Overlap in Consistent Cycles

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Abstract: A (directed) cycle C in a graph Γ is called consistent provided there exists an automorphism of Γ, acting as a 1-step rotation of C. A beautiful but not well-known result of J.H. Conway states that if Γ is arc-transitive and has valence d, then there are precisely d − 1 orbits of consistent cycles under the action of Aut(Γ).

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In this paper, we extend the definition of consistent cycles to those which admit a $k$-step rotation, and call them $1/k$-consistent. We investigate $1/k$-consistent cycles in view of their overlap. This provides a simple proof of the original Conway’s theorem, as well as a generalization to orbits of $1/k$-consistent cycles. A set of illuminating examples are provided. © 2007 Wiley Periodicals, Inc. J Graph Theory 55: 55–71, 2007

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1. INTRODUCTION

Consider the three pictures of the Petersen Graph shown in Figure 1.

In each figure, pay particular attention to the outside cycle: a pentagon, a hexagon, and a nonagon, respectively. The first two differ in an important way from the third; the first two diagrams have symmetry which rotates the outer pentagon or hexagon one step. While the third picture is also very symmetric and does have a $120^\circ$ rotation, there is no symmetry of the graph rotating the nonagon by a single step.

We use the word consistent (see Section 2 for formal definition) to describe a cycle such as the pentagon and hexagon in Figure 1. That is, some symmetry of the graph is consistent with one-step rotation in the cycle.

The remarkable theorem of Biggs and Conway [1] may be the best-kept secret of graph symmetry: it shows that if a graph $\Gamma$ is symmetric (i.e., arc-transitive) and has valence $d > 1$, then there are exactly $d-1$ orbits of consistent (directed) cycles under the action of $\text{Aut}(\Gamma)$. The proof given in [1] is incomplete and an expansion of it quickly becomes very technical, as in [3].

In this paper we introduce some simple ideas for understanding consistent cycles, give a simple proof of the Biggs–Conway result, generalize it to the case of $1/k$-consistent cycles (such as the nonagon in Fig. 1), and locally arc-transitive automorphism groups, and prove some other related results.
2. DEFINITIONS

In this paper, we consider three closely related structures in a connected graph \( \Gamma \), each of which can be called a “cycle” in some context. First, a cycle is often defined as a sequence \( \alpha = [v_0, v_1, v_2, \ldots, v_{r-1}] \), \( r \geq 3 \), of pairwise distinct vertices of \( \Gamma \) such that each vertex \( v_i \) is adjacent to \( v_{i+1} \), the addition being modulo \( r \). This is, in fact, how [1] defines a cycle. We will use the word cyclet to mean such a sequence, and we will think of it as a rooted, directed cycle. To be explicit, a cyclet \( \alpha = [v_0, v_1, v_2, \ldots, v_{r-1}] \) is distinct from its shift \( \alpha^1 = [v_1, v_2, \ldots, v_{r-1}, v_0] \) and its inverse \( \alpha^{-1} = [v_{r-1}, v_{r-2}, \ldots, v_2, v_1, v_0] \).

Second, consider the equivalence relation on the set of cyclets generated by the shift relationship. Equivalence classes under this relation will be called directed cycles. The directed cycle containing \( \alpha \) will be called \( \tilde{\alpha} \); that is, the directed cycle \( \tilde{\alpha} \) is the set of all \( t \)-shifts \( \alpha^t = [v_t, v_{t+1}, v_{t+2}, \ldots, v_0, v_1, \ldots, v_{t-1}] \) of \( \alpha \), for \( t = 0, 1, \ldots, r - 1 \).

Finally, a cycle is an equivalence class of cyclets under the relation generated by shift and inverse relationships. This is equivalent to thinking of a cycle, as we often do, as a connected subgraph of \( \Gamma \) in which each vertex has valence 2. Thus, the cyclet \( \alpha \) and its inverse cyclet \( \alpha^{-1} \) induce the same cycle \( \tilde{\alpha} \), but distinct directed cycles \( \tilde{\alpha} \) and \( \tilde{\alpha}^{-1} \), respectively.

A symmetry or automorphism of \( \Gamma \) is a permutation of the vertices which preserves edges. The symmetries of \( \Gamma \) form a group, \( \text{Aut}(\Gamma) \), under composition, and we will write the application of a symmetry \( g \) to a vertex \( v \) using superscripts: \( v^g \). An \( s \)-arc in \( \Gamma \) is a sequence \( \alpha = [v_0, v_1, v_2, \ldots, v_s] \) of vertices of \( \Gamma \) in which every two consecutive vertices are adjacent, and every three consecutive vertices are distinct. A 1-arc is an arc or a dart and a 0-arc can be viewed as a vertex.

For a subgroup \( G \) of \( \text{Aut}(\Gamma) \), we say that \( G \) is \( s \)-arc-transitive or that \( \Gamma \) is \((G, s)\)-arc transitive provided that \( G \) acts transitively on the set of all \( s \)-arcs in \( \Gamma \). If \( s = 1 \), we say that \( G \) is arc-transitive, sometimes dart-transitive. If a subgroup \( G \) of \( \text{Aut}(\Gamma) \) has the property that for each vertex \( v \), the stabilizer \( G_v \) is transitive on \( s \)-arcs whose initial vertex is \( v \), we say that \( G \) is locally \( s \)-arc-transitive.

If \( G \) is locally arc-transitive on \( \Gamma \), there are two possibilities. The first is that \( \Gamma \) is symmetric, that is, arc-transitive. The second is that \( \Gamma \) is bipartite, \( G \) is transitive on each of the bipartition sets (but not on the set of all vertices), and \( G \) is transitive on edges. In this case, vertices of one set may be of a valence different than that for the other set.

Let \( G \) be a group of symmetries of the graph \( \Gamma \). A cyclet \( \alpha = [v_0, v_1, v_2, \ldots, v_{r-1}] \) is \( G \)-consistent provided there is \( g \in G \) such that \( v_i^g = v_{i+1} \) for all \( i \), the addition being modulo \( r \). Such a symmetry \( g \) is called a shunt for \( \alpha \). Observe that if a cyclet \( \alpha \) is consistent, so are all of its shifts, and so are all of their inverses. If \( \alpha \) is consistent, we will say the same for the directed cycle \( \tilde{\alpha} \) and the cycle \( \tilde{\alpha}^{-1} \).

More generally, a cyclet \( \alpha = [v_0, v_1, v_2, \ldots, v_{r-1}] \) is \((G, \frac{1}{k})\)-consistent provided there is \( g \in G \) such that \( v_i^g = v_{i+k} \) for all \( i \), the addition again being modulo \( r \). Such a symmetry \( g \) is called a \( \frac{1}{k} \)-shunt for \( \alpha \). Note that we do not require that \( k \) is
a divisor of \( r \). Moreover, if \( k' = \gcd(k, r) \), then \( \alpha \) is \((G, \frac{1}{k'})\)-consistent if and only if it is \((G, \frac{1}{r})\)-consistent.

Observe that if a cycle \( \alpha \) is \((G, \frac{1}{k})\)-consistent, so are all of its shifts, as well as the inverse. Hence we may call the cycle \( \tilde{\alpha} \), and the directed cycle \( \vec{\alpha} \), \((G, \frac{1}{k})\)-consistent whenever \( \alpha \) is \((G, \frac{1}{k})\)-consistent. When \( G \) is clear from the context, the reference to \( G \) will be omitted.

With this notation in hand, we can restate the theorem of Biggs and Conway in this form:

**Theorem 2.1 [1].** *If \( \Gamma \) is a graph and \( G \) is any arc-transitive subgroup of \( \text{Aut}(\Gamma) \), and if vertices in \( \Gamma \) have valence \( d \), then there are exactly \( d - 1 \) \( G \)-orbits of \( G \)-consistent cyclets in \( \Gamma \).*

It follows that there are exactly \( d - 1 \) \( G \)-orbits of \( G \)-consistent directed cycles, as well.

The main result of this paper is Theorem 4.3 which proves the Biggs–Conway result as a Corollary, as well as the following:

1. If \( G \) is arc-transitive on \( \Gamma \) and \( \Gamma \) has girth at least \( 2k + 1 \), then there are exactly \( (d - 1)^k \) \( G \)-orbits of \((G, \frac{1}{k})\)-consistent cyclets.
2. If \( \Gamma \) is locally arc-transitive, but not arc-transitive, and its vertices have degrees \( d \) and \( d' \), and if \( k \) is even, and \( \Gamma \) has girth at least \( 2k + 1 \), then there are exactly \( 2(d - 1)^k (d' - 1)^k \) \( G \)-orbits of \((G, \frac{1}{k})\)-consistent cyclets.

The main organizational tool for the theorem is the idea of maximal overlap among cyclets and orbits of cyclets.

### 3. OVERLAP

Let \( \Gamma \) be a graph, and let \( \alpha = [a_0, a_1, \ldots, a_{r-1}] \) and \( \beta = [b_0, b_1, \ldots, b_{s-1}] \) be two cyclets in \( \Gamma \). Then we define the overlap (actually, this is the maximal initial overlap) of \( \alpha \) and \( \beta \) to be \(-1\) if \( a_0 \neq b_0 \) and

\[
m(\alpha, \beta) = \max\{t : a_i = b_i \text{ for } i = 0, 1, 2, \ldots, t\}
\]

otherwise. Moreover, if \( \mathcal{A} \) and \( \mathcal{B} \) are two sets of cyclets in \( \Gamma \), then we define the overlap of \( \mathcal{A} \) and \( \mathcal{B} \) to be:

\[
m(\mathcal{A}, \mathcal{B}) = \max\{m(\alpha, \beta) : \alpha \in \mathcal{A}, \beta \in \mathcal{B}, \alpha \neq \beta\}
\]

We call the overlap \( m(\mathcal{A}, \mathcal{A}) \) the self-overlap of \( \mathcal{A} \). Note that the self-overlap of any set \( \mathcal{A} \) is strictly less than the length of the largest cyclet in \( \mathcal{A} \).

In this section, we prove some basic properties of this overlap function. We will restrict our attention to the case where the sets of cyclets are orbits.
under the action of some subgroup $G \leq \text{Aut}(\Gamma)$. The group $G$ will be fixed throughout the section and the reference to it will be omitted. In particular, the $G$-orbits of cyclets will be called simply \textit{orbits of cyclets}. The first two lemmas are simple but useful observations, and the proofs are left to the reader.

**Lemma 3.1.** Let $A$ and $B$ be two orbits of cyclets in $\Gamma$ and let $a_0$ be an arbitrary element of $A$. Then $m(A, B) = \max\{m(a_0, \beta) : \beta \in B, \beta \neq a_0\}$.

**Lemma 3.2.** Let $G$ be a group of automorphisms of $\Gamma$ acting transitively on the darts, and let $A$ and $B$ be two $G$-orbits of cyclets in $\Gamma$. Then $m(A, B) \geq 1$.

**Lemma 3.3.** Let $\alpha, \beta$, and $\gamma$ be arbitrary cyclets in a graph $\Gamma$. If $m(\alpha, \beta) < m(\beta, \gamma)$, then $m(\alpha, \gamma) = m(\alpha, \beta)$. Similarly, if $A, B,$ and $C$ are orbits of cyclets in $\Gamma$ such that $m(A, B) < m(B, C)$, then $m(A, C) = m(A, B)$.

**Proof.** Let $r = m(\alpha, \beta)$ and $s = m(\beta, \gamma)$. Since $r \leq s$, the first $r$ edges of $\alpha$ belong to $\beta$ as well as to $\gamma$, implying that $m(\alpha, \gamma) \geq r$. On the other hand, since $r < s$, edge $r + 1$ of $\beta$ is in $\gamma$ but not in $\alpha$. Hence $m(\alpha, \gamma) \leq r$ and the first part of lemma is proved.

Let now $\beta_0, \gamma_0$ be cyclets in $B, C$ such that $m(\beta_0, \gamma_0) = m(B, C)$. Then $m(\alpha, \beta_0) \leq m(A, B) < m(B, C) = m(\beta_0, \gamma_0)$ for every $\alpha \in A$. Therefore, by the first part of lemma, $m(\alpha, \gamma_0) = m(\alpha, \beta_0)$ for every $\alpha \in A$, and so $\max\{m(\alpha, \gamma_0) : \alpha \in A\} = \max\{m(\alpha, \beta_0) : \alpha \in A\}$. The result now follows from Lemma 3.1.

By definition of the overlap, given two orbits $A$ and $B$ of cyclets, there exist $\alpha \in A$ and $\beta \in B$ realizing the overlap of $A$ and $B$ (i.e., $m(\alpha, \beta) = m(A, B)$). For a larger collection of orbits, it is not clear at all that the analog would hold. Before we prove that it does, consider the following example:

**Example 3.4.** Consider the graph shown in Figure 2, the octahedron, together with its full automorphism group. Observe that the cyclets $\alpha = [x, y, z]$, $\beta = [x, y, s, t]$, and $\gamma = [x, y, z, s, t, u]$ are consistent relative to the following shunts: the $120^\circ$ rotation of the left-hand side diagram in Figure 2, the $90^\circ$ rotation about $uz$-axis of the middle diagram in Figure 2, and the $60^\circ$ rotation of the right-hand side diagram in Figure 2, respectively. They give rise to three different orbits $A, B,$ and $C$, respectively. Observe that $m(\alpha, \beta) = 1 = m(A, B)$, $m(\alpha, \gamma) = 2 = m(A, C)$,
and \( m(\beta, \gamma) = 1 = m(B, C) \). Hence, the cyclers \( \alpha, \beta, \) and \( \gamma \) realize the overlaps of \( \mathcal{A}, B, \) and \( C \).

Let \( \Gamma \) be a graph, \( G \leq \text{Aut}(\Gamma) \), and \( \mathcal{F} \) a family of \( G \)-orbits of cyclers in \( \Gamma \). A set \( S \) of representatives, one from each element of \( \mathcal{F} \) will be called compact provided that \( m(\alpha, \beta) = m(\alpha^G, \beta^G) \) for each pair of distinct elements \( \alpha, \beta \in S \). For example, the set \( \{\alpha, \beta, \gamma\} \) in Example 3.4 is a compact set of representatives of the family of all orbits of consistent cyclers in the octahedron.

**Lemma 3.5.** Every family of pairwise distinct orbits of cyclers has a compact set of representatives.

**Proof.** Let \( \Gamma \) be a graph, let \( G \leq \text{Aut}(\Gamma) \), and let \( \mathcal{F} \) be a family of pairwise distinct \( G \)-orbits of cyclers in \( \Gamma \). We need to show that \( \mathcal{F} \) has a compact set of representatives. The proof is by induction on the size of \( \mathcal{F} \). The statement clearly holds if \( |\mathcal{F}| = 2 \). Now suppose that \( |\mathcal{F}| \geq 3 \) and assume that the statement holds for families of size less than \( |\mathcal{F}| \). Choose distinct elements \( \mathcal{A}, B \in \mathcal{F} \) so that \( m(\mathcal{A}, B) \geq m(\mathcal{A}', \mathcal{Y}) \) for all distinct \( \mathcal{A}', \mathcal{Y} \in \mathcal{F} \), and let \( \mathcal{F}' = \mathcal{F} \setminus \{\mathcal{A}\} \). By induction hypothesis, \( \mathcal{F}' \) has a compact set \( \mathcal{S}' \) of representatives. For \( \mathcal{X} \in \mathcal{F}' \), let \( \alpha(\mathcal{X}) \in \mathcal{S}' \) denote the representative of \( \mathcal{X} \), and let \( \beta = \alpha(B) \). By Lemma 3.1, we may choose \( \alpha_0 \in \mathcal{A} \) so that \( m(\mathcal{A}, B) = m(\alpha_0, \beta) \). We claim that \( \mathcal{S} = \mathcal{S}' \cup \{\alpha_0\} \) is a compact set of representatives of \( \mathcal{F} \). It suffices to show that \( m(\mathcal{A}, \mathcal{X}) = m(\alpha_0, \alpha(\mathcal{X})) \) for every \( \mathcal{X} \in \mathcal{F}' \setminus \{B\} \).

Fix \( \mathcal{X} \in \mathcal{F}' \setminus \{B\} \). Let us first show that for every \( \xi \in \mathcal{X} \), \( m(\alpha_0, \xi) = m(\xi, \beta) \). Namely, if \( m(\alpha_0, \xi) < m(\xi, \beta) \), then by Lemma 3.3, \( m(\alpha_0, \beta) = m(\alpha_0, \xi) \). On the other hand, by our choice of \( \alpha_0 \) and \( \beta \), it follows that \( m(\alpha_0, \beta) \geq m(\xi, \beta) \). Hence \( m(\alpha_0, \xi) < m(\xi, \beta) \leq m(\alpha_0, \beta) = m(\alpha_0, \xi) \), a contradiction. By swapping the roles of \( \mathcal{A} \) and \( \mathcal{B} \), we show that \( m(\alpha_0, \xi) > m(\xi, \beta) \) also leads to a contradiction, implying that \( m(\alpha_0, \xi) = m(\xi, \beta) \) for every \( \xi \in \mathcal{X} \), as claimed. By taking the maximum of \( m(\alpha_0, \xi) \) over all \( \xi \in \mathcal{X} \) and using Lemma 3.1, we see that \( m(\mathcal{A}, \mathcal{X}) = m(B, \mathcal{X}) \). By induction hypothesis, the latter equals \( m(\beta, \alpha(\mathcal{X})) \), and by what we have just shown, also to \( m(\alpha_0, \alpha(\mathcal{X})) \). Therefore \( m(\mathcal{A}, \mathcal{X}) = m(\alpha_0, \alpha(\mathcal{X})) \), as required.  

**4. CONSISTENT CYCLETs**

Observe that every symmetry \( g \) of a graph \( \Gamma \) which maps a vertex \( v \) to an adjacent vertex is a shunt for a consistent cyclet \( [v, v^g, v^{g^2}, \ldots] \), provided that \( v^{g^2} \neq v \). The following lemma generalizes this observation to the case of \( \frac{1}{2} \)-consistent cyclers. The proof is technical but straightforward, and is left to the reader. The lemma and the theorem which follows it use the hypothesis that the girth of the graph is at least \( 2k + 1 \). A result of this hypothesis is that if \( \alpha = [u_0, u_1, \ldots, u_r] \) is an \( r \)-arc and \( \beta = [v_0, v_1, \ldots, v_s] \) is an \( s \)-arc where \( r, s \leq k \), and \( v_0 = u_0, u_r = v_s \), then \( \alpha = \beta \).

**Lemma 4.1.** Let \( k \) be a positive integer, let \( \Gamma \) be a graph of girth at least \( 2k + 1 \), let \( [y_0, y_1, \ldots, y_k, y_{k+1}] \) be a \((k+1)\)-arc in \( \Gamma \), and let \( g \) be an automorphism of \( \Gamma \) which maps the arc \((y_0, y_1)\) to the arc \((y_k, y_{k+1})\). Then there exists a unique
cyclot starting in $y_0$ for which $g$ is a $\frac{1}{k}$-shunt. Moreover, this cyclot is of the form 
$[y_0, \ldots, y_{k-1}, y_0, \ldots, y_{k-1}, y_0, \ldots, \{y_0, \ldots, y_{k-1}\}]$.

To facilitate the formulation and proof of our generalization of the Biggs–Conway result, we introduce the following function $\delta$.

**Definition 4.2.** For a graph $\Gamma$, a locally arc-transitive group $G \leq \text{Aut}(\Gamma)$, and a positive integer $\ell$, let

$$\delta_{(\Gamma, G)}(\ell) = \begin{cases} 
(d - 1)^\ell; & \text{if } G \text{ is arc-transitive,} \\
2(d - 1)^{(d' - 1)/2}; & \text{if } G \text{ is not arc-transitive and } \ell \text{ is even,} \\
0; & \text{if } G \text{ is not arc-transitive and } \ell \text{ is odd,}
\end{cases}$$

where $d$ and $d'$ are the valences of two adjacent vertices in $\Gamma$. (Note that $d = d'$ if $G$ is arc-transitive.)

The following Theorem is the primary result of this paper:

**Theorem 4.3.** Let $k$ be a positive integer, let $\Gamma$ be a graph of girth at least $2k + 1$, and let $G$ be a locally arc-transitive subgroup of $\text{Aut}(\Gamma)$. Then there are exactly $\delta_{(\Gamma, G)}(k)$ $G$-orbits of $\frac{1}{k}$-consistent cycles.

**Proof.** Let $U$ be an orbit of $G$ on $V(\Gamma)$. Let $d$ be the valency of a vertex in $U$ and $d'$ the valency of an adjacent vertex. Note that $U = V(\Gamma)$ if $G$ is arc-transitive, and that $U$ is a bipartition set if $G$ is not arc-transitive. In the latter case, there are no automorphisms in $G$ which move a vertex from $U$ to a vertex from $V(\Gamma) \setminus U$.

This implies that for an odd number $k$ (and not arc-transitive group $G$), there are no $\frac{1}{k}$-consistent cycles in $\Gamma$, and so, we may henceforth assume that $G$ is arc-transitive or $k$ is even. Furthermore, if $G$ is not arc-transitive, then for each $\frac{1}{k}$-consistent cyclot $\alpha$ starting in $U$, its shift $\alpha'$ is $\frac{1}{k}$-consistent cyclot starting in $V(\Gamma) \setminus U$ and vice versa.

To simplify the notation, let $\delta' = \delta_{(\Gamma, G)}(k)$ if $G$ is arc-transitive and $\delta' = \frac{1}{2} \delta_{(\Gamma, G)}(k)$ if it is not. It follows from the above discussion that it suffices to show that the number of orbits of $\frac{1}{k}$-consistent cycles starting in $U$ is $\delta'$.

Let $\{A_1, \ldots, A_t\}$ be the family of $G$-orbits of $\frac{1}{k}$-consistent cycles whose initial vertex is in $U$. By Lemma 3.5, we can choose $\alpha_i \in A_i$ such that $m(\alpha_i, \alpha_j) = m(A_i, A_j)$ for $1 \leq i, j \leq t$. Since $G$ is locally arc-transitive, it follows that $m(\alpha_i, \alpha_j) \geq 1$ for $1 \leq i, j \leq t$. Let $v_0$ and $v_1$ be the first two vertices of $\alpha_1$ (and thus of every $\alpha_i$).

Consider the set $\mathcal{T}$ of $(k - 1)$-arcs $[x_0, x_1, \ldots, x_{k-1}, v_0, v_1]$ such that $[x_0, x_1, \ldots, x_{k-1}, v_0, v_1]$ is a $(k + 1)$-arc. Since we are assuming that $k$ is even if $G$ is not arc-transitive, and since the girth of $\Gamma$ is at least $2k + 1$, it is easy to see that $|\mathcal{T}| = \delta'$.

We shall first show that $t \leq \delta'$. Suppose the contrary: $t > \delta'$. Then there exist $i, j \in \{1, \ldots, t\}$, $i \neq j$, such that the last $k$ vertices of $\alpha_i$ and $\alpha_j$ are the same. Let $g_i$ and $g_j$ be $\frac{1}{k}$-shunts for $\alpha_i$ and $\alpha_j$, and let $\beta_i$ and $\beta_j$ be the images of $\alpha_i$ and $\alpha_j$.
under $g_i^{-1}$ and $g_j^{-1}$, respectively. Then $\beta_i \in A_i, \beta_j \in A_j$, and $m(\beta_i, \beta_j) > m(\alpha_i, \alpha_j)$, which is a contradiction. Hence $t \leq 8$.

Suppose now that $t < 8$. Then there is at least one $(k-1)$-arc $[x_0, x_1, \ldots, x_{k-1}] \in T$ which is not a “tail” of $\alpha_i$ for any $i$, $1 \leq i \leq t$ (where by a “tail” of a cyclet, we mean the $k$-tuple of the last $k$ vertices in the cyclet).

Moreover, since $G$ is either arc-transitive, or $k$ is even and $G$ is transitive on all arcs starting in a vertex from $U$, there exists an automorphism $g_1 \in G$ which maps $(x_0, x_1)$ to $(v_0, v_1)$. Hence, Lemma 4.1 applies, with $x_0, x_1, v_0, v_1$ playing the roles of $y_0, y_1, y_2, y_{k+1}$, respectively. Then there exists a cyclet $[x_0, \ldots, x_{k-1}, v_0, v_1, \ldots]$ for which $g_1$ is a $\frac{1}{k}$-shunt. Then, of course, $g_1$ is a $\frac{1}{k}$-shunt for $[v_0, v_1, \ldots, x_0, \ldots, x_{k-1}]$, as well.

Among all $(G, \frac{1}{k})$-consistent cyclets of the form $[v_0, v_1, \ldots, x_0, \ldots, x_{k-1}]$ choose one (say $\tau$) which maximizes the sum $\sum_{i=1} m(\alpha_i, \tau)$. Let $g \in G$ be a $\frac{1}{k}$-shunt for $\tau$. The cyclet $\tau$ belongs to exactly one of the $G$-orbits $A_1, \ldots, A_s$. Without loss of generality, we may assume that $\tau \in A_1$. Let $h \in G$ be such that $\tau^{gh} = \alpha_1$ and let $m = m(\alpha_1, \tau)$. Note that $m \geq 1$. Moreover, by our choice of $\alpha_i \in A_i$, we see that $m(\alpha_i, \tau) \leq m(\alpha_i, \alpha_i)$ for every $i \neq 1$.

Next, consider the automorphism $gh \in G$. Since $g$ is a $\frac{1}{k}$-shunt for $\tau$, it maps $x_0$ to $v_0$ and $x_1$ to $v_1$. Since $h$ fixes $v_0$ and $v_1$, it follows that $x_0^{gh} = v_0$ and $x_1^{gh} = v_1$. Therefore, in view of Lemma 4.1, there exists a $\frac{1}{k}$-consistent cyclet $\tau'$ of the form $[v_0, v_1, \ldots, x_0, \ldots, x_{k-1}]$ for which $gh$ is a $\frac{1}{k}$-shunt.

Recall that $m = m(\alpha_1, \tau)$ and let $\alpha_1 = [v_0, v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+\ell}]$. Then $\tau = [u_0, u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+\ell}]$ for some $u_i$ with $u_i = v_i$ for $i \leq m$ and $u_{m+1} \neq v_{m+1}$. Observe that $\ell \geq k$, for otherwise there would exist a closed walk $[v_m, v_{m+1}, \ldots, v_{m+\ell}, v_0, u_{m+1}, \ldots, u_{m+\ell}, v_m]$ of length less than $2k+1$, contradicting our assumption on the girth of $\Gamma$.

We will now show that $m(\alpha_1, \tau') = m + k$. Let $\tau' = [v_0, w_1, \ldots, w_{\ell}]$ for some $w_i$. Choose $i \in \{0, \ldots, m\}$. Observe that, since $g$ is a $\frac{1}{k}$-shunt for $\tau$, we see that $v_i^{gh} = u_i^{gh} = u_{i+k}$. Further, since $\tau^{gh} = \alpha_1$, we also see that $u_j^{gh} = v_j$ for $j \in \{1, \ldots, m+\ell\}$. Therefore, $v_i^{gh} = u_i^{gh} = v_i$ for all $i \in \{0, \ldots, m\}$. In particular, since $gh$ is a $\frac{1}{k}$-shunt for $\tau'$, it follows that $w_k = v_0^{gh} = v_k$.

If $w_i \neq v_i$ for some $i \in \{1, \ldots, k-1\}$, then the sequence of vertices $[v_0, v_1, \ldots, v_k, w_{k-1}, \ldots, u_1, v_0]$ contains a cycle of length less than $2k+1$, contradicting our assumption on girth of $\Gamma$. Hence, $w_i = v_i$ for all $i \in \{0, \ldots, k\}$.

Similarly, $w_{k+1} = w_1^{gh} = v_1^{gh} = v_{k+1}$. We can repeat this argument several times, and finally obtain that $w_{k+i} = v_{k+i}$ for all $i \in \{1, \ldots, m\}$. This shows that $m(\alpha_1, \tau') \geq m + k$. On the other hand, if $m(\alpha_1, \tau') > m + k$, then $v_{k+m+1} = w_{k+m+1} = w_{m+1}^{gh} = v_{m+1}^{gh}$. Observe also that $u_{m+1}^{gh} = u_{m+k+1}^{gh} = v_{m+k+1}$, implying that $u_{m+1} = v_{m+1}$, which contradicts the fact that $m(\alpha_1, \tau) = m$. This shows that $m(\alpha_1, \tau') = m + k$, and in particular, $\tau' \neq \alpha_i$.

Next, let us show that $m(\tau', \alpha_i) \geq m(\tau, \alpha_i)$ for all $i \geq 2$. Recall that $\tau = [v_0, v_1, \ldots, v_m, u_{m+1}, \ldots, u_{m+\ell}]$ and $\tau' = [v_0, v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+k}, w_{m+k+1}, w_{\ell}]$ (where $u_{m+1} \neq v_{m+1}$ and...
\[ w_{m+k+1} \neq v_{m+k+1} \]. If \( m(\tau', \alpha_j) < m(\tau, \alpha_j) \) for some \( j \geq 2 \), then \( \alpha_j = [v_0, v_1, \ldots, v_m, u_{m+1}, \ldots, \] \) implying that \( m(\alpha_1, \alpha_j) = m(\tau, \alpha_j) \) for all \( i \geq 2 \). This now implies that \( \sum_{i=1}^t m(\alpha_i, \tau') < \sum_{i=1}^t m(\alpha_i, \tau) \), contradicting the choice of \( \tau \).

**Remarks**

1. Since every graph has girth at least three, Theorem 4.3 can be applied with \( k = 1 \) without any restriction on the girth, and thus yields the Conway–Biggs original result from [1] (or see Theorem 2.1).
2. It follows easily from the above proof that even if the condition on the girth of \( \Gamma_1 \) in Theorem 4.3 fails to be true, the number of \( \frac{1}{k} \)-consistent cyclets is still bounded above by \( \delta_{(\Gamma, G)}(k) \).

We call a cyclet *precisely* \( \frac{1}{k} \)-consistent (relative to a group \( G \)) if it is \( \frac{1}{k} \)-consistent but not \( \frac{1}{\ell} \)-consistent for any \( \ell \leq k \). Note that if a cyclet is \( \frac{1}{k} \)-consistent and precisely \( \frac{1}{k} \)-consistent, then \( \ell \mid k \). At this point, a natural question arises: How many orbits of precisely \( \frac{1}{k} \)-consistent cyclets are there?

Let \( k, \Gamma, G, d, \) and \( d' \) be as in Theorem 4.3, that is, let \( k \) be a positive integer, let \( \Gamma \) be a graph of girth at least \( 2k + 1 \), let \( G \) be a locally arc-transitive subgroup of \( \text{Aut}(\Gamma) \), and let \( d \) and \( d' \) be the valences of two adjacent vertices in \( \Gamma \). Let \( f(k) \) denote the number of \( G \)-orbits of precisely \( \frac{1}{k} \)-consistent cyclets in \( \Gamma \). Then, in view of Theorem 4.3, we see that

\[
\sum_{\ell \mid k} f(\ell) = \delta_{(\Gamma, G)}(k)
\]

where \( \delta_{(\Gamma, G)} \) is the function from Definition 4.2. By Möbius Inversion Formula (see [2, Theorem 10.4]), we obtain this lemma:

**Lemma 4.4.** With the above notation and assumptions, the following holds:

\[
f(k) = \sum_{\ell \mid k} \mu \left( \frac{k}{\ell} \right) \delta_{(\Gamma, G)}(\ell),
\]

where \( \mu \) denotes the Möbius function.

### 5. ORBITS OF DIRECTED CYCLES

Recall that a directed cycle \( \vec{\alpha} \) is \( \frac{1}{k} \)-consistent (relative to a group \( G \leq \text{Aut}(\Gamma) \)) if an underlying cyclet \( \alpha \) is \( \frac{1}{k} \)-consistent. The group \( G \) acts on the set of all \( \frac{1}{k} \)-consistent directed cycles in a natural way.

Suppose that \( \vec{A} \) is an orbit of \( \frac{1}{k} \)-consistent directed cycles, and let \( \mathcal{A} \) be the set of all cyclets which underlie members of \( \vec{A} \). Then \( \mathcal{A} \) is a union of \( G \)-orbits of
\(1/k\)-consistent cyclets. Suppose that \(\alpha \in A\) is precisely \(1/\ell\)-consistent for some divisor \(\ell\) of \(k\). Then it is easy to see that every element of \(A\) is precisely \(1/\ell\)-consistent, and that the shifts \(\alpha, \alpha^1, \alpha^2, \ldots, \alpha^{\ell-1}\) form a complete set of representatives of \(G\)-orbits of cyclets contained in \(A\). In particular, \(A\) is a union of \(\ell\) distinct \(G\)-orbits of \(1/k\)-consistent cyclets (each being a “shift” of a fixed one).

**Theorem 5.1.** Let \(k\) be a positive integer, let \(\Gamma\) be a graph of girth at least \(2k + 1\), and let \(G\) be a locally arc-transitive subgroup of \(\text{Aut}(\Gamma)\). Then the number of \(G\)-orbits of directed \(1/k\)-consistent cycles in \(\Gamma\) is exactly

\[
\frac{1}{k} \sum_{\ell | k} \varphi \left( \frac{k}{\ell} \right) \delta_{(\Gamma, G)}(\ell)
\]

where \(\varphi\) denotes the Euler totient (and the summation is taken over all positive divisors of \(k\)).

**Proof.** First, we will need a formula from number theory (see [2, (10.9), page 93], for example) relating the totient and the Möbius function:

\[
\sum_{r|m} \frac{1}{r} \mu(r) = \frac{\varphi(n)}{n}.
\]

By Lemma 4.4 it follows that the number \(N\) of \(G\)-orbits of \(1/k\)-consistent directed cycles in \(\Gamma\) is

\[
N = \sum_{m | k} \frac{1}{m} f(m) = \sum_{m | k} \frac{1}{m} \sum_{\ell | m} \mu \left( \frac{m}{\ell} \right) \delta_{(\Gamma, G)}(\ell).
\]

Hence

\[
N = \sum_{\ell | k} \sum_{r | \frac{k}{\ell}} \frac{1}{r} \mu(r) \delta_{(\Gamma, G)}(\ell) = \sum_{\ell | k} \frac{1}{\ell} \delta_{(\Gamma, G)}(\ell) \sum_{r | \frac{k}{\ell}} \frac{1}{r} \mu(r).
\]

By the formula cited above, we get

\[
N = \sum_{\ell | k} \frac{1}{\ell} \delta_{(\Gamma, G)}(\ell) \frac{k}{\ell} \varphi \left( \frac{k}{\ell} \right) = \frac{1}{k} \sum_{\ell | k} \varphi \left( \frac{k}{\ell} \right) \delta_{(\Gamma, G)}(\ell),
\]

as required. \(\Box\)

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6. THE COMPACT DIAGRAM

Recall Example 3.4 and Figure 2, where a compact set of cyclets in the octahedron was shown. We want to represent this set by the diagram shown in Figure 3, called a compact diagram.

In this diagram, the root vertex, on the far left, represents the common beginning vertex $v_0$ of all cyclets in the set ($x$ in this case). Each path from root to leaf corresponds to one cyclet of the set and vice versa. Edges in such a path show edges in the cyclet up to the first of its edges not contained in any of the others. For example, the cyclet $\gamma$ is represented by the path $xyzs$ because $zs$ is the first edge in $\gamma$ not used in $\alpha$ or $\beta$. We say in this case that $z$ is the branching point for $\gamma$ (and its orbit $C = \gamma^G$ under $G$). The distance between the root and the branching point of $C$ is called the branching index of $C$ and denoted by $b(C)$.

Observe that these definitions extend naturally to the general case: The resulting compact diagram is always a tree (by definition), whose vertices are labeled by (not necessarily distinct) vertices of the original graph $\Gamma$. Note that, if $F$ is the family of all orbits of consistent cyclets in $\Gamma$, then for every $A \in F$, it follows that

$$b(A) = \max\{m(A, B) : B \in F \setminus \{A\}\}.$$  \hspace{1cm} (*)

As a further example, consider the compact diagram shown in Figure 4. This diagram shows a situation which might occur in an arc-transitive graph of valence 5 (it is unknown, as yet, if there is a graph which does have this diagram). The four
orbits $A, B, C, D$ of consistent cyclets correspond to the leaves at the right end of the diagram. Orbits $A, B$ share their first $b$ edges and so $m(A, B) = b$, similarly, $m(C, D) = c$ and the overlap is $a$ for any other pair of orbits. Also $b(A) = b(B) = b$ and $b(C) = b(D) = d$.

We introduce a notion of constructability of orbits. Abbreviate $m(A, A)$ by $m(A)$. Consider an orbit $A$ with $m(A) = m$. Let $\alpha_1 = [v_0, v_1, v_2, \ldots v_{m-1}]$, $\alpha_2 = [u_0, u_1, u_2, \ldots u_{m-1}]$ be representatives of $A$ so that $m(A) = m(\alpha_1, \alpha_2)$. Then $v_i = u_i$ for $0 \leq i \leq m$. There is some $g \in G$ such that $\alpha_i^g = \alpha_2$. Thus $v_i^g = u_{m+1}$. Let $t$ be a shunt for $\alpha_1$. Then $s = gt$ sends $v_0$ to $v_1$ to $v_2 \ldots v_m$ to $v_{m+1}$ and then $v_{m+1}$ to $u_{m+1}$, which is not in $\alpha_1$. Thus the orbit $\beta$ of $v_0$ under $s$ is a consistent cycle whose overlap with $\alpha_1$ is exactly $m + 1$. Thus, it is not in orbit $A$ and so must be in some other orbit $B$. We say in this situation, that $B$ is constructible from $A$, that $A$ constructs $B$; we write “$A \rightarrow B$”. Since there may be many choices for $g$ and $t$, one orbit $A$ may construct one or many other orbits. It can even construct the “trivial” orbit, consisting of “2-cycles,” that is, one edge traversed once in each direction.

**Theorem 6.1.** Let $A$ and $B$ be two orbits of consistent cyclets. Then $m(A) = b(A) - 1$, and the following five statements are equivalent:

1. $A \rightarrow B$;
2. $m(A) < m(A, B)$;
3. $m(A) + 1 = m(A, B)$;
4. The branching point of $A$ lies on the path of $B$;
5. $b(A) = m(A, B)$.

**Proof.** Let us first show that (1), (2), and (3) are equivalent. Clearly (3) implies (2). It is also clear from the construction that if $A \rightarrow B$ (via some $\alpha_1, \alpha_2 \in A$, $\beta \in B$), then $m(A) + 1 = m(\alpha_1, \beta)$. Thus $m(A) < m(A, B)$, which shows that (1) implies (2).

Next, we show that (2) implies (3) and (1): Choose $\alpha = [v_0, v_1, v_2, \ldots v_{n-1}] \in A, \beta \in B$ so that $m(\alpha, \beta) = m(A, B)$. Let $t_\alpha$ be a shunt for $\alpha$ and $t_\beta$ be a shunt for $\beta$. Let $m = m(A, B) - 1$. Then $g = t_\beta t_\alpha^{-1}$ is a symmetry which fixes $v_0, v_1, v_2, \ldots v_m$ but moves $v_{m+1}$. Since $\alpha$ and $\alpha^g$ are both in $A$ and $m(\alpha, \alpha^g) = m$, it follows that $m(A) \geq m$, and by hypothesis (2), $m(A) = m$. Thus (2) implies (3). Moreover, $\alpha, \alpha^g \in A$ together with $t_\alpha$ and $g$ satisfy the hypotheses of the construction, and the cyclet obtained by the construction is the orbit of $v_0$ under $gt_\alpha = t_\beta t_\alpha^{-1} \alpha = t_\beta$, which is exactly $\beta$. This shows that (2) implies (1), and completes the proof that (1), (2), and (3) are equivalent.

We will now show that $m(A) = b(A) - 1$. Note first that there is always an orbit, say $C$, such that $A \rightarrow C$. Since (1) implies (3), we see that $m(A) = m(A, C) - 1$, and by Equation (*) above, it follows that $m(A) \leq b(A) - 1$. Suppose that $m(A) < b(A) - 1$. Then, by Equation (*), there exists an orbit $D$ such that $m(A, D) > m(A, C)$. Since $m(A) = m(A, C) - 1$, it follows that $m(A) < m(A, D)$. By equivalence of (2) and (3), we obtain $m(A) = m(A, D) - 1$, implying

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that \( m(A, C) = m(A, D) \), which is a contradiction. Hence \( m(A) = b(A) - 1 \), as required.

It is now clear that (5) is equivalent to (3). As it is also obvious that (4) and (5) are equivalent, this completes the proof of the theorem.

Thus, compact diagram introduced above contains all the information about the overlaps and the self-overlaps of orbits of consistent cyclets. For instance, in Figure 4, \( m(A) \) must be \( b - 1 \) because \( A \)'s branching point is at a vertex at distance \( b \) from the root.

7. THE UNDIRECTED CYCLES AND CHIRALITY

In previous sections, we were mainly interested in oriented objects, such as cyclets and directed cycles, and their orbits. We would now like to turn our attention to the orbits of consistent (undirected) cycles.

Consider the \( G \)-orbit \( \vec{A} \) of \((G, \frac{1}{k})\)-consistent directed cycle \( \vec{a} \). The orbit of the inverse directed cycle \( \vec{a}^{-1} \) will then be called the inverse orbit and denoted by \( \vec{A}^{-1} \). Clearly, the orbit \( \vec{A} \) is symmetric (in the sense that \( \vec{A} = \vec{A}^{-1} \)) if and only if \( \vec{a} \) is symmetric (in the sense that \( \vec{a} \) and \( \vec{a}^{-1} \) belong to the same \( G \)-orbit). An orbit \( \vec{A} \) which is not symmetric will be called chiral.

Note that if \( k \geq 2 \), the symmetry of \( \vec{A} \) does not necessarily imply that the corresponding cyclets \( \alpha \) and \( \alpha^{-1} \) belong the same \( G \)-orbit. Namely, it could happen that none of the automorphisms \( g \in G \) that map \( \vec{a} \) to \( \vec{a}^{-1} \) fixes the initial vertex of \( \alpha \), and still some automorphism \( g \in G \) maps \( \alpha \) to a shift of \( \alpha^{-1} \). We will see an instance of such a phenomenon in Example 8.2.

The orbit of undirected cycles \( \bar{A} \) arising from a cycle \( \bar{a} \) will be called symmetric if \( \bar{A} \) is symmetric, and chiral otherwise. Note that \( \bar{A} \mapsto \bar{A} \) induces a one-to-one correspondence between symmetric orbits of directed cycles and symmetric orbits of undirected cycles, and a two-to-one correspondence between the chiral orbits. Hence, if \( c \) and \( s \) are the numbers of chiral and symmetric orbits of \( \frac{1}{k} \)-consistent cycles, respectively, then \( 2c + s \) is the number of orbits of \( \frac{1}{k} \)-consistent directed cycles, given in Theorem 5.1

8. EXAMPLES

In this section, we present a set of illuminating examples.

Example 8.1. Let us go back to the initial example of this paper, the Petersen graph, together with its full automorphism group \( G \). By Theorem 5.1, there are 2 orbits of consistent directed cycles; these are precisely the set \( \mathcal{A}_5 \) of all directed 5-cycles, and the set \( \mathcal{A}_6 \) of all directed 6-cycles, respectively. Both of these orbits are symmetric, and give rise to two distinct orbits of symmetric undirected consistent
cycles. (See Figure 1.) Further, Theorem 5.1 implies that there are exactly
\[ \frac{1}{2} \sum_{\ell \geq 2} \varphi \left( \frac{2}{\ell} \right) 2^\ell = \frac{1}{2} (1 \cdot 2 + 1 \cdot 2^2) = 3 \]
orbits of \( \frac{1}{2} \)-consistent directed cycles. Two of them are, of course, \( \vec{A}_5 \) and \( \vec{A}_6 \). The third orbit comes from the directed 8-cycle depicted as the rim cycle in Figure 5. This orbit is also symmetric.

Finally, since the girth of the Petersen graph is 5, Theorem 5.1 cannot be applied for \( k \geq 3 \); and if it were applied, it would indeed give an incorrect number of \( \frac{1}{2} \)-consistent directed cycles. It can be seen that, besides the orbits \( \vec{A}_5 \) and \( \vec{A}_6 \), there is only one additional orbit of (exactly) \( \frac{1}{2} \)-consistent directed cycles; namely the one arising from the nonagon shown in Figure 1. However, the formula of Theorem 5.1 would yield the number 4. The “missing” orbit is the one arising from the degenerate \( \frac{1}{2} \)-consistent directed “cycle” made up from a 3-arc followed by its reverse (say \( [1, 2, 3, 4, 3] \)).

Example 8.2. Let us continue with the previous example of the Petersen graph but taking \( G \) to be the index-2 subgroup of \( \text{Aut}(\Gamma) \), isomorphic to the alternating group \( A_5 \). As before, Theorem 5.1 implies that there are 2 orbits of consistent directed cycles, and one additional orbit of exactly \( \frac{1}{2} \)-consistent directed cycles. However, in contrast with the previous example, the set \( \vec{A}_5 \) of all directed 5-cycles splits into two orbits of consistent directed 5-cycles, one generated by the “outer” cyclet \( \alpha_1 = [1, 2, 3, 4, 5] \), and the other generated by the cyclet \( \alpha_2 = [1, 2, 7, 10, 5] \). Consequently, the directed 6-cycles cannot be consistent here, but they constitute the orbit \( \vec{A}_6 \) of precisely \( \frac{1}{2} \)-consistent directed cycles. All of these orbits are symmetric. However, while the cycllets \( \alpha_1 \) and \( \alpha_2 \) are such that they belong to the same \( G \)-orbit as their respective inverses \( \alpha_1^{-1} \) and \( \alpha_2^{-1} \), this is not the case for the orbit \( \vec{A}_6 \). Finally, the 8-cycle, which was \( \frac{1}{2} \)-consistent in the previous case, is not \( \frac{1}{2} \)-consistent here (but is, of course, \( \frac{1}{4} \)-consistent).

Example 8.3. Consider the complete graph \( K_5 \). We will be interested in the orbits of directed and undirected consistent cycles with respect to the three qualifying
subgroups of $\text{Aut}(K_5)$: $S_5$, $A_5$, and the affine group $\text{Aff}(1, 5)$ of order 20. Since the graph has valence 4, there should be three orbits of consistent directed cycles in each case.

1. In the action of $S_5$ on $K_5$, all directed 3-cycles are consistent and symmetric, as are all directed 4-cycles and all directed 5-cycles. Hence they give rise to three distinct orbits of undirected cycles.

2. In the action of $A_5$ on $K_5$, directed 3-cycles and directed 5-cycles are consistent, but not directed 4-cycles. However, in this action, there are two orbits of directed 5-cycles: $[1, 2, 3, 4, 5]$ belongs to one, and $[1, 2, 3, 5, 4]$ belongs to the other. All these consistent directed cycles are symmetric, and thus give rise to three distinct orbits of undirected cycles.

3. The action of the affine group $G = \text{Aff}(1, 5)$ may be thought of as generated by the permutations $(1, 2, 3, 4, 5)$ and $(1, 3, 4, 2)$. We will find it most convenient to think of $G$ as the rotation group of the regular map $\{4, 4\}_{2,1}$, shown in Figure 6. This figure shows a map on a torus in the form of rectangle with identifications as indicated by the vertex labels. Since the group has order 20, no directed 3-cycle can be consistent. There is one orbit of consistently directed 5-cycles, corresponding to the horizontal and vertical lines. There are two orbits of consistently directed 4-cycles: one contains $[1, 3, 4, 2]$ and the other contains $[1, 2, 4, 3]$. Thus, these orbits are inverse to each other, one consisting of the faces in clockwise order and the other of faces in counterclockwise order. They both induce the same orbit of undirected cycles. Hence there are exactly two orbits of consistent cycles in this case, one containing the 5-cycle $[1, 2, 3, 4, 5]$, and the other containing the face $[1, 2, 4, 3]$.

**Example 8.4.** Let us now consider the incidence graph of the Fano plane, also known as the Heawood graph $H$, shown in Figure 7. Note that the vertex $x \in \mathbb{Z}_7$ is adjacent to the vertex $y'$ if and only if $y - x \in \{1, 2, 4\}$. The girth of $H$ is 6.

In this example, we will be interested in the orbits of directed and undirected consistent cycles with respect to the full automorphism group $\text{Aut}(H) \cong \text{PGL}(2, 7)$, acting transitively on the darts of $H$, and its index-2 subgroup $G = \text{Aut}^+(H) \cong \text{PSL}(2, 7)$, corresponding to the symmetry group of the Fano plane, which is locally arc-transitive but not arc-transitive.

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By Theorem 5.1, there should be two orbits of consistently directed cycles with respect to $\text{Aut}(H)$. These are the orbits of $\vec{\alpha} = [0, 2', 1, 3', 2, 4']$ and $\vec{\beta} = [0, 2', 1, 5', 4, 6', 2, 4']$. Both of these orbits are symmetric, and thus give rise to two distinct orbits of undirected cycles. The orbit of precisely $\frac{1}{2}$-consistent directed cycles, predicted by Theorem 5.1, is the orbit of the directed hamiltonian cycle $\vec{\gamma} = [0, 2', 1, 3', 2, 4', 3, 5', 4, 6', 5, 0', 6, 1']$. This orbit is, of course, symmetric.

Now, consider the group $G = \text{Aut}^+(H)$. Since it is locally arc-transitive but not arc-transitive, it admits no consistent cycles. By Theorem 5.1, the number of $\frac{1}{2}$-consistent directed cycles is

$$\frac{1}{2} \left( 0 + 2 \cdot (3 - 1)^2 \right) = 4.$$  

Of course, two of them are the $G$-orbits of the cycles $\vec{\alpha}$ and $\vec{\beta}$, which were consistent for $\text{Aut}(H)$. Since the $\frac{1}{2}$-shunt for the directed hamiltonian cycle $\vec{\gamma}$ belongs to $G$, its $G$-orbit is the third $\frac{1}{2}$-consistent directed cycle for $G$. However, since none of the symmetries of $H$ which map $\vec{\gamma}$ to $\vec{\gamma}^{-1}$ belongs to $G$ (that is, they all switch the two bipartition sets), this orbit is chiral with respect to $G$, even though it was symmetric with respect to $\text{Aut}(H)$. The fourth orbit is therefore the reverse of the latter. It follows that there are precisely three orbits of undirected $\frac{1}{2}$-consistent cycles: two of them are symmetric, and the one containing the hamiltonian cycle is chiral (with respect to $G$).

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After this paper had been submitted to the journal, William Kantor sent us a manuscript of a paper, in which he revisited the original result of Conway, and proved it elegantly in a purely group-theoretical setting. He did not, however, attempt to generalize the result in any of the ways presented here.

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