if the translates of a compact set under the action of a group of homeomorphisms cover the whole space, then the space has either one end or two ends, or the end boundary is a Cantor set. A crucial assumption on the space in order to obtain a Cantor set as boundary is local compactness. The graph theoretic analogue of locally compact spaces are locally finite graphs. For these graphs, a set of vertices is compact if and only if it is finite. If the translates of a finite set under a group action cover the whole graph, then the graph is called almost transitive. The graph theoretic analogue to Hopf's theorem holds for connected, locally finite, almost transitive graphs. Abels observed in [**1**] that Hopf's assumptions on the group action can be weakened. It suffices to assume that all end points are accumulation points of an orbit of the group. In Section 8 we give a simple proof for this theorem for the case of locally finite graphs. In an exercise we give an example which shows that there are group actions which satisfy the assumptions in the Theorem of Abels but which do not satisfy the assumptions in the Theorem of Hopf.

Halin proved in [**11**] that for every graph automorphism g on a connected graph which does not fix a finite set of vertices there is a line (2-sided infinite path of distinct vertices) which is invariant under a power of g. Such a line is called g-periodic. For any vertex x the sequence $(g^n(x))_{n\in\mathbb{N}}$ converges to an end which contains a subray of the g-periodic line. This end is called the direction $\mathcal{D}(g)$ of g. The direction of g^{-1} also contains a subray of the g-periodic line. The concept of direction in locally finite graphs only makes sense for automorphisms which do not fix a finite set of vertices. There are various ways of defining ends to be thick or thin. But no matter which definition we use, in locally finite graphs we get the result that the direction of g is thick, if $\mathcal{D}(g) = \mathcal{D}(g^{-1})$, and the directions $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ are thin if $\mathcal{D}(g) \neq \mathcal{D}(g^{-1})$. We prove such a theorem for one of these definitions of thickness in Section 9. Automorphisms act on the set of ends in a natural way. The directions $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ are the only ends which are fixed by g.

2. Graphs

Definition 1. A *graph* X is a pair (VX, EX) such that VX is any set and $EX \subset \{\{x,y\} \mid x,y \in VX\}$. The elements of VX are called *vertices* and the elements of EX are called *edges*. Vertices x and y are *adjacent*, or *neighbours*, if $\{x,y\}$ is an edge. The *degree* for a vertex is the cardinaltity of its neighbours. If all vertices have the same degree d then we call the graph *regular* (or d-regular). A graph is *locally finite* if all vertices have only finitely many neighbours.

With $\mathbb{Z}$ we denote the set of integers, and we set $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 1\}$ and $\mathbb{N}_0 = \{n \in \mathbb{Z} \mid n \geq 0\}$.

A *path of length* $n \in \mathbb{N}_0$ from a vertex x to a vertex y is an $(n+1)$-tuple

$$(x = z_0, z_1, \dots, z_n = y)$$

of vertices such that x_{i-1} and x_i are adjacent for $1 \leq i \leq n$. The *distance* $d_X(x,y)$ between x and y is the length of the shortest path from x to y. Let A and B be non-empty sets of vertices. Then

$$d_X(A,B) = \min\{d_X(y,z) \mid y \in A,\ z \in B\},$$

and $d_X(x,A)$ is defined as $d_X(\{x\},A)$. A path from x to y of length $d_X(x,y)$ is called *geodesic*. The path $(z_0, z_1, \dots, z_n)$ is called *closed* if $z_0 = z_n$. A *tree* is a connected graph without closed paths. The *concatenation of paths*

$$\pi_1 = (x_0, x_1, \dots, x_m = y_0) \quad \text{and} \quad \pi_2 = (x_m = y_0, y_1, \dots, y_n)$$

is the path

$$\pi_1 \circ \pi_2 = (x_0, x_1, \dots, x_m = y_0, y_1, \dots, y_n).$$

A set of vertices C is called *connected* if any pair of vertices in C can be connected by a path in X which is contained in C. The graph X is *connected* if VX is connected.

Example 1. Figure 1 shows the 3-regular Petersen graph X where

$$VX = \{x \in \mathbb{N} \mid 1 \leq n \leq 10\} \quad \text{and}$$

$$EX = \{\{i, i+1\}, 1 \leq i \leq 9, \{1,5\}, \{1,8\}, \{2,10\}, \{3,7\}, \{4,9\}, \{6,10\}\}.$$

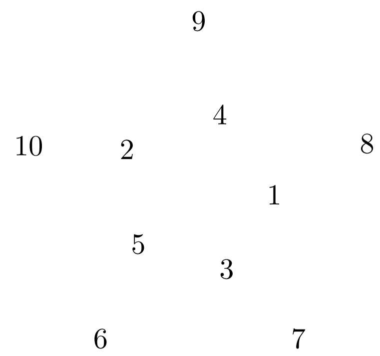

Figure 1

In *digraphs* (or *directed graphs*) the edges are not sets of vertices but ordered pairs of vertices. An edge (x,y) in a directed graph has the *origin vertex* x and the *terminal* *vertex* y. A digraph is said to be *undirected* if (x, y) is an edge whenver (y, x) is an edge.

Note that if X is a connected graph then (VX, d_X) is a metric space.

DEFINITION 2. The *ball* in a graph X with center o and radius r is the set

$$B_X(o, r) = \{x \in VX \mid d_X(o, x) \le r\}.$$

The *diameter* of a set of vertices A is

$$\operatorname{diam}_X A = \max\{d_X(x, y) \mid x, y \in A\}.$$

The set A is *bounded* if its diameter is finite.

EXERCISE 1. Let X be the Petersen graph from Example 1. Determine $d_X(\{5, 6\}, \{8, 9\})$, $\operatorname{diam}_X VX$ and $B_X(5, r)$, for $r = 0, 1, 2, 3$.

LEMMA 1. A set of vertices in a connected locally finite graph is bounded if and only if it is finite.

PROOF. Finite sets are bounded. Let o be any vertex. If X is locally finite then the ball $B_X(o, 1)$ is finite. Suppose $B_X(o, n)$ is finite. Because X is locally finite, it follows that $B_X(o, n + 1)$ is also finite. Thus every ball is finite and consequently every bounded set is finite. □

The following exercise shows that connected graphs are just metric spaces with some additional properties.

EXERCISE 2. Let (M, d) be a metric space such that the metric d has only integer values. Suppose that for any x and y with $d(x, y) = n$ there is a sequence $x = z_0, z_1, \ldots, z_n = y$ such that $d(z_i, z_{i+1}) = 1$ for all i. Set $VX = M$, $EX = \{\{x, y\} \mid d(x, y) = 1\}$ and $X = (VX, EX)$. Show that $d(x, y) = d_X(x, y)$, for all $x, y \in M$.

3. Ends of trees

Ends of general graphs will be introduced as sets of rays in Section 5. In the case of trees the situation is simpler. Ends of trees can be considered as rays which originate in a fixed vertex.

DEFINITION 3. Let T be a tree. We fix a vertex o and call it the *origin vertex* of T. Let $\Omega_o T$ be the set of rays which originate in o. Let a and b be any elments of $VT \cup \Omega_o T$. If a is a vertex then let $\pi(a)$ be the geodesic path from o to a. If a is a ray then we set $\pi(a) = a$. Analogously we define $\pi(b)$. Then $\pi(a) = (o = a_0, a_1, a_2, \ldots)$ and $\pi(b) = (o = b_0, b_1, b_2, \ldots)$, which are finite or infinite sequences. Let k be the maximal integer such that $a_i = b_i$, $0 \le i \le k$. Then $a_k = b_k$ is called the *confluent* of a and b (notation: $a \wedge b$) with respect to o. We define

$$u_o(a, b) = \begin{cases} 0 & \text{if } a = b, \\ \frac{1}{1 + d_T(o, a \wedge b)} & \text{if } a \ne b. \end{cases}$$

LEMMA 2 (see Lemma 8). Let T be a tree. Then $(VT \cup \Omega T, u_o)$ is an ultrametric space.

PROOF. Positive definiteness and symmetry follow from the definition. We have to show that the strong triangle inequality

$$u_o(a,b) \leq \max\{u_o(b,c), u_o(a,c)\}$$

holds for all a, b and c in $VT \cup \Omega T$. We may assume that $u_o(a,b)$ is the maximal of these three distances. By Definition 3, $a \wedge c$ is element of

$$\pi(a) = (o = x_0, x_1, \ldots, x_k = a \wedge b, x_{k+1}, \ldots).$$

Let $a \wedge c$ be the vertex x_l. If $l \leq k-1$ then $u_o(a,c)$ would be greater than $u_o(a,b)$. Hence $l \geq k$. If $l = k$ then $a \wedge c = a \wedge b$ which implies $u_o(a,c) = u_o(a,b)$ and the strong triangle inequality is satisfied. The remaining possibility is $l > k$. In this case $a \wedge b$ is also the confluent of b and c, and the strong triangle inequality is again satisfied. □

DEFINITION 4. The topology generated by u_o on $VT \cup \Omega T$ is called the end topology of T.

For vertices x and y of T we write $x \leq_o y$ whenever x lies on a geodesic from o to y. This definies a partial order on VT. Let $C_o(x)$ be the set of vertices y such that $x \leq y$. We define

$$D_o(x) = \{a \in VT \cup \Omega X \mid x \wedge a \in C_o(x)\}.$$

Then $D_o(x)$ is the set of vertices in $C_o(x)$ together with all rays that originate in o and contain x. In other words,

$$D_o(x) = \{a \in VT \cup \Omega X \mid u_o(a,o) = u_o(x,o)\}.$$

Let ω be an end. Then

$$D_o(x) = B_{X,o}(\omega, u_o(o,x))$$

LEMMA 3. The end topology of a tree does not depend on the choice of o. That is, let o and o' be any two vertices, then the topologies induced by u_o and $u_{o'}$ are isomorphic.

PROOF. □

EXERCISE 3.

(i) Show that any set of vertices of a tree is open and closed in the end topology.
(ii) Show that the end topology of a tree is totally disconnected.

4. Cayley graphs

DEFINITION 5. Let G be a group and let S be a subset of G which does not contain the neutral element. Set $S^{\pm} = S \cup \{s^{-1} \mid s \in S\}$ (that is, $S^{\pm}$ is symmetric). The *undirected Cayley graph* $\mathrm{Cay}(G,S)$ has vertex set $VX = G$ and vertices (or group elements) x and y are adjacent if $x^{-1}y$ is in $S^{\pm}$. The *directed Cayley graph* $\vec{\mathrm{Cay}}(G,S)$ has vertex set $VX = G$ and (x,y) is an edge if $x^{-1}y$ is in S.

Vertices x and y are adjacent if and only if there is an s in $S^{\pm}$ such that $xs = y$. A Cayley graph $\mathrm{Cay}(G,S)$ is connected if and only if S generates G.

Definition 6. Let X be an undirected graph. Then we write $\vec{X}$ for the directed graph with $V\vec{X} = VX$ and $E\vec{X} = \{(x, y) \mid \{x, y\} \in EX\}$. Note that one edge $\{x, y\}$ in EX corresponds to two edges (x, y) and (y, x) in $E\vec{X}$. Let D be a directed graph with a map $c : ED \to S$. Such a map is called a *colouring* and S is the set of *colours*. The colouring is *regular* if for every vertex x and every colour $s \in S$ there is exactly one edge e with origin vertex x such that $c(e) = s$ and exactly one edge f with terminal vertex x such that $c(f) = s$.

Theorem 1. *A connected graph X is isomorphic to a Cayley graph of some group $G = VX$ with generating set S if and only if $\vec{X}$ has a regular edge colouring $c : EX \to S$ which satisfies the following* closed path property*:*

- *If there is a closed directed path $(x_0, x_1, \dots, x_n = x_0)$ with colours $s_i = c(x_{i-1}, x_i)$, $1 \le i \le n$, then every directed path $(y_0, y_1, \dots, y_n = y_0)$ with the same colours $s_i = c(y_{i-1}, y_i)$ is also closed.*

Proof. Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph. Then (i) is satisfied if we let o be the neutral element. The colouring $c : E\vec{X} \to S$ with $c : (x, y) \mapsto x^{-1}y$ is regular. Let $\pi = (x_0, x_1, \dots, x_n = x_0)$ be a closed directed path with colours $s_i = c(x_{i-1}, x_i)$, $1 \le i \le n$. Then $s_1 s_2 \cdots s_n = 1$. Hence $y s_1 s_2 \cdots s_n = y$ and the path which starts in y and has the same colouring as π is also closed.

Let $\vec{X}$ be a directed graph with a regular colouring $c : EX \to S$ which satisfies the closed path property. Let x be a vertex and let s be a colour. Then there is exactly one neighbour z of x such that $c(x, z) = s$. Let s^{-1} be the colour of (z, x). By the closed path property, $c(u, v) = s$ implies $c(v, u) = s^{-1}$, for any adjacent vertices u and v. Hence the element s^{-1} is well defined for all $s \in S$.

Let x be a vertex and let $(s_1, s_2, \dots, s_n)$ be a word whose letters are colours. Since c is regular, there is a unique path $(x = x_0, x_1, x_2, \dots, x_n = y)$ such that $c(x_{i-1}, x_i) = s_i$. We write $y = x(s_1, s_2, \dots, s_n)$ for this unique vertex which is determined by x and the word $(s_1, s_2, \dots, s_n)$. For the following we fix a vertex o. Let $(s_1, s_2, \dots, s_m)$ and $(t_1, t_2, \dots, t_n)$ be words of colours such that $o(s_1, s_2, \dots, s_m) = o(t_1, t_2, \dots, t_n)$. Then $o(s_1, s_2, \dots, s_m, t_n^{-1}, t_{n-1}^{-1}, \dots, t_1^{-1}) = o$. By the closed path property, we see that $x(s_1, s_2, \dots, s_m, t_n^{-1}, t_{n-1}^{-1}, \dots, t_1^{-1}) = x$, for any vertex x. Colour regularity implies $x(s_1, s_2, \dots, s_m, t_n^{-1}, t_{n-1}^{-1}, \dots, t_2^{-1}) = x(t_1)$. By induction we get $x(s_1, s_2, \dots, s_m) = x(t_1, t_2, \dots, t_n)$. Hence, for any vertices x and y and for a word $\alpha = (s_1, s_2, \dots, s_m)$ which describes a path from o to y (that is, $y = o(s_1, s_2, \dots, s_m)$), we obtain a vertex $x(s_1, s_2, \dots, s_m)$ which does not depend of the choice of α as long as α describes a path from o to y. We denote the vertex $x(s_1, s_2, \dots, s_m)$ with $x \circ y$, and $\circ$ is a well defined map $VX \times VX \to VX$.

To complete the proof of the theorem we show that $(VX, \circ)$ is a group with neutral element o. By the definition of $\circ$ we have $o \circ x = x$ for any vertex x, and by the closed path property we have $x \circ o = x$. Let $x = o(s_1, s_2, \dots, s_m)$, $y = o(t_1, t_2, \dots, t_n)$ and $z = o(u_1, u_2, \dots, u_k)$ be any three vertices. Then $x \circ (y \circ z)$ and $(x \circ y) \circ z$ are both equal to the vertex $o(s_1, s_2, \dots, s_m, t_1, t_2, \dots, t_n, u_1, u_2, \dots, u_k)$. Hence the operation $\circ$ is associative. The inverse element of a vertex $x = o(s_1, s_2, \dots, s_m)$ is the vertex $x^{-1} = o(s_m^{-1}, s_{m-1}^{-1}, \dots, s_1^{-1})$. Thus $(VX, \circ)$ is a group, and this group is generated by the neighbours $o(s_i)$ of o. If we identify the colours s_i with the vertices $o(s_i)$ then $G = VX$ is generated by S and X is the Cayley graph $\mathrm{Cay}(G, S)$. □

DEFINITION 7. An *isomorphism* between graphs X and Y is a bijective function $g : VX \to VY$ such that any vertices x and y in VX are adjacent in X if and only if $g(x)$ and $g(y)$ are adjacent in Y. Graphs X and Y are called *isomorphic* if there is an isomorphism between X and Y. An isomorphism $g : VX \to VX$ is called an *automorphism*. The group of all automorphisms is denoted with $\mathrm{Aut}(X)$. Its group operation is the concatenation of functions. A group G *acts on a graph* X if there is a homomorphism $h : G \to \mathrm{Aut}(X)$. Instead of $h(g)$ we will usually just write g if this causes no confusion. A group G acts *freely* on X if the only element which fixes a vertex is the identity.

What is the difference between considering an action of a group G on a graph and considering the automorphism group $\mathrm{Aut}(X)$? It can happen that there are two different elements g_1 and g_2 of G which have the same action on X. In other words, there is one automorphism $h(g_1) = h(g_2)$ which corresponds to different group elements g_1 and g_2. In this case, the kernel of h is not trivial, it contains more than only the neutral element of G. On the other hand, it may happen that there are automorphisms f in $\mathrm{Aut}(X)$ which do not correspond to any element of G. Then $h^{-1}(f) = \emptyset$.

Why group actions instead of the automorphism group? Suppose we are interested in the structure of a group G. If we can find a graph X on which G acts in the certain way then we can derive algebraic properties from geometric properties of the graph, but without the assumption that G is the whole automorphism group.

DEFINITION 8. A group G acts *transitively* on a graph X if $G(x) = \{g(x) \mid g \in G\} = VX$ for some vertex x of X. A graph which has a transitive group action is called *transitive*.

Note that if $G(x) = VX$ for some vertex x then $G(y) = VX$ for any vertex y. A graph X is transitive if and only if $\mathrm{Aut}(X)$ acts transitively on X. As a consequence of Theorem 1 we obtain the following corollary.

EXERCISE 4.

(i) Find a regular graph with 7 vertices which is not transitive.
(ii) Show that there is no regular transitive graph with less than 7 vertices.

Theorem 1 implies the following.

COROLLARY 1. Let X be a directed graph with a regular colouring $c : EX \to S$ which is invariant under a transitive action of a group G with a trivial kernel and let S be a subset of G. Then X is the Cayley graph $\mathrm{Cay}(G, S)$.

PROOF. The closed path property is satisfied, because G acts transitively and the action preserves the colouring. By Theorem 1, X is a Cayley graph. To show that this Cayley graph is in fact the Cayley graph $\mathrm{Cay}(G, S)$, we have to use the assumption that the kernel of the action of G on X is trivial. Let o be a vertex and let g be an element of G. Let $(s_1, s_2, \ldots s_n)$ be the colours of a path from o to $g(o)$. For any vertex x, the image $g(x)$ is the vertex $x \circ (s_1, s_2, \ldots s_n)$, see proof of Theorem 1. Since we have assumed that the kernel of the action is trivial (only contains the neutral element), g is the only element of G which maps the vertex o to the vertex $g(o)$. Hence this yields a bijection between G and VX which is a group isomorphism between G and $(VX, \circ)$ (see proof of Theorem 1). Thus X is the Cayley graph of G with respect to S. □

EXERCISE 5. For the Platonic bodies let e be the number of edges of a face and let d be the number of faces in each vertex. Note that d is also the number of edges in each vertex. There are five Platonic bodies, the Tetrahedron ($e = d = 3$), the Cube ($e = 4$, $d = 3$), the Octahedron ($e = 3$, $d = 4$), the Dodecahedron ($e = 5$, $d = 3$) and the Icosahedron ($e = 3$, $d = 5$). Set $G = \langle a, b \mid a^2 = b^d = (ab)^e = 1 \rangle$ and $S = \{a, b\}$. These five polyhedrons correspond to regular graphs whose vertices have degree d. A Platonic body with parameters e_1 and f_1 is said to be *dual* to the Platonic body with parameters e_2 and f_2 if $e_1 = f_2$ and $f_1 = e_2$.

(i) Draw the Cayley graphs $\mathrm{Cay}(G, S)$ for the parameters of the five Platonic bodies. These graphs are Platonic bodies with "chopped off" vertices.

(ii) What happens if we consider other values for the parameters e and d? If the resulting group is infinite then draw a finite part of the Cayley graph. Without proof: Which of these graphs can you use as a model for tiling your bathroom? Which of them are dual to each other?

EXERCISE 6. Show that the Icosahedron is a Cayley graph by constructing a regular edge colouring which satisfies the closed path property.

LEMMA 4. A group G acts on any of its Cayley graphs by left multiplication. On the directed Cayley graph $\vec{\mathrm{Cay}}(G, S)$ the left multiplication is a colour preserving automorphism.

PROOF. Let $X = \mathrm{Cay}(G, C)$ be a Cayley graph and let h be an element of G. The left multiplication l_h is the map $G \to G$ where $g \mapsto hg$. Let g_1 and g_2 be elements of G. Then $g_1 \neq g_2$ implies $hg_1 \neq hg_2$ and l_h is injective. Let g be any element of G, then $h^{-1}g$ is again in G and $l_h(h^{-1}g) = g$. Thus l_h is surjective and consequently a bijection.

Recall that elements g_1 and g_2 are adjacent if and only if there is an s in S such that $g_1 s = g_2$. This equation is equivalent to $l_h(g_1)s = l_h(g_2)$. Hence l_h is an automorphism of the Cayley graph and it preserves the edge colouring of S. □

If G is commutative then the right multiplication also acts on the Cayley graphs of G as graph automorphisms. But for non-commutative groups this is not true in general.

THEOREM 2 (Theorem of Sabidussi, see [**18**]). *A graph X is a Cayley graph of a group G if and only if there is a transitive and free action of G on X.*

PROOF. By Lemma 4, the left multiplication is a transitive action on any Cayley graph. If a left multiplication l_h on a Cayley graph has a fixed point g then $hg = g$ and h must be the neutral element. Hence this action is free.

Let G be a group which acts transitively and freely on an undirected digraph $\vec{X}$. Let o be any vertex. Since G acts transitively, we can find a set $S = \{s_i \mid i \in I\}$ of group elements such that $\{s_i(o) \mid i \in I\}$ is the set of neighbours of o. We set $e_i = (o, s_i(o))$ and write $g((x, y))$ instead of $(g(x), g(y))$, which the image of an edge (x, y) under the action of g.

Next we construct a regular edge colouring of $\vec{X}$. Let s_i be the colour of e_i. Suppose there are distinct edges e_i and e_j which can be mapped to the same edge e be distinct group elements g_i and g_j. This means that $g_i(e_i) = e$ and $g_j(e_j) = e$. Hence $g_j^{-1}g_i(o) = o$. Since G acts freely, $g_j^{-1}g_i$ must be the neutral element of G and consequently $g_j = g_i$. A contradiction. Hence each edge e is image of at most one edge e_i. For each vertex x there is a g in G such that $g(o) = x$, because G acts transitively. The element g acts as a graph automorphism. Thus the edges which originate in o are mapped bijectively onto the edges which originate in x. Each edge e of the graph is the image of exactly one edge e_i. This defines an edge colouring $c : EX \to S$. In the following keep in mind that elements of S are elements of the group G and colours at the same time. If there is a g such that $g(o, s_i(o)) = e$ then $c(e)$ is defined as s_i. For a colour $s \in S$, the element s^{-1} is the colour of the edge $(o, s^{-1}(o))$ which is the colour of $(s(o), o)$. For each s in $S^{\pm} = S \cup \{s^{-1} \mid s \in S\}$, there is exactly on edge originating in o with the colour s and exactly on edge terminating in o with the colour s. Note that we do not exclude the case $s = s^{-1}$. It follows that the colouring c is regular. The kernel of a homomorphism $G \to \mathrm{Aut}(X)$ of a free group action only contains the neutral element. By Corollary 1, X is the Cayley graph $\mathrm{Cay}(G, S)$. □

Exercise 7. Show that the Petersen graph (see Example 1) is no Cayley graph. Hint: For a regular edge colouring of a regular graph whose vertices have degree 3, there are two cases. Case 1. $S = \{a, b, c\}$ and $a^2 = b^2 = c^2 = 0$. Up to permutation, a cycle of length 5 has to have the form (x, y, x, y, z) where $x, y, z \in S$. Case 2. $S = \{a, b\}$, $a^2 = 0$ and $b^2 \neq 1$. Cycles of length 5 are (up to permutation) of the form (x, x, x, x, x), (a, x, x, x, x), (a, x, a, x, x) or (a, x^{-1}, a, x, x) where $x \in \{b, b^{-1}\}$. Use the closed path property to lead these cases to a contradiction. You can also find a solution for this exercise by using the fact that up to isomorphy there are only two groups with 10 elements.

5. End spaces

Definition 9. A *ray* is an infinite path $(x_0, x_1, \ldots)$ of distinct vertices. A set of vertices F *separates* sets of vertices A and B if any path from any vertex in A to any vertex in B contains an element of F. Two rays are *equivalent* (or *vertex equivalent*) if they cannot be separated by a finite set of vertices.

The *boundary* θC (or *vertex boundary*) of $C \subset VX$ is the set of vertices in $VX \setminus C$ which are adjacent to a vertex in C. If θC is finite then C is called a *cut* (or *vertex cut*). A ray R *lies in* a set of vertices C if all but finitely many vertices of R are elements of C.

Exercise 8. Show that the following holds for any graph.

(i) Vertex equivalence of rays is an equivalence relation on the set of all rays.
(ii) Two rays R_1 and R_2 are vertex equivalent if and only if there is a third ray which has infinitely many vertices in common with R_1 as well as with R_2.

Definition 10. The equivalences classes on the set of rays are called *ends* (or *vertex ends*). An end *lies in* a set of vertices C (or C *contains* ω) if all rays of ω lie in C. The set of ends which lie in C is denoted by ΩC. We write ΩX instead of ΩVX for the set of all ends. A set of vertices F *separates* ends ω_1 and ω_2 if F separates any ray in ω_1 from any ray in ω_2.

If a ray R lies in a cut C then the end which R belongs to lies also in C. Each pair of distinct ends is separated by a finite set of vertices.

Lemma 5. The set

$$\mathcal{B}(X) = \{C \cup \Omega C \mid C \text{ is a cut.}\}$$

is closed under finite intersection.

Proof. To show that the intersection of finitely many elements of $\mathcal{B}(X)$ is again in $\mathcal{B}(X)$ it suffices to show that the intersection of two elements $C \cup \Omega C$ and $D \cup \Omega D$ of $\mathcal{B}(X)$ is again in $\mathcal{B}(X)$. Let C and D be cuts. First we show that $C \cap D$ is also a cut. Let x be an element of $\theta(C \cap D)$. Then x is adjacent to some y in $C \cap D$. Since x is in $VX \setminus (C \cap D)$, x is either element of $VX \setminus C$ or element of $VX \setminus D$. If $x \in VX \setminus C$ then $x \in \theta C$, because x is adjacent to $y \in C$. If $x \in VX \setminus D$ then $x \in \theta D$, because x is adjacent to $y \in D$. Thus x is in $\theta C \cup \theta D$, and consequently $\theta(C \cap D) \subset \theta C \cup \theta D$. Since θC and θD are finite, the boundary $\theta(C \cap D)$ is also finite which means that $C \cap D$ is a cut.

Let R be a ray which lies in $C \cap D$. Then R lies in C and in D. If R lies in C and in D then R lies in $C \cap D$. Hence $\Omega(C \cap D) = \Omega C \cap \Omega D$. We conclude,

$$(C \cup \Omega C) \cap (D \cup \Omega D) = (C \cap D) \cup (\Omega C \cap \Omega D) = (C \cap D) \cup \Omega(C \cap D),$$

where $C \cap D$ is a cut and $(C \cap D) \cup \Omega(C \cap D)$ is an element of $\mathcal{B}(X)$. □

Definition 11. The topology on $VX \cup \Omega X$ generated by the base $\mathcal{B}(X)$ is called *(vertex) end topology* of X.

Let F_n denote the free group of rank n.

Exercise 9. Show that the infinite Cayley graphs $\mathrm{Cay}(G, S)$ with $S = \{a, b\}$ correspond to the graphs drawn below by finding a suitable edge colouring. How many ends do they have? See Figure 2.

(a) $G = \mathbb{Z} = F_1 = \langle a, b \mid a^2 = b \rangle$
(b) $G = \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}) = \langle a, b \mid ab = ba, b^2 = 1 \rangle$
(c) $G = \mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$
(d) $G = F_2 = \langle a, b \rangle$

Exercise 10. Find groups G' with generating set S' such that the Cayley graph $\mathrm{Cay}(G', S')$ is the same as in Exercise 9 but such that G' is not isomorphic to the corresponding group G in Exercise 9.

Exercise 11. Set (i) $x_n = (ab)^n$, (ii) $x_n = ab^n$, (iii) $x_n = a^n b$, (iv) $x_n = ba^n$ and (v) $x_n = a^{(-1)^n n}$. Which of the sequences $(x_n)_{n \in \mathbb{N}}$ are convergent in the end topology of the Cayley graphs (a), (b), (c) and (d) of Exercise 9? How many cluster points do they have and which of them coincide?

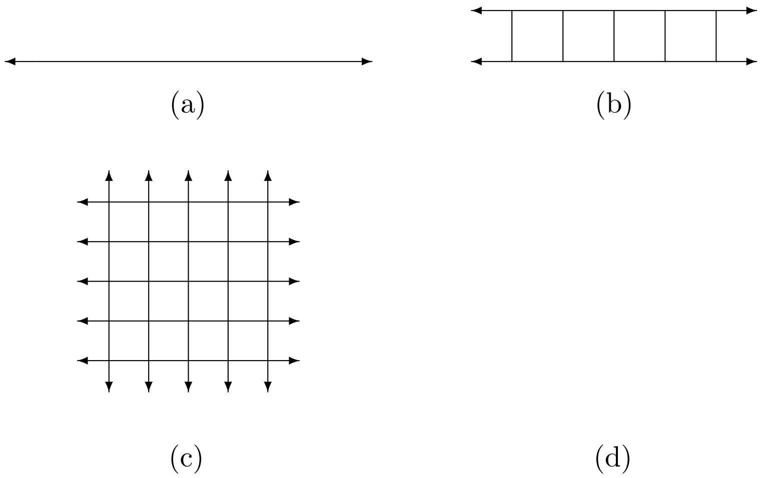

Figure 2

6. Compactness and separation

THEOREM 3. *The vertex end topology of a locally finite graph is T_4 and has a countable base. All singletons are closed.*

PROOF. First we prove that all singletons $\{\alpha\}$, $\alpha \in VX \cup \Omega X$, are closed. We will do this by showing that for any other element $\beta \in VX \cup \Omega X$, $\beta \neq \alpha$, there is an element $U_\beta = C \cup \Omega C$ of the base $\mathcal{B}(X)$ which contains β but which does not contain α. Once we have proved this, the complement of α in $VX \cup \Omega X$ is open, because it is the union of the open sets U_β, $\beta \neq \alpha$. And this means that $\{\alpha\}$ is closed.

So let us distinguish:

(i) If β is a vertex then we choose $U_\beta = \{\beta\}$. The singleton $\{\beta\}$ of the vertex β is a cut, because X is locally finite.

(ii) If β is an end and α is a vertex then let C be the component of $VX \setminus \{\alpha\}$ which contains the end β. Then $U_\beta = C \cup \Omega C$ is an element of the base which contains β but not α.

(iii) Let α and β be both ends. By the definition of an end, α and β can be separated by a finite set of vertices F. Let C be the component of $VX \setminus F$ which contains β, then $U_\beta = C \cup \Omega C$ contains β but not α.

By what we have said at the beginning of this proof, this now implies that $\{\alpha\}$ is closed.

Next we show that the end topology on $VX \cup \Omega X$ is regular. Let A be a closed set which does not contain some element $\alpha \in VX \cup \Omega X$. We have to show that there are disjoint open sets U and V such that $\alpha \in U$ and $A \subset V$. Morover, we will show that U and V can be chosen as elements of the base $\mathcal{B}(X)$. If α is a vertex then $U = \{\alpha\}$ and $V = (VX \cup \Omega X) \setminus \{\alpha\} = VX \setminus \{\alpha\} \cup \Omega(VX \setminus \{\alpha\})$. Let α be an end. Since the complement $(VX \cup \Omega X) \setminus A$ is open, it is a union of elements of the base. One of these elements, say $C \cup \Omega C$, contains α. We set $U = C \cup \Omega C$ and $V = (VX \cup \Omega X) \setminus U = VX \setminus C \cup \Omega(VX \setminus C)$. Because $U \subset (VX \cup \Omega X) \setminus A$, we have $A \subset V$. Note that since X is locally finite, $VX \setminus C$ is a cut and therefore V is also an element of the base. The base elements U and V are disjoint, α is an element of U and A is a subset of V. Thus the end space is regular.

We have seen that all singletons are closed and that the topology is regular. This implies that it is Hausdorff.

By Lemma 1, all balls are finite. Thus VX is countable and there are only countably many finite sets of vertices F. Using the assumption that X is locally finite, we see that for each such set F there are only finitely many cuts C such that $\theta C = F$. Hence the base $\mathcal{B}(X)$ is countable. A regular space with a countable base is normal, see Theorems 10 and 11. A normal Hausdorff space is T_4, see Section 10.3. □

Lemma 6. Let C be a set of vertices such that $VX \setminus C$ is connected and such that C contains the boundary of a set of vertices D. Then either $D \subset C$ or $VX \setminus D \subset C$.

Proof. Suppose there are vertices $x \in D \setminus C$ and $y \in (VX \setminus D) \setminus C$. Since the complement of C is connected there is a path π from x to y which is completely contained in $VX \setminus C$. This path contains vertices of D and vertices of $VX \setminus D$. Thus π must contain a vertex of θD which is a contradiction to the assumption $\theta D \subset C$. □

Definition 12. A *subpath* of a path $(x_0, x_1, \ldots, x_n)$ or a ray $(x_0, x_1, \ldots)$ is a finite subsequence $(x_i, x_{i+1}, \ldots, x_{i+k})$. A *geodesic ray* is a ray whose subpaths are all geodesic. The components in the complement of a ball $B_X(o, r)$ are called *radial cuts* with center o and radius r. The set of all radial cuts with center o and radius r is denoted by $\mathcal{C}_o(r)$. Given an end ω, we write $C_o(\omega, r)$ for the radial cut in $\mathcal{C}_o(r)$ which contains ω.

Note that given o, r and ω then $C_o(\omega, r)$ is well defined. But for a given radial cut, o, r and ω are not necessarily uniquely determined.

Lemma 7. Let X be a locally finite graph and let there be a descending sequence $C_0 \supset C_1 \supset C_2 \supset \ldots$ of non-empty radial cuts C_r with center o and radius r. Then there is a geodesic ray R which originates in o and lies in all cuts C_n. The end which contains R is the only end which lies in all these radial cuts.

Proof. Let S_n be the set of geodesic paths from an element of θC_n to o. The paths in S_n have length n. By Lemma 1, any ball is finite. Since S_n is contained in $B_X(o, n)$, this implies that S_n only contains finitely many paths. Set $S = \bigcup_{n \geq 1} S_n$. Then S is infinite. Hence there is a path π_1 in S_1 which is subpath of infinitely many elements of S. One of the finitely many paths in S_2 which contains π_1 is itself again subpath of infinitely many elements of S. Let us denote this path by π_2. By induction we obtain a sequence of paths $(\pi_n)_{n \geq 1}$ such that $\pi_n \in S_n$ and π_n is a subpath of π_{n+1}. Hence

$$R = \bigcup_{n \geq 1} \pi_n$$

is a geodesic ray which lies in all cuts C_n. Let us denote the end which contains R with ω. Suppose there is another end η which lies in all these cuts C_n. There is a finite set of vertices F which separates ω from η. And there is a ball $B_X(o, n)$ which contains F. Hence η cannot lie in C_i, for all $i \geq n$. □

Theorem 4. *The end topology of a locally finite graph is compact. Every sequence of distinct elements of $VX \cup \Omega X$ has a subsequence which converges to an end.*

Proof. Let $\xi = (z_k)_{k \in \mathbb{N}}$ be a sequence of distinct elements of $VX \cup \Omega X$. Fix some vertex o. Since X is locally finite there are only finitely many radial cuts with center

o and radius $r = 0$. One of these cuts must contain infinitely many elements of the sequence ξ. Let us denote this cut by C_0. There are only finitely many cuts with center o and radius $r = 1$ which are contained in C_0. One of these cuts must contain infinitely many elements of ξ. Let us denote this cut by C_1. By induction, we obtain an infinite descending sequence $C_0 \supset C_1 \supset C_2 \supset \ldots$ of radial cuts C_r with center o and radius r, all of which contain infinitely many elements of ξ. Thus there is a subsequence $\xi = (z_{k_n})_{n \in \mathbb{N}}$ of ξ such that $z_{k_n} \in C_n \cup \Omega C_n$. Lemma 7 says that there is an end ω which lies in all these cuts C_n.

Let U be a neighbourhood of ω. Then U contains an element $D \cup \Omega D$ of the base $\mathcal{B}(X)$ such that $\omega \in \Omega D$. Since θD is finite, there is an n_0 such that $\theta D \subset VX \setminus C_{n_0}$. The complement $VX \setminus (VX \setminus C_{n_0}) = C_{n_0}$ is connected and we can apply Lemma 6. Thus $D \subset VX \setminus C_{n_0}$ or $VX \setminus D \subset VX \setminus C_{n_0}$. Because ω lies in both cuts C_{n_0} and D, the intersection $C_{n_0} \cap D$ is not empty. Hence $D \subset VX \setminus C_{n_0}$ is impossible and we have $VX \setminus D \subset VX \setminus C_{n_0}$ which is equivalent to $C_{n_0} \subset D$. And we also have $\Omega C_{n_0} \subset \Omega D$. Thus $z_{k_n} \in U$, for $n \geq n_0$, $(z_{k_n})_{n \in N}$ converges to ω and the end topology is sequentially compact. Every sequentially compact space with a countable base is compact (see Theorems 11 and 12) $\square$

The end topology on $VX \cup \Omega X$ is a compactification of the discrete topology on VX and the set of ends ΩX is a compact boundary of VX. In locally finite graphs, singletons of vertices are open. Thus VX is open and ΩX is closed. Closed subsets of compact sets are compact.

In non-locally finite graphs, the vertex end topology is a sequentially compact and compact T_0-space, see [**14**, Lemma 5, Theorem 1]. But ΩX is not always a closed subset of $VX \cup \Omega X$ and ΩX is not always compact.

7. Metrisation of the end topology

The end topology of a locally finite graph is metrizable, because any regular T_1 space with a countable base is metrizable, see Theorem 13. But we can also give an explicit metric which induces the end topology.

Let us fix some vertex o. Let a and b be elements of $VX \cup \Omega X$. Then there is a smallest integer $r_o(a, b) \geq 0$ such that $B_X(o, r_o(a, b))$ separates a from b. Let f be a positive, strictly decreasing function $f : \mathbb{N}_0 \to [0, \infty)$ such that $\lim f(n) = 0$ for $n \to \infty$. And then we set

$$u_o(a, b) = \begin{cases} 0 & \text{if } a = b, \\ f(r_o(a, b)) & \text{if } a \neq b. \end{cases} \tag{1}$$

For instance, $f : n \mapsto 1/(1 + n)$ or $f : n \mapsto e^{-n}$.

The function u_o will turn out to be a metric we are looking for, a metric that generates the end topology.

Exercise 12. Let $F_2 = \langle a, b \rangle$ be the free group of rank 2 as in Exercise 9d, see also Exercise 11. We consider the Cayley graph $\mathrm{Cay}(F_2, S)$ where $S = \{a, b\}$. Let ω be the end which contains the ray $(a, ab, aba, abab, ababa \ldots)$. Let x_1 be the neutral element of F_2, we set $x_2 = ababba$, $x_3 = baba$ and $x_4 = \omega$. Let u_o be defined by (1) for the function $f : n \mapsto 1/(1 + n)$. Determine the values for $u_{X, x_i}(x_j, x_k)$, where $1 \leq i, j, k \leq 4$ are distinct integers.

Since f is strictly decreasing, we have

$$u_o(a,b) \leq u_o(c,d) \iff r_o(a,b) \geq r_o(c,d), \tag{2}$$

for all a, b, c and d in $VX \cup \Omega X$.

LEMMA 8. Let X be a connected locally finite graph. Then $(VX \cup \Omega X, u_o)$ is an ultrametric space.

PROOF. Positive definiteness and symmetry follow from the definition of u_o. We have to show that the strong triangle inequality

$$u_o(a,b) \leq \max\{u_o(b,c), u_o(a,c)\}$$

is satisfied for all a, b and c in $VX \cup \Omega X$. We may assume that $a \neq b$ and that

$$u_o(a,b) = \max\{u_o(a,b), u_o(b,c), u_o(a,c)\},$$

because otherwise the strong triangle inequality holds anyway. By (2), this is equivalent to

$$r_o(a,b) = \min\{r_o(a,b), r_o(b,c), r_o(a,c)\}. \tag{3}$$

If $r_o(a,b) = r_o(a,c)$ then $u_o(a,b) = u_o(a,c)$ and the strong triangle inequality is satisfied. Otherwise we have $r_o(a,b) < r_o(a,c)$. This implies that neither a nor c is a vertex in $B_X(o, r_o(a,b))$.

Let C be the component of $VX \setminus B_X(o, r_o(a,b))$ such that $c \in C \cup \Omega C$. Since $B_X(o, r_o(a,b))$ does not separate a and c, the element a is also in $C \cup \Omega C$. But b is not in C, because $B_X(o, r_o(a,b))$ separates a from b. Hence $B_X(o, r_o(a,b))$ separates b from c. This implies $r_o(b,c) \leq r_o(a,b)$. By (3), we have $r_o(b,c) = r_o(a,b)$ which implies $u_o(b,c) = u_o(a,b)$ and the strong triangle inequality is satisfid. □

Let $O_o(a,\rho) = \{b \in VX \cup \Omega X \mid u_o(a,b) < \rho\}$ denote the open ball with respect to the ultrametric u_o with center a and radius ρ. Recall that $C_o(\omega, r)$ denotes the radial cut with center o and radius r which contains ω, see Definition 12.

LEMMA 9. Let X be a locally finite connected graph, let $\omega \in \Omega X$ and $a \in VX \cup \Omega X$. Then

$$O_o(\omega, u_o(\omega,a)) = C_o(\omega, r_o(\omega,a)) \cup \Omega C_o(\omega, r_o(\omega,a)).$$

PROOF. By (2), the ball $O_o(\omega, u_o(\omega,a))$ is the set of elements b in $VX \cup \Omega X$ such that $r_o(\omega,b) > r_o(\omega,a)$. These are the elements b which are not separated from ω by $B_X(o, r_o(\omega,a))$. And these are precisely the elements of $C_o(\omega, r_o(\omega,a)) \cup \Omega C_o(\omega, r_o(\omega,a))$. □

LEMMA 10. For any vertex x, the singleton $\{x\}$ is open in the end topology as well as in the topology generated by u_o. Let ω be a end which is element of a subset A of $VX \cup \Omega X$. Then ω is an inner point of A with respect to the end topology if and only if ω is an inner point of A with respect to the topology generated by u_o.

PROOF. A vertex x is separated from any element a of $(VX \cup \Omega X) \setminus \{x\}$ by $B_X(o, d_X(o,x))$, because this ball contains x. In other words, $u_0(x,a) > f(d_X(o,x))$, and $O_o(x, f(d_X(o,x))) = \{x\}$. Hence $\{x\}$ is open in the topology which is induced by u_o. Since X is locally finite, $\{x\}$ is cut. Hence $\{x\} \in \mathcal{B}(X)$ and $\{x\}$ is open in the end topology.

Let the end ω be an inner point of $A \subset VX \cup \Omega X$ with respect to the ultrametric u_o. That is, there is an $\epsilon > 0$ such that $O_o(\omega, \epsilon) \subset A$. Let $(x_0, x_1, x_2, \ldots)$ be a ray in ω. Then

$$\lim_{n\to\infty} r_o(\omega, x_n) = \infty. \quad \text{which implies} \quad \lim_{n\to\infty} f(r_o(\omega, x_n)) = 0.$$

There is an n such that $u_o(\omega, x_n) < \epsilon$. Using Lemma 9, we get

$$C_o(\omega, r_o(\omega, x_n)) \cup \Omega C_o(\omega, r_o(\omega, x_n)) = O_o(\omega, u_o(\omega, x_n)) \subset O_o(\omega, \epsilon) \subset A.$$

Since $C_o(\omega, r_o(\omega, x_n)) \cup \Omega C_o(\omega, r_o(\omega, x_n))$ is an element of the base $\mathcal{B}(X)$, this means that ω is an inner point of A with respect to the end topology.

Let the end ω be an inner point of $A \subset VX \cup \Omega X$ with respect to the end topology. Then there is an element $C' \cup \Omega C'$ of $\mathcal{B}(X)$ such that $\omega \in C' \cup \Omega C' \subset A$. If $C' = VX$ then $A = VX \cup \Omega X$ and every point in A is an inner point because $VX \cup \Omega X$ is open. If $VX \neq C'$ then $\theta C' \neq \emptyset$ and there is a finite connected set F which contains $\theta C'$. The set $C = C' \setminus F$ is also an element of $\mathcal{B}(X)$ which contains ω. The complement $VX \setminus C$ is connected. Since θC is finite, there is an integer r such that $\theta C \subset B_X(o, r)$. This implies $C_o(\omega, r) \subset C$. Hence $\theta C_o(\omega, r+1) \subset C_o(\omega, r) \subset C$. The complement $VX \setminus C$ is connected. We can now apply Lemma 6. So either $C_o(\omega, r+1) \subset C$ or $VX \setminus C_o(\omega, r+1) \subset C$. The latter is equivalent to $C \cup C_o(\omega, r+1) = VX$ which is impossible, because neither C nor $C_o(\omega, r+1)$ contains θC. Hence $C_o(\omega, r+1) \subset C$. Let a be an element of $C_o(\omega, r+1)$. Then $r_o(\omega, a) > r+1$. Set $\epsilon = u_o(\omega, a)$. By Lemma 9,

$$O_o(\omega, \epsilon) = C_o(\omega, r_o(\omega, a)) \cup \Omega C_o(\omega, r_o(\omega, a)) \subset C_o(\omega, r+1) \cup \Omega C_o(\omega, r+1) \subset A$$

which means that ω is an inner point of A with respect to u_o. □

Lemma 10 implies the following theorem:

THEOREM 5. *The ultrametric u_o of a connected locally finite graph X induces the end topology on $VX \cup \Omega X$.*

COROLLARY 2. The topology induced by the ultrametric u_o does not depend on the choice of o.

COROLLARY 3. The end topoogy of a connected locally finite graph is strongly 0-dimensional.

PROOF. The end topology of a connected locally finite graph is induced by an ultrametric. See Theorem 14. □

8. The Theorem of Abels and Hopf

Let A be a subset of $VX \cup \Omega X$. Then ∂A denotes the topological boundary in the end topology (see Section 10.3).

EXERCISE 13. Let X be a locally finite graph and let A be a subset of $VX \cup \Omega X$. Show that $\partial A \subset \Omega X$.

EXERCISE 14. Let X be the one-side infinite comb (see Figure 3). That is,

$$VX = \{(x, y) \mid x, y \in \mathbb{N}_0\} \quad \text{and}$$

$$EX = \{\{(x, y), (x, y+1)\} \mid x, y \in \mathbb{N}_0\} \cup \{\{(x, 0), (x+1, 0)\} \mid x \in \mathbb{N}_0\}.$$

Since X is a tree, there is a one-to-one correspondence between the ends of X and the rays which originate in a given vertex, say $(0,0)$. These rays are

$$\begin{aligned} R &= \{(0,0),(1,0),(2,0),(3,0),\ldots\} \\ S_0 &= \{(0,0),(0,1),(0,2),(0,3),\ldots\} \\ S_1 &= \{(0,0),(1,0),(1,1),(1,2),(1,3),\ldots\} \\ S_2 &= \{(0,0),(1,0),(2,0),(2,1),(2,2),(2,3),\ldots\} \quad \text{etc.} \end{aligned}$$

Let η be the end containing R and let ω_n be the end containing S_n. Then

$$\Omega X = \{\eta, \omega_0, \omega_1, \omega_2, \ldots\},$$

see Figure 3. Determine ∂A_i for

(i) $A_1 = \{(2,y),(3,y) \mid y \in \mathbb{N}_0\}$,
(ii) $A_2 = \{(x,x) \mid x \in \mathbb{N}_0\}$,
(iii) $A_3 = \{(x,y) \mid x,y \in \mathbb{N}_0, x \geq y\}$ and
(iv) $A_4 = \{(x,y) \mid x,y \in \mathbb{N}_0, x \leq y\}$.

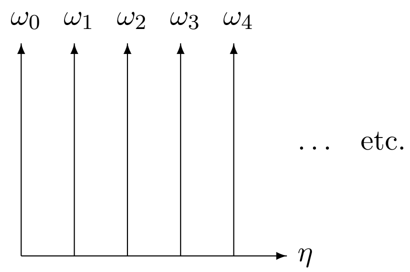

Figure 3

Let G act on a graph X and let x be a vertex. Then we write $G(x)$ to denote the orbit $\{g(x) \mid g \in G\}$.

LEMMA 11. Let a group G act on a connected locally finite graph X such that $\partial G(x) = \Omega X$ for some vertex x. Then $\partial G(y) = \Omega X$ for any vertex y.

PROOF. First note that $\partial G(y) \subset \Omega X$, see Exercise 13. We want to prove $\Omega X \subset \partial G(y)$. Let ω be an end. In order to show that $\omega \in \partial G(y)$, we have to show that every element $C \cup \Omega C$ of the base $\mathcal{B}(X)$ which contains ω also contains some element $g(y)$ of the orbit $G(y)$. Set

$$C' = \{z \in C \mid d_X(z, VX \setminus C) > d_X(x,y)\}.$$

The difference $C \setminus C'$ is bounded and bounded sets in locally finite graphs are finite. This implies that C' is a cut and that $\Omega C' = \Omega C$. Hence $C' \cup \Omega C'$ is also an open neighbourhood of ω. Since $\partial G(x) = \Omega X$, there is an element $g'(x)$ in C'. By the definition of C',

$$d_X(g'(x), VX \setminus C) > d_X(x,y) = d_X(g'(x), g'(y)).$$

This implies $g'(y) \in C$. Hence $C \cup \Omega C$ contains an element of the orbit $G(y)$, and this was what we had to show to complete the proof of the Lemma. $\square$

THEOREM 6. *Let a group G act on an infinite locally finite connected graph X such that every end is an accumulation point of the orbits of G. Then X has 1 or 2 ends, or ΩX is a Cantor set.*

A similar theorem was first proved by Hopf in [**13**] for Freudenthal's end compactification in [**6**]. He assumed that there is a group action and a compact set such that translates of the compact set cover the whole space. The graph theoretic analogue are almost transitive graphs (see below) where the translates of a finite set of vertices cover the whole graph. Abels observed in [**1**] that this assumption can be weakened. He just assumed that all boundary points are accumulation points of an orbit. Hence Theorem 6 is the graph theoretic version of the theorem of Abels.

PROOF OF THEOREM 6. Let ω_1 and ω_2 be any distinct ends. Then there is a finite set of vertices F such that ω_1 and ω_2 lie in different components C_1 and C_2 of $VX \setminus F$. The base elements $C_1 \cup \Omega C_1$ and $C_2 \cup \Omega C_2$ are disjoint, open and closed in the end topology on $VX \cup \Omega X$. Hence ΩX is totally disconnected.

Since X is infinite, there is at least one end. Suppose there are three distinct ends ω_1, ω_2 and ω_3 and suppose that ΩX is not perfect. A T_1 space which is not perfect has an isolated point. For such an isolated end η there is an open and closed element $C \cup \Omega C$ of the base $\mathcal{B}(X)$ such that $\Omega C = \{\eta\}$. We can choose C such that $VX \setminus C$ is connected. Then $C \neq VX$ and $\theta C \neq \emptyset$. There is a finite connected set F which separates ω_1, ω_2 and ω_3 from each other. Set

$$C' = \{x \in C \mid d_X(x, \theta C) \geq \operatorname{diam}_X F\}.$$

Let x be a vertex of θC. By Lemma 11, there is a g in G such that $g(x) \in C'$. This implies $g(F) \subset C$. Let D_i be the component of $VX \setminus F$ which contains ω_i where $i = 1, 2, 3$. Then $g(\theta D_i) \subset C$. The complements $VX \setminus D_i$ are connected, because F is connected. The same holds for $VX \setminus g(D_i)$ and $g(F)$. Hence we can apply Lemma 6, and either $g(D_i) \subset C$ or $g(VX \setminus D_i) \subset C$. Since η is the only end which lies in C, the inclusion $g(D_i) \subset C$ can only be satisfied for at most one i. Thus there are at least two distinct indices i_a and i_b such that $g(VX \setminus D_{i_a}) \subset C$ and $g(VX \setminus D_{i_b}) \subset C$ which is equivalent to $g(D_{i_a}) \supset VX \setminus C$ and $g(D_{i_b}) \supset VX \setminus C$. But D_{i_a} and D_{i_b} are disjoint, and therefore $g(D_{i_a})$ and $g(D_{i_b})$ are also disjoint. Hence the assumption that there is an isolated end leads to a contradiction, which proves that ΩX is perfect. We have seen before that ΩX is metrizable and compact. Thus ΩX is a Cantor set. □

Theorem 6 applies to transitive graphs such as Cayley graphs. A graph X is *transitive* if $\operatorname{Aut}(X)(x) = \{g(x) \mid g \in \operatorname{Aut}(X)\} = VX$ for any vertex x. Then $\partial \operatorname{Aut}(X)(x) = \partial VX = \Omega X$.

A group G is said to act *almost transitively* on a graph X if it has only finitely many orbits. And G is said to act *metrically almost transitive* on X if there is an integer r such that $d_X(x, G(y)) \leq r$ for any vertices x and y. Equivalently, if there is a ball B such that

$$\bigcup_{g \in G} g(B) = VX.$$

In locally finite graphs, almost transitivity is equivalent to metrical almost transitivity. For locally finite graphs with an almost transitive group action, the condition $\partial G(x) =$

ΩX of the Theorem 6 is satisfied. The following example shows that $\partial G(x) = \Omega X$ does not necessarily imply that G acts almost transitive, or metrically almost transitive.

EXERCISE 15. Set $G = F_2 = \langle a, b\rangle$ and $X = \mathrm{Cay}(G, \{a, b\})$. Let G' be the commutator subgroup of G. (i) Show that $\partial G' = \Omega G$. Hint: Use Corollary 5. (ii) Show that for every integer r there is a ball with radius r which is disjoint from G'. Note that G', considered as set of vertices, is the same as $G'(1)$, considered as orbit of the group action.

9. Thick ends, thin ends and directions of group actions

DEFINITION 13. Let G act on a graph X. An element g of G is called *elliptic* if it fixes a finite set of vertices.

LEMMA 12. The following are equivalent for connected locally finite graphs,

(i) the element g is not elliptic,
(ii) there is a vertex x such that the orbit of g in x (this is the set $\{g^i(x) \mid i \in \mathbb{Z}\}$) is infinite, and
(iii) for any vertex x, the orbit of g in x is infinite.

PROOF. The implications (i) $\implies$ (iii) and (iii) $\implies$ (ii) are clear. Let g have an infinite orbit on x. By Lemma 1, this orbit is unbounded. Let F be a finite set of vertices. Then there is an m such that $d_X(x, g^m(x)) > 2(d_X(x, F) + \mathrm{diam}_X(F))$. By the triangle inequality we get

$$d_X(x, g^m(x)) \le d_X(x, F)+\mathrm{diam}_X(F)+d_X(F, g^m(F))+\mathrm{diam}_X(g^m(F))+d_X(g^m(F), g^m(x)).$$

Note that $\mathrm{diam}_X(g^m(F)) = \mathrm{diam}_X(F)$ and $d_X(g^m(F), g^m(x)) = d_X(x, F)$. Hence

$$d_X(F, g^m(F)) \ge d_X(x, g^m(x)) - 2(d_X(x, F) + \mathrm{diam}_X(F)) > 0,$$

F and $g(F)$ are disjoint and F is not fixed by g. □

DEFINITION 14. A two-sided infinite path $(\ldots, x_{-1}, x_0, x_1, \ldots)$ of distinct vertices is called a *line*. A non-elliptic automorphism g has a *periodic line* L (or: L is *g-periodic*) with period $p \ge 1$ if $g^p(L) = L$.

A non-elliptic automorphism g acts on a periodic line $L = (\ldots, x_{-1}, x_0, x_1, \ldots)$ with period p as non-trivial translation in the sense that there is a positive integer k such that $g^p(x_i) = x_{i+k}$.

THEOREM 7 (Theorem 7 in [**11**]). *Every non-elliptic automorphism of a connected graph has a periodic line.*

PROOF. Let g be non-elliptic and let d be the minimal distance $d_X(x, g^n(x))$ for all vertices x and all integers $n \ge 1$. Choose a vertex x and an integer p such that $d_X(x, g^p(x)) = d$. Let π_0 be a path from x to $g^p(x)$ of length d, set $v_i = g^{ip}(x)$, for $i \in \mathbb{Z}$. Note that $d > 0$ and $v_i \ne v_k$ whenever $i \ne k$, because g is non-elliptic. We define $\pi_i = g^{ip}(\pi_0)$, which is a path from v_i to v_{i+1}, and

$$L = \ldots \circ \pi_{-1} \circ \pi_0 \circ \pi_1 \circ \ldots$$

which is a two-sided infinite path $L = (\ldots, x_{-1}, x_0, x_1, \ldots)$. This infinite path is invariant under g^p. To show that L is a line we have to show that $x_i \ne x_j$ whenever $i \ne j$.

Assume that L is not a line. Then there are integers l and m, $m \geq 1$, such that the path $\pi_l \circ \pi_{l+1} \circ \ldots \circ \pi_{l+m}$ contains a closed subpath. We may assume that m is minimal with this property. If $m \geq 2$ then $\pi_l \circ \pi_{l+1} \circ \ldots \pi_{l+m-1}$ and $\pi_{l+1} \circ \pi_{l+1} \circ \ldots \pi_{l+m}$ consist of distinct vertices. But π_l has a vertex y in common with π_{l+m}, where

$$y \neq v_{l+m}. \tag{4}$$

If $m = 1$ then π_l and π_{l+1} have the vertex v_{l+1} in common. But since $\pi_l \circ \pi_{l+1}$ contains a closed subpath, there is another vertex y which is element both paths π_l and π_{l+1}. And this vertex y satisfies (4).

Back to the general case $m \geq 1$. The vertices y and $g^{mp}(y)$ are both elements of π_{l+m}. If one of the vertices y and $g^{mp}(y)$ is no end vertex of π_{l+m} (which means that either y or $g^{mp}(y)$ is not in $\{v_{l+m}, v_{l+m+1}\}$) then $d_X(y, g^{mp}(y)) < d$. This contradicts the assumptions above. Hence

$$\{y, g^{mp}(y)\} = \{v_{l+m}, v_{l+m+1}\}. \tag{5}$$

Putting together (4) and (5), we get $y = v_{l+m+1}$ and $g^{mp}(y) = v_{l+m}$. Hence

$$g^{(m+1)p}(y) = g^p(g^{mp}(y)) = g^p(v_{l+m}) = v_{l+m+1} = y.$$

Since $m \geq 0$, this implies that g is elliptic. Hence the assumtion that L does not consist of distinct vertices leads to a contradiction. □

Note that Lemma 1 implies that all rays and lines in locally finite graphs have no infinite bounded subsets. Such rays and lines are called *metric*. A *metric translation* is a non-elliptic element g which has a metric periodic line. All non-elliptic elements on a locally finite graph are metric translations.

LEMMA 13. Let g be a non-elliptic element of a group which is acting on a connected locally finite graph X. Let R be a subray of a g-periodic line with period p such that $g^p(R) \subset R$. Then the end ω which contains R is invariant under g.

PROOF. Let x be the initial vertex of R. If $g^p(R) \subset R$ then $g^{ip}(x) \in R$ and $g(g^{ip})(x) = g^{ip+1}(x) \in g(R)$ for all $i \geq 0$. Suppose the end ω which contains R is not invariant under g. Then there is a ball $B(x, r)$ which separates R from $g(R)$. Let C be the component of $VX \setminus B(x, r)$ which contains ω and let C' be the component of $VX \setminus B(x, r)$ which contains $g(\omega)$. All but finitely many vertices $g^{ip}(x)$ are in C and all but finitely many vertices $g^{ip+1}(x)$ are in C'. Because R and $g(R)$ are both metric, there is an integer $m \geq 1$ such that $g^{mp}(x) \in C \setminus B_X(x, d_X(x, g(x)) + r)$ and $g^{mp+1}(x) \in C' \setminus B_X(x, d_X(x, g(x)) + r)$. Every path from $g^{mp}(x)$ to $g^{mp+1}(x)$ must go through $B_X(x, r)$ and has consequently a length which is greater than $d_X(x, g(x))$. Thus $d_X(g^{mp}(x), g^{mp+1}(x)) > d_X(x, g(x))$, which is a contradiction. □

DEFINITION 15. Let G act on a graph X and let R be a ray such that $g^n(R) \subset R$ for some non-elliptic element g of G. Then we call the end $\mathcal{D}(g)$ which contains R the *direction* of g. If $\mathcal{D}(g) = \mathcal{D}(g^{-1})$ then g is called *parabolic*, and if $\mathcal{D}(g) \neq \mathcal{D}(g^{-1})$ then we call g *hyperbolic*.

By Lemma 13, the direction of a non-elliptic element is well defined.

DEFINITION 16. The *thickness of an end* ω is the smallest integer $\mu(\omega)$ such that there is a descending sequence of cuts $(C_n)_{n\in\mathbb{N}}$ which contain ω, such that

$$\operatorname{diam}_X(\theta C_n) \le \mu(\omega) \quad \text{and} \quad \bigcap_{n\in\mathbb{N}} C_n = \emptyset.$$

An end ω is called *thin* if $\mu(\omega) < \infty$ and *thick* if $\mu(\omega) = \infty$.

EXAMPLE 2. The graph in Exercise 9c has one thick end. All the other graphs in Exercise 9 have only thin ends.

LEMMA 14. Let C be a cut and let g be an element such that $g(C \cup \theta C) \subset C$. Then g is non-elliptic, $\mathcal{D}(g) \neq \mathcal{D}(g^{-1})$, $\mathcal{D}(g)$ lies in C, $\mathcal{D}(g^{-1})$ lies in $VX \setminus C$ and $\mu(\mathcal{D}(g))$ and $\mu(\mathcal{D}(g^{-1}))$ are both less or equal $\operatorname{diam}_X C$.

PROOF. We have $C \uplus \theta C \supseteq C \supseteq g(C) \uplus g(\theta C)$, where $\uplus$ denotes a disjoint union. By induction we get

$$C \uplus \theta C \;\supseteq\; C \;\supseteq\; g(C) \uplus g(\theta C) \;\supseteq\; g(C) \;\supseteq\; g^2(C) \uplus g^2(\theta C) \;\supseteq\; g^2(C) \;\supseteq\; \dots \tag{6}$$

and the sets θC, $g(\theta C)$, $g^2(\theta C), \dots$ are pairwise disjoint. Then

$$d_X(\theta C, g^m(\theta C)) \ge m,$$

because a path from a vertex x in θC to a vertex y in $g^m(\theta C)$ has to contain vertices of $g^i(\theta C)$ for $i = 0, \dots, m$. Together with (6) this implies

$$d_X(\theta C, g^m(\theta C \cup C)) \ge m.$$

For any x in θC, we have $d_X(x, g^m(x)) \ge m$. The orbit of x under g is infinite. Lemma 12 implies that g in non-elliptic. By Theorem 7, there is a g-periodic line L with period p. Let y be an element of L. Set $m = d_X(x, y) + 1$ then

$$d_X(g^m(x), \theta C) \ge d_X(x, y) + 1 = d_X(g^m(x), g^m(y)) + 1$$

which implies that $g^m(y)$ is in C, and $g^n(y)$ is in C whenever $n \ge m$. Let $i_0 p$ be greater or equal m, where $i_0 \in \mathbb{Z}$. Then the vertices $g^{ip}(y)$, for $i \ge i_0$, are in C and they are elements of L. The cut C contains infinitely many elements of L, θC is finite and L consists of distinct vertices. Hence C contais a subray of L and $\mathcal{D}(g)$ is in C.

Set

$$C_n = g^{np}(C \cup \theta C))$$

for $n \in \mathbb{N}$. Let z be any vertex in VX. There is a positive integer m such that

$$d_X(\theta C, g^{mp}(C \cup \theta C)) > d_X(\theta C, z).$$

Thus z is not in $g^{mp}(C \cup \theta C)$. This implies $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$ and $\mu(\mathcal{D}(g)) \le \operatorname{diam}_X \theta C$.

The inclusion $g(C \cup \theta C) \subset C$ implies

$$VX \setminus C \subset g(VX \setminus (C \cup \theta C)) \quad \text{and} \quad g^{-1}(VX \setminus C) \subset VX \setminus (C \cup \theta C).$$

Set $C' = VX \setminus (C \cup \theta C)$. Note that $\theta C' = \theta C$, $VX \setminus C = C' \cup \theta C'$ and

$$g^{-1}(C' \cup \theta C') = g^{-1}(VX \setminus C) \subset VX \setminus (C \cup \theta C) = C'.$$

By considering g^{-1} instead of g and C' instead of C, we can now use same arguments as before. Hence $\mathcal{D}(g^{-1})$ is in C' and $\mu(\mathcal{D}(g^{-1})) \le \operatorname{diam}_X C' = \operatorname{diam}_X C$. □

THEOREM 8. *Let X be locally finite and connected and let g be a non-elliptic element. Then $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ are the only elements which are fixed by g.*

PROOF. Let ω be an end which is fixed by g and which is neither $\mathcal{D}(g)$ nor $\mathcal{D}(g^{-1})$. By Theorem 7, there is a g-periodic line L. There is a finite connected set F which separates ω from both directions $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ and contains an element of L. Note that the case $\mathcal{D}(g) = \mathcal{D}(g^{-1})$ is not excluded. Let A be the component of $VX \setminus F$ which contains ω. The set F is bounded, g is a metric translation and $F \cup A$ has only a finite intersection with L. Hence there is a k such that $g^k(F) \cap (F \cup A) = \emptyset$. Set $C = VX \setminus (F \cup A)$. Since the complement of C is connected and C contains the boundary of $g(A)$, which is a subset of $g(F)$, we can apply Lemma 6. Thus either $g(A) \subset C$ or $VX \setminus g(A) \subset C$. If $g(A) \subset C$ then ω is not fixed by g. If $VX \setminus g(A) \subset C$ then $VX \setminus C = A \cup F \subset g(A)$ and $g^{-1}(A \cup F) \subset A$. By Lemma 14, the direction $\mathcal{D}(g^{-1})$ must lie in A, which is impossible since A has only a finite intersection with the g-periodic line L. We see that the assumption that ω is fixed by g leads to a contradiction. □

THEOREM 9. *Let g be a non-elliptic element of a group G acting on a locally finite connected graph X. If $\mathcal{D}(g) = \mathcal{D}(g^{-1})$ then $\mathcal{D}(g)$ is thick, if $\mathcal{D}(g) \neq \mathcal{D}(g^{-1})$ then $\mathcal{D}(g)$ is thin.*

In [**11**], Theorem 9 was stated in a stronger version, including statements on maximal sets of disjoint periodic lines. For us, the present version of the theorem with a shorter proof will be sufficient.

There are various definitions of thickness of ends. At least for transitive locally finite transitive graphs they all yield the same distinction between thick and thin ends. Theorem 9 holds for any of these definitions of thickness. In [**11**], Halin defined the thickness of an end ω as the maximal number of disjoint rays in ω. Woess used a modification of Definition 16. Definition 16 was used in [**15**] to classify graphs which are quasi-isometric (roughly isometric) to a tree.

PROOF OF THEOREM 9. Suppose that $\mathcal{D}(g) \neq \mathcal{D}(g^{-1})$. Let F be a connected set which separates $\mathcal{D}(g)$ from $\mathcal{D}(g^{-1})$. Since g is non-elliptic, there is a k such that $g^k(F) \cap F = \emptyset$. Let C be the component of $VX \setminus F$ which contains $g^k(F)$. One of the directions $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ lies in C, the other in $VX \setminus C$. Since $VX \setminus C$ is connected we can apply Lemma 6. Hence $g^k(C) \subset C$ or $g^k(VX \setminus C) \subset C$. The latter is impossible, because then g^k would not fix the direction which lies in $VX \setminus C$. Thus $g^k(\theta C \cup C) \subset g^k(F \cup C) \subset C$, we can apply Lemma 14 and see that $\mu(\mathcal{D}(g))$ and $\mu(\mathcal{D}(g^{-1}))$ are both thin.

Suppose $\mathcal{D}(g)$ is thin. Then there is a descending sequence of cuts $(C_n)_{n \in \mathbb{N}}$ which contain $\mathcal{D}(g)$, such that $\operatorname{diam}_X \theta C_n \leq \mu(\mathcal{D}(g))$ and $\bigcap_{n \in \mathbb{N}} C_n = \emptyset$. Let L be a g-periodic line with period p. Since $\bigcap_{n \in \mathbb{N}} C_n = \emptyset$, there is a cut C_m which does not contain the whole line L. If $L \setminus C_m$ is infinite then $VX \setminus C_m$ must contain a subray of L, because θC_m is finite and because L consists of distinct vertices. Then θC_m separates $\mathcal{D}(g)$ from $\mathcal{D}(g^{-1})$ and $\mathcal{D}(g) \neq \mathcal{D}(g^{-1})$. If $L \setminus C_m$ is finite then set $d = \operatorname{diam}_X L \setminus C_m$. There is an integer i such that $d_X(g^{ip}(\theta C_m), \theta C_m) > d$. If $L \setminus g^{ip}(C_m)$ is infinite then $g^{ip}(\theta C_m)$ separates $\mathcal{D}(g)$ from $\mathcal{D}(g^{-1})$ and $\mathcal{D}(g) \neq \mathcal{D}(g^{-1})$. If $L \setminus g^{ip}(C_m)$ is finite then $d_X(g^{ip}(\theta C_m), \theta C_m) > d$ implies that $\operatorname{diam}_X L \setminus g^{ip}(C_m) > d$. But

$$\operatorname{diam}_X L \setminus g^{ip}(\theta C_m) = \operatorname{diam}_X g^{-ip}(L \setminus g^{ip}(\theta C_m)) =$$

$$\operatorname{diam}_X g^{-ip}(L) \setminus \theta C_m = \operatorname{diam}_X L \setminus \theta C_m$$

which is a contradiction. □

EXERCISE 16. Set $G = \mathbb{Z}^2 * \mathbb{Z} = \langle a, b, c \mid ab = ba \rangle$ and $S = \{a, b, c\}$. Let ω be an end of $\mathrm{Cay}(G, S)$. The infinite reduced words over $S \cup S^{-1}$ correspond to rays in ω originating in the neutral element. For which words is ω thick, and for which words is ω thin?

10. Appendix A - Background from topology

10.1. Ordered sets. A *partial order* $\leq$ on a set X is a relation that satisfies the following three properties for all x and y in X:

(i) $x \leq x$ (reflexivity),
(ii) if $x \leq y$ and $y \leq x$ the $x = y$ (antisymmetry) and
(iii) if $x \leq y$ and $y \leq z$ the $x \leq z$ (transitivity).

A *partial order* is called a *total order* if furthermore,

(iv) $x \leq y$ or $y \leq x$ for all x and y in X (transitivity).

10.2. Topological spaces. A *topology* τ on a set X is a family of subsets of X with the following properties:

(1) Any union of elements of τ is in τ.
(2) Any finite intersection of elements of τ is in τ.
(3) $\emptyset$ and X are in τ.

We say that (X, τ) is a *topological space*, or abbreviated "X is a topological space" when no confusion can result about τ.

The elements of τ are called *open* sets and the complements $X \setminus O$ of open sets O are called *closed*. The *interior* A° of a subset A of X is the union of all open subsets of A. The closure $\overline{A}$ of A is the intersection of all closed sets which contain A, and $\partial A = \overline{A} \cap \overline{X \setminus A}$ is called the *(topological) boundary* of A. Then $\partial A = \overline{A} \setminus A^\circ$.

The set A is *dense* in X if $\overline{A} = X$. A set A is *countable* if there exists an injective function $A \to \mathbb{N}$. A topological space with a countable dense subset is *separable*.

A set U is a *neighbourhood* of x if there is an open set O such that $x \in O \subset U$. In the *discrete topology* all sets are open and closed.

Let (X, τ) be a topological space and Z be a subset of X. Then

$$\tau|_Z = \{O \cap Z \mid O \in \tau\}$$

is a topology which we call the *relative topology* for Z.

A *base* of a topology is a collection $\mathcal{B}$ of sets such that any open set is a union of elements of $\mathcal{B}$. Note that $\mathcal{B}$ is a base if and only if whenever O is an open set and x an element of O then there is a $B \in \mathcal{B}$ such that $x \in B \subset O$.

10.3. Separation.

- A topological space is a T_0*-space* if one point out of any two distinct points has a neighbourhood which does not contain the other point.
- A topological space is a T_1*-space* if any two distinct points have a neighbourhood which does not contain the other point.
- A topological space is a *Hausdorff space* (or T_2*-space*) if any two distinct points have disjoint neighbourhoods which do not contain the other point.
- A space is called *regular space* if whenever A is a closed set and x a point which is not in A then there are disjoint open sets U and V such that $x \in U$ and $A \subset V$. A regular T_1 space is called a T_3*-space*.
- A space is called *normal* if whenever A and B are a closed sets then there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$. A regular T_1-space is called a T_4*-space*.

For a topological space we have the implications

$$T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0.$$

Not every regular space is T_1 or T_2. For example consider the indiscrete topology $(X, \{\emptyset, X\})$. A regular space where all singletons (that is, sets which contain exactly one element) are closed is T_2.

10.4. Convergence and covers. A sequence $\xi = (x_n)_{n\in\mathbb{N}}$ in a topological space X is said to *converge* to $x \in X$ (notation: $\lim \xi = x$) if for every neighbourhood U of x there is an integer n_0 such that $x_n \in U$ for all $n \geq n_0$. A point x is an *cluster point of a sequence* $(x_n)_{n\in\mathbb{N}}$ if for any integer n_0 and for any neighbourhood U of x there is an integer $n \geq n_0$ such that $x_n \in U$. A point x is an *accumulation point of a set* F if any neighbourhood of x contains an element of F which is different from x.

An *open cover* of a set A is a collection of open sets whose union contains A. A subset of an open cover of A which is again a cover of A is called a *subcover*. A set F is *compact* if every open cover of F has a finite subcover. A topological space (X, τ) is *compact* if X is a compact set. If every cover of X has a countable subcover then we call (X, τ) a *Lindelöf space*. A topological space X is *sequentially compact* if every sequence has a convergent subsequence.

The reader should be warned that topological terms are sometimes used differently. Some authors request certain separation properties in some definitions which other authors do not request. As in [**19**], we use the definitions without assuming additional separation axioms. In [**4**] for example, a compact space has to be Hausdorff, a Lindelöf space has to be regular, and a regular space has to be T_1.

Theorem 10 (Theorem 3.8.2 in [**4**], Theorem 16.8 in [**19**]). *A regular Lindelöf space is normal.*

Theorem 11 (Theorem 3.8.1 in [**4**], Theorem 16.9 in [**19**]). *A space with a countable base is Lindelöf.*

Theorem 12 (Theorem 3.10.30 in [**4**], Exercise 17G.2 and Definition 17.1 in [**19**]). *A sequentially compact Lindelöf space is compact.*

The reader should be warned that some authors who consider compact or squentially compact spaces always include the assumtion that the space is Hausdorff. A standard text book with this approach is the book of Engelkind [**4**].

For the vertex end topology in locally finite graphs, this does not make a difference, since it is Hausdorff anyway. But when we study non-localy finite graphs, then this end topology is not always Hausdorff.

10.5. Metrizable spaces. A *metric space* is a pair (M, d) consisting of a set M together with a function $d : M \to \mathbb{R}$ such that for any x, y and z in M:

(1) $d(x, x) = 0$ and $d(x, y) > 0$, whenever $x \neq y$,
(2) $d(x, y) = d(y, x)$
(3) $d(x, z) \leq d(x, y) + d(y, z)$.

Condition (1) is called *positive definiteness*, (2) *symmetry* and (3) is the *triangle inequality*. The function d is called *metric*. An *ultrametric* is a metric which satisfies the *strong triangle inequality*,

(4) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

LEMMA 15. Let (M, d) be a metric space and let

$$O_d(x, \epsilon) = \{y \in M \mid d(x, y) < \epsilon\}$$

be the open ball with center x and radius ϵ. Let τ_d be the set of subsets A of M such that for each $x \in A$ there is an $\epsilon > 0$ such that $O_d(x, \epsilon) \subset A$. Then τ_d is a topology and (M, τ_d) is a T_4 space.

The *topology of a metric space* (M, d) is the topology τ_d mentioned in Lemma 15.

A topological space (X, τ) is *metrizable* if there exists a metric d on X such that $\tau = \tau_d$.

THEOREM 13 (Theorem 4.3.6 and Corollary 4.1.16 in [**4**], Theorem 23.1 in [**19**]). *A T_1-space X is metrizable and separable if and only if it is regular and has a countable base.*

A space is *totally disconnected* if any two points have disjoint open and closed neighbourhoods. A set U is a *neighbourhood* of a set A if there is an open set O such that $A \subset O \subset U$. We call a space *strongly 0-dimensional* if any two disjoint closed sets have disjoint open and closed neighbourhoods.

THEOREM 14 (Theorem II in [**9**], Exercise 7.3.F in [**4**]). *A metrizable space is metrizable with an ultrametric if and only if it is strongly 0-dimensional.*

Singletons in metric spaces are closed. Hence every strongly 0-dimensional metric space is totally disconnected. This implies the following.

COROLLARY 4. An ultrametric space is totally disconnected.

10.6. Cantor sets.

DEFINITION 17. Two topological spaces X and Y are *homeomorphic* if there is a continuous bijection $f : X \to Y$ whose inverse function f^{-1} is also continuous. The function f is called *homeomorphism*.

A set is called *perfect* if it is the set of all its accumulation points. A *Cantor set* is a non-empty, totally disconnected, perfect and metrizable Hausdorff space.

THEOREM 15 (Exercise 6.2.A.(c) in [**4**], Theorem 30.3 in [**19**]). *All Cantor sets are homeomorphic.*

EXAMPLE 3. Let $A_1 = [0, 1]$ be the real unit interval equipped with the usual topology of real numbers. This is the topology given by the metric $d(x, y) = |x - y|$. We obtain A_2 by removing the open interval $(\frac{1}{3}, \frac{2}{3})$, and A_3 is obtained by removing the intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ from A_2. In general, A_n is obtained from A_{n-1} by removing the open middle third from each of the 2^{n-1} closed intervals of A_{n-1}. Then $\bigcap A_n$ is a Cantor set.

11. Appendix B - Background from group theory

11.1. Free groups and homomorphisms.

DEFINITION 18. A *group* $(G, *)$ is a set G, containing an element 1, together with a function $* : G \times G \to G$, called *group operation*, such that for all a, b and c in G,

(i) $1 * g = g$,
(ii) $(a * b) * c = a * (b * c)$ and
(iii) there is an element g^{-1} of G such that $g * g^{-1} = g^{-1} * g = 1$.

The element 1 is called *neutral element*, (ii) is the *associativity* and the element g^{-1} is the *inverse element* of g.

A group has only one neutral element and every element of the group has only one inverse element. Sometimes we will just write ab instead of $a*b$ if it is clear which group operation we mean.

DEFINITION 19. A *word over a set* S is an n-tuple, $n \in \mathbb{N}_0$, of elements of S. The elements of S are called *letters*.

Let $S = \{x_i \mid i \in I\}$, where I is any non-empty set of indices and let $S^{-1} = \{x_i^{-1} \mid i \in I\}$ be a set which is disjoint with S. The letters x_i and x_i^{-1} are called *inverse to each other*. A *reduced word* over $S^{\pm} = S \cup S^{-1}$ is a word such that no consecutive letters are inverse to each other. Let $F(S)$ be the set of reduced words over $S^{\pm}$. Let $(y_1, \ldots y_m)$ and $(z_1, \ldots z_n)$ be elements of $F(S^{\pm})$. If y_m and z_1 are not inverse to each other we define

$$(y_1, \ldots y_m) * (z_1, \ldots z_n) = (y_1, \ldots y_m, z_1, \ldots z_n).$$

If y_m and z_1 are inverse to each other then there is a maximal integer k, $0 \leq k \leq \max\{m, n\}$, such that y_{m-i} and z_i are inverse to each other for $0 \leq i \leq k$, and we set

$$(y_1, \ldots y_m) * (z_1, \ldots z_n) = (y_1, \ldots y_{m-k-1}, z_{k+1}, \ldots z_n).$$

This defines a map $* : F(S) \times F(S) \to F(S)$. We call $(F(S), *)$ the *free group with base* S. The *rank* of a free subgroup is the number of elements of a base.

The free group with base S is a group. Its neutral element is the empty word. The inverse element of a word $(y_1, \ldots y_m)$ of $F(S)$ is the element $(y_m^{-1}, \ldots y_1^{-1})$.

DEFINITION 20. Let $(G, *)$ be a group. A subset H of G is called *subgroup* if $(H, *)$ is a group.

LEMMA 16. The intersection of subgroups is a subgroup.

DEFINITION 21. Let S be a subset of a group G. Then the intersection of all subgroups which contain S is the subgroup which is *generated* by S. The set S is called *generating set* if it generates G. A group is *cyclic* if it is generated by a singleton.

Let g be an element of G. Then $gH = \{g * h \mid h \in H\}$ is called a *left coset* of H and $Hg = \{h * g \mid h \in H\}$ is called a *right coset*. A subgroup H of G is *normal* if its left and right cosets coincide. In other words, if $gH = Hg$ for any g in G.

Let $(G, *)$ and $(H, \circ)$ be groups. A function $h : G \to H$ is a *homomorphism* if $h(a*b) = h(a) \circ h(b)$ for all a and b in G. We call $\text{kernel}(h) = h^{-1}(1) = \{g \in G \mid h(g) = 1\}$ the *kernel* of h. The kernel is called *trivial* if it only contains the neutral element.

If there is a bijective homomorphism $f : G \to H$ then G and H are called *isomorphic* and f is called a *group isomorphism*. An isomorphism $G \to G$ is called *group automorphsim*.

LEMMA 17. A kernel of a group homomorphism $f : G \to H$ is a normal subgroup of G.

The following theorem describes the *universal property* of a free group. It is often used a definition for free groups. The equivalence of our Definition 19 and the statement of the following theorem has to be proved anyway, see Theorem 11.1 in [**17**].

THEOREM 16. *Let S be a subset of a group F. If for every group G and every function $f : S \to G$ there is a unique homomorphism $\varphi : F \to G$ such that φ extends f (that is, $\phi(s) = f(s)$ for any $s \in S$) then F is a free group with base S.*

11.2. Quotient groups and presentations.

DEFINITION 22. For subsets A and B of a group G we call

$$AB = \{ab \mid a \in A,\ b \in B\}$$

the *product of sets*.

THEOREM 17. *Let H be a normal subgroup of G and let G/N be the set of all cosets of N (since N is normal, the left and the right cosets coincide). Then G/N is a group with respect to the product of sets as group operation.*

DEFINITION 23. The group G/N of Theorem 17 is called *quotient group* (or *factor group*). Let S be a subset of a group G and let $R = \{r_i \mid i \in I\}$ be a set of words over S. The *normal subgroup of F generated by R* is the intersection of all subgroups of G which are normal and for which for each $r_i = (s_{i,1}, s_{i,2}, \ldots, s_{i,n(i)})$, $i \in I$, the product $s_{i,1}s_{i,2} \ldots s_{i,n(i)}$ is the neutral element.

Let $\{r_i \mid i \in I\}$ be a set of words over $S = \{s_j \mid j \in J\}$ where I and J are any sets of indices. A group G has *generators* S and *relations* of R if G is isomorphic to F/N where F is the free group with base S and N is the normal subgroup of F generated by the relations of R. Then $\langle s_j,\ j \in J \mid r_i,\ i \in I\rangle$ is called a *presentation* of G.

Heuristically we can say that $\langle s_j,\ j \in J \mid r_i,\ i \in I\rangle$ is the "largest" group with generators s_j in which the relations $r_i = 1$ are satisfied.

11.3. The commutator subgroup.

DEFINITION 24. Let a and b be elements of a group G. Then $aba^{-1}b^{-1}$ is called *commutator* of a and b. The subgroup G' of G which is generated by the set of commutators is called the *commutator subgroup*.

The set of of commutators is not necessarily a subgroup. Hence the commutator subgroup may conatin more elements than the set of commutators. For an example and more details see [**17**, Exercise 2.43].

THEOREM 18 (Theorem 2.23 in [**17**]). *The commutator subgroup G' is a normal subgroup of G. If H is a normal subroup of G then G/H is abelian if and only if $G' \leq H$.*

In this sense, G' is the smallest normal subgroup of G such that G/G' is abelian.

DEFINITION 25. A *long commutator* of G is an element g of G of which can be expressed in the form

$$a_1 a_2 \dots a_n a_1^{-1} a_2^{-1} \dots a_n^{-1}, \quad a_i \in G. \tag{7}$$

We call n the *length of the realization* (7).

THEOREM 19 (Theorem 3 in [**21**]). *Any product of n commutators is a long commutator which has a realization of length $2n$. Any long commutator with a realization of length $2n$ or of length $2n+1$ is a product of n commutators.*

COROLLARY 5. The commutator subgroup is the set of long commutators.

PROOF. Let L be the set of long commutators. Every element of the commutator subgroup G' is a product of commutators. By Theorem 19, such a product is in L. Hence $G' \subset L$. Theorem 19 also says that every element of L is a product of commutator. Hence $L \subset G'$. This implies $L = G'$. □

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