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Free groups and ends of graphs

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Abstract Let X be a connected graph. An automorphism of X is said to be *parabolic* if it leaves no finite subset of vertices in X invariant and fixes precisely one end of X and *hyperbolic* if it leaves no finite subset of vertices in X invariant and fixes precisely two ends of X. Various questions concerning dynamics of parabolic and hyperbolic automorphisms are discussed.

The set of ends which are fixed by some hyperbolic element of a group G acting on X is denoted by $\mathcal{H}(G)$. If G contains a hyperbolic automorphism of X and G fixes no end of X, then G contains a free subgroup F such that $\mathcal{H}(F)$ is dense in $\mathcal{H}(G)$ with respect to the natural topology on the ends of X.

As an application we obtain the following: A group which acts transitively on a connected graph and fixes no end has a free subgroup whose directions are dense in the end boundary.

Keywords Automorphisms of graphs · Ends of graphs · Free groups

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Introduction

The well known "Tits alternative" says that a finitely generated linear group over a field is either almost soluble (contains a soluble subgroup of finite index) or contains a non-Abelian free subgroup. Numerous other results of this type are known, saying that a group satisfying certain conditions has a non-Abelian free subgroup. Such results for groups acting on graphs can for instance be found in [15, 16, 22, 24] and [26]. Sometimes it is shown that the free subgroup is in some sense "large", see for instance the theory of tree lattices, [3]. Recent results about the existence of free subgroups in topological groups and applications to free subgroups in the automorphism groups of trees can be found in [1, 2].

The ends of a graph can be viewed as the points of a boundary of the graph representing "different" ways of going to infinity. The set of ends is equipped with a topology in a natural way. The automorphism group has a natural action on the ends. Automorphisms of connected graphs are of three different types. The first one leaves some finite non-empty subset of vertices invariant (elliptic automorphisms), the next one fixes a unique end of the graph and has no finite orbits (parabolic automorphisms) and the last one fixes precisely two ends and has no finite orbits (hyperbolic automorphisms). Hyperbolic automorphisms can be thought of as translating the whole graph away from one of the fixed ends towards the other one, which is called the direction of the automorphism. For a parabolic automorphisms the corresponding image might be that the automorphism rotates around the end. The fixed end of a parabolic automorphisms is called the direction of the automorphism.

Let G be a group acting on a graph X. The set of directions of G is the set of all those ends of X that occur as directions of parabolic and hyperbolic automorphisms in G. The main result in this paper says that if G is a group acting on a connected graph X and fixes no end of X then G contains a free subgroup such that the set of directions of this free subgroup is dense in the set of directions of G. The special case when X is a locally finite tree and G a unimodular group acting with finitely many orbits follows from the results on tree lattices in [3].

The ends of a graph are related to various other concepts that have been shown to be useful in group theory, e.g. the boundary of a hyperbolic metric space, convergence groups and the Floyd boundary (see [16]). It would be interesting to know if results similar to the main result of this paper hold in these cases.

The first section contains definitions and preliminary results. The second section contains the proof of the main theorem. We consider free subgroups F of G such that for every vertex v in X and every element f of F the distance in the graph X between v and $f(v)$ is larger than a constant k that depends on the graph X and the action of G on X. Zorn's Lemma applies to this class of free subgroups. The difficult part of the proof is to show that a maximal free subgroup in this class has the property that its directions are dense in the directions of the underlying group G. For this, a ping-pong argument is used. A consequence of the main theorem is the following: A group which acts transitively on a connected graph and fixes no end has a free subgroup whose directions are dense in the end boundary.

Because of the generality of the result, the proof is more technical and difficult than it would be for locally finite graphs or trees. The theory of Dunwoody's structure trees would allow one to deduce the main result for locally finite graphs from the tree case.

The final section contains some comments relating the main theorem to results about fixed point properties of groups acting on graphs and also about possible extensions to other structures. In particular, the theorem is also valid for metric ends of graphs as defined in [17] and for $\mathbb{R}$-trees.

1 Preliminaries on ends and automorphisms

1.1 Ends

A *graph* X is a pair (VX, EX), where VX denotes the set of *vertices* and EX is the set of *edges*. An edge is a set consisting of two vertices. Vertices x and y are *adjacent* if $\{x, y\} \in EX$. A *path* from x to y of length n is a finite sequence of distinct vertices $x = z_0, z_1, \ldots, z_{n-1}, z_n = y$ such that z_i and z_{i+1} are adjacent for all i. A *ray* in a graph X is a sequence $x_1, x_2, \ldots$ of distinct vertices such that x_i and x_{i+1} are adjacent for all i. A *tail* of a ray $x_1, x_2, \ldots$ is a subray of the type $x_n, x_{n+1}, \ldots$. A *double ray* is a two way infinite sequence of distinct vertices $\ldots, x_{-1}, x_0, x_1, x_2, \ldots$ so that x_i and x_{i+1} are adjacent for all i. Let A denote a set of vertices in X. The subgraph *spanned* by A is a graph having A as a vertex set and the edge set is the set of all edges in X whose end vertices belong to A. We say that a set A of vertices is *connected* if the subgraph spanned by A is connected (any two vertices can be connected by a path which is contained in A). The *connected components* (or just *components*) of a set of vertices are its maximal connected subsets. A connected graph comes naturally equipped with a metric which is defined as the length of the shortest path between two vertices. We say a set of vertices is *bounded* if it has finite diameter.

Definition 1 Let X be a connected graph.

(i) The *ends* (or more precisely *vertex ends*) of X are defined as equivalence classes of rays such that two rays R_1 and R_2 are equivalent if whenever S is a finite set of vertices then infinitely many vertices from both R_1 and R_2 belong to the same connected component of $VX \setminus S$. The set of ends of X is denoted by ΩX.
(ii) The *edge ends* of X are defined as equivalence classes of rays such that two rays R_1 and R_2 are equivalent if whenever S is a finite set of edges then infinitely many vertices from both R_1 and R_2 belong to the same connected component when the edges in S have been removed from X. The set of edge ends of X is denoted by $\Omega_E X$.

The idea of defining ends as equivalence classes of rays comes from Halin. In his paper [11, §1] two rays R_1 and R_2 are said to belong to the same end if and only if there is the third ray R_3 that intersects both infinitely often. It is easy to show that this equivalence relation is the same as the relation defined in the first part of Definition 1. On the other hand, the use of connected components when finitely many vertices have been removed is more in line with Freudenthal's ideas from [8–10].

For locally finite graphs the notions of vertex and edge ends coincide. The results in this paper hold for both vertex and edge ends and the same proofs work in both cases. The proofs presented here are written with vertex ends in mind.

For a set C of vertices in X, define $C^* = VX \setminus C$. The *vertex boundary* NC of C is defined as the set of all vertices in C^* which are adjacent to a vertex in C. If C is a set with finite vertex boundary and C contains a tail of a ray R that is in the end ω then C will contain a tail from every ray in ω. Thus we are justified in saying that the end ω *lives in* C. The set of ends that live in C will be denoted by ΩC. Two ends ω_1 and ω_2 are said to be *separated* by a finite set S of vertices if ω_1 and ω_2 live in distinct components of $VX \setminus S$. Similarly, we say that S separates three ends if no two of them live in the same component of $VX \setminus S$.

Next we state two simple results from [18] which will be used repeatedly.

Lemma 1 ([18, Lemma 2.8]) *Let C be a set of vertices in a connected graph X such that C^* is connected and such that C contains the boundary of a set of vertices D. Then either $D \subseteq C$ or $D^* \subseteq C$.*

Corollary 1 ([18, Corollary 2.9]) *Let C be a set of vertices in a connected graph X such that C^* is connected and let g be an automorphism such that $g(NC) \subseteq C$. Then either $g(C) \subseteq C$ or $g(C^*) \subseteq C$.*

The set of ends is equipped with a totally disconnected topology in a natural way. A basis for this topology is formed by all sets ΩC where NC is finite. When X is a locally finite connected graph then ΩX is compact. The situation is more complicated for graphs which are not locally finite, see [17]. Note that a set A of ends is dense in ΩX if whenever C is an infinite set with finite vertex boundary containing an end then some end from A lives in C.

1.2 Ends and automorphisms

The automorphism group of X, denoted by $\mathrm{Aut}(X)$, acts on the set of rays and the equivalence relations used to define the edge and vertex ends of X are preserved. Thus we get an action of $\mathrm{Aut}(X)$ on the sets of ends ΩX.

Fundamental results about automorphisms of infinite graphs come from Halin's paper [12]. The special case of automorphisms groups of trees had been investigated before by Tits in [23]. Halin's results are phrased in terms of vertex ends. The results and their proof carry over almost verbatim to edge ends.

The automorphisms of a connected graph X are of two basic types: $g \in \mathrm{Aut}(X)$ is *elliptic* (called *type 1* in [12]) if there is a finite set of vertices S of X that is invariant under g and *non-elliptic* (called *type 2* in [12]) if there is no finite invariant set of vertices. If g is non-elliptic then there is a double ray L in X, and a positive integer n, such that L is invariant under g^n and g^n acts as a non-trivial translation on L (see [12, Theorem 7]).

Proposition 1 (Cf. [12, Theorem 9]) *Let X be a connected graph and g an automorphism of X. Then one of the following three possibilities holds*:

(i) *g leaves some finite non-empty set of vertices of X invariant. Then g is called an* elliptic *automorphism.*
(ii) *g leaves no finite non-empty set of vertices invariant and fixes a unique end. Then G is called* parabolic.
(iii) *g leaves no finite non-empty set of vertices invariant and fixes precisely two ends. Then G is called* hyperbolic.

1.3 Directions of automorphisms

Suppose that g is a parabolic or hyperbolic automorphism of a connected graph X. Let L be a double ray $\ldots, x_{-1}, x_0, x_1, x_2, \ldots$ in X such that g^{n_0} acts on L as a non-trivial translation. Suppose that there is a positive integer l_0 such that $g^{n_0}(x_i) = x_{i+l_0}$ for all integers i. Define the *direction* of g, denoted $\mathcal{D}(g)$, as the end that the ray $x_0, x_1, x_2, \ldots$ belongs to. The ray $x_0, x_{-1}, x_{-2}, \ldots$ belongs to the end $\mathcal{D}(g^{-1})$. The ends $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ are fixed by g. If g is a parabolic automorphism then $\mathcal{D}(g) = \mathcal{D}(g^{-1})$ but if g is hyperbolic then $\mathcal{D}(g) \neq \mathcal{D}(g^{-1})$.

We start by discussing a result characterizing the hyperbolic automorphisms.

Theorem 1 (Cf. [14, Theorem 2.7]) *Let X be a connected graph and g an automorphism of X. The automorphism g is hyperbolic if and only if there is a finite set of vertices S, a component C of $VX \setminus S$ and an integer k_0 such that $g^{k_0}(C \cup S) \subseteq C$.*

Conversely, assume that g is a hyperbolic automorphism of X and C is a connected set of vertices such that NC is finite and $\mathcal{D}(g)$ lives in C but $\mathcal{D}(g^{-1})$ does not live in C. Then there is number k_0 such that $g^{k_0}(C \cup NC) \subseteq C$.

The second part of the above theorem is not stated explicitly in [14] but is evident from the proof of Theorem 2.7 in [14], where no special properties of C other than those in the assumptions of Theorem 1 above are assumed.

Suppose that g is a hyperbolic automorphism of X. Let S, C and k_0 be as in the theorem above. We may assume that $g^{n_0}(C \cup S) \subseteq C$ (if not, then we can replace n_0 with $n_0 k_0$). The sets S and $g^{jn_0}(C)$ belong to different components of $VX \setminus g^{kn_0}(S)$ if $1 \leq k \leq j$. Hence a path from a vertex in S to a vertex in $g^{jn_0}(C)$ must contain vertices from each of the disjoint sets $S, g^{n_0}(S), g^{2n_0}(S), \ldots, g^{jn_0}(S)$ and thus has length at least $j+1$ and thus the distance from S to x is at least $j+1$. This implies that $\bigcap_{j \geq 0} g^{jn_0}(C) = \emptyset$. Analogously, a path from a vertex x in $VX \setminus g^{-n_0 j}(C)$ to S must contain vertices from each of the sets $g^{-n_0 j}(S), \ldots, g^{-2j}(S), g^{-j}(S)$ and S. Then $d(x, S) \geq j$ and $\bigcap_{j \geq 0}(VX \setminus g^{-n_0 j}(C)) = \emptyset$ which implies $\bigcup_{j \geq 0} g^{-n_0 j}(C) = VX$. Suppose there is a number p_0 such that if $p \geq p_0$ then $x_p \notin C$. Because of the way g^{n_0} acts on L, we see that $x_p \notin g^{-jn_0}(C)$ for $j \geq 0$. This contradicts the fact that $\bigcup_{j \geq 0} g^{-jn_0}(C) = VX$. From this we conclude that C must contain a ray of the form $x_i, x_{i+1}, x_{i+2}, \ldots$. Similarly, we see that C^* must contain a ray of the form $x_j, x_{j-1}, x_{j-2}, \ldots$. Thus S separates the ends $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$. We can also note that $\mathcal{D}(g)$ lives in C.

At this point it is also worth mentioning the following two properties. Firstly, if g is a hyperbolic automorphism (parabolic automorphism) and k a positive number then g^k is also a hyperbolic automorphism (parabolic automorphism) and $\mathcal{D}(g^k) = \mathcal{D}(g)$. Secondly, if h is some automorphism of X then hgh^{-1} is a hyperbolic automorphism (parabolic automorphism) of X and $\mathcal{D}(hgh^{-1}) = h(\mathcal{D}(g))$.

Definition 2 Let X be a connected graph and G a group acting on X. Define $\mathcal{D}(G)$ to be the set of all directions $\mathcal{D}(g)$ for some $g \in G$ and define $\mathcal{H}(G)$ to be the set of all directions $\mathcal{D}(g)$ for some hyperbolic automorphism $g \in G$. Let $\mathcal{P}(G)$ denote the set of directions of the parabolic elements of G.

The sets $\mathcal{D}(G)$, $\mathcal{H}(G)$ and $\mathcal{P}(G)$ are invariant under the action of G on ΩX.

The results in the next Theorem are not surprising and have been partially anticipated in the literature. For example, the case of a hyperbolic automorphism of a locally finite graph is treated in [25, Lemma 2.4].

Theorem 2 *Let X be a connected graph and g a hyperbolic or parabolic automorphism of X. Suppose C is a set such that NC is finite and $\mathcal{D}(g)$ lives in C. If S is a finite set of vertices then there is an integer m_0 such that $g^m(S) \subseteq C$ for all $m \geq m_0$.*

When g is a parabolic automorphism and ω an end of X, then one can find an integer m_0 such that $g^m(\omega)$ lives in C for all $m \geq m_0$ and the sequence $\{g^m(\omega)\}_{n \in \mathbb{N}}$ converges to $\mathcal{D}(g)$.

If g is an hyperbolic automorphism and ω an end not equal to $\mathcal{D}(g^{-1})$, then one can find an integer m_0 such that $g^m(\omega)$ lives in C for all $m \geq m_0$ and the sequence $\{g^m(\omega)\}_{n \in \mathbb{N}}$ converges to $\mathcal{D}(g)$.

Proof To begin with we prove the first part of the theorem in the case where the set S contains only one element v. Let L be a double ray $\ldots, x_{-1}, x_0, x_1, x_2, \ldots$ in X such that some power g^{k_0} of g acts like a translation on L. Let P be a path from x_0 to v. Note that each orbit of g is infinite and both the path P and the set NC contain only finitely many vertices. Hence it is possible to find a number m_1 such that if $m \geq m_1$ then $g^{mk_0}(P)$ does not intersect NC. The end $\mathcal{D}(g)$ belongs to C so by choosing m_1 big enough we can guarantee that if $m \geq m_1$ then $g^{mk_0}(x_0)$ is in C. But if $g^{mk_0}(x_0)$ is in C and $g^{mk_0}(P)$ does not contain a vertex from NC then $g^{mk_0}(v)$ is in C. Next we set $v_i = g^i(v)$ for $i = 0, 1, \ldots, k-1$. Applying the above we can find a number m_1 such that if $m \geq m_1$ then $g^{mk_0}(v_i)$ is in C for $i = 0, 1, \ldots, k-1$. But $g^{mk_0}(v_i) = g^{mk_0+i}(v)$ and hence we conclude that if $m_0 = m_1 k_0$ then $g^m(v)$ is in C for every $m \geq m_0$. It follows that if S is a finite set of vertices then there is a number m_0 such that if $m \geq m_0$ then $g^m(S) \subseteq C$.

Assume that g is a hyperbolic automorphism of X. Suppose that ω is an end of X and that ω is not equal to either $\mathcal{D}(g)$ or $\mathcal{D}(g^{-1})$ (the case when $\omega = \mathcal{D}(g)$ is trivial). Let S be a finite set separating ω from the ends $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$. We can by adding vertices to S assume that S contains NC. Let C', C'' and D denote the components of $VX \setminus S$ that contain $\mathcal{D}(g)$, $\mathcal{D}(g^{-1})$ and ω, respectively. Because $NC \subseteq S$, we have $C' \subseteq C$. Suppose that m_0 is a number such that if $m \geq m_0$ then $g^m(S) \subseteq C' \subseteq C$. Note that g^{-1} is also a hyperbolic automorphism and thus it is possible to assume that m_0 has been chosen large enough so that $g^{-m}(S) \subseteq C''$ for all $m \geq m_0$. Hence $S \subseteq g^m(C'')$. The end $g^m(\omega)$ is contained in the component $g^m(D)$ of $VX \setminus g^m(S)$. Since $N(g^m(D)) \subseteq g^m(S) \subseteq C$ we can conclude, by Corollary 1, that $g^m(D) \subseteq C$ or $g^m(D)^* \subseteq C$. The latter is impossible since $g^m(D)^*$ includes the set $g^m(C'')$ which by the above contains S and is thus not contained in C. Therefore $g^m(D) \subseteq C$ and we conclude that $g^m(\omega)$ is in C. It also follows that the sequence $\{g^m(\omega)\}_{n\in\mathbb{N}}$ converges to $\mathcal{D}(g)$.

Suppose now that g is parabolic. Suppose that ω is an end of X and that ω is not equal to $\mathcal{D}(g)$. Let S be a finite set separating ω from $\mathcal{D}(g)$ and assume that S contains NC. Thus the component C' of $VX \setminus S$ that contains $\mathcal{D}(g)$ is contained in C. Since $g^m(NC') \subseteq g^m(S) \subseteq C'$ we again conclude that either $g^m(C') \subseteq C'$ or $g^m(C'^*) \subseteq C'$. The first possibility would by Theorem 1 imply that g is a hyperbolic automorphism. We are assuming that g is parabolic, so $g^m(C'^*) \subseteq C' \subseteq C$. The set C'^* contains D and thus $g^m(D) \subseteq C$ and the end $g^m(\omega)$ lives in C. As above it follows that $\{g^m(\omega)\}_{n\in\mathbb{N}}$ converges to $\mathcal{D}(g)$. □

Corollary 2 *Let X be a connected graph and g a hyperbolic automorphism of X. Suppose S is a finite set separating $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ and let C denote the component of $VX \setminus S$ containing $\mathcal{D}(g)$. Let C' be the component of $VX \setminus S$ that contains $\mathcal{D}(g^{-1})$. Then*

$$\bigcap_{n\geq 0} g^n(C) = \emptyset \quad \textit{and} \quad \bigcup_{n\geq 0} g^n(C') = VX.$$

Proof Let v be a vertex in X. By the theorem above there is a number m_0 such that if $m \geq m_0$ then $g^{-m}(v) \in C'$ and thus $v \in g^m(C')$. This implies that $\bigcup_{n\geq 0} g^n(C') = VX$. Since $C \subseteq VX \setminus C'$ it is now evident that $\bigcap_{n\geq 0} g^n(C) = \emptyset$. □

1.4 Density properties of directions

The following theorem shows how one can construct hyperbolic automorphisms.

Theorem 3 ([13]) *Let X be a connected graph. Suppose C is an infinite connected subset of X and NC is finite. Let S be a connected subset containing NC. If there are elements*

g, h in $\mathrm{Aut}(X)$ *such that* $g(S) \subseteq C$ *and* $h(S) \subseteq VX \setminus (C \cup S)$ *then there is a hyperbolic automorphism* f *in* $\mathrm{Aut}(X)$ *such that* $\mathcal{D}(f) \in \Omega C$*. If* $f = g$ *or* $f = h^{-1}$ *will not do then one can take* $f = g^{-1}h^{-1}$.

Corollary 3 ([6, Corollary 3]) *Let X be a connected graph with at least two ends. Suppose G is a group acting transitively on X. Then $\mathcal{H}(G)$ is dense in ΩX.*

Next we state a general result about permutation groups that will be useful later.

Lemma 2 (Cf. [20, Lemma 2.3], see also [5, Theorem 6.6]) *Let G be a permutation group acting on a set X. Suppose all the orbits of G are infinite. If S is a finite subset of X then there is an element g in G such that $S \cap g(S) = \emptyset$.*

This lemma also implies that if S' and S'' are two finite subsets of X then there is an element $g \in G$ such that $g(S') \cap S'' = \emptyset$ (apply the Lemma to the set $S = S' \cup S''$).

Corollary 4 *Let G be a group acting on a connected graph X. Suppose that G fixes no end of X and that $\mathcal{H}(G)$ has at least three elements. Then all the orbits of G on ΩX are infinite. If S is a finite set of ends then there is an element $g \in G$ such that $S \cap g(S) = \emptyset$.*

Proof Suppose $g \in G$ is a hyperbolic automorphism. As noted above, g fixes only the ends $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ and these are also precisely the ends fixed by g^n for all $n \neq 0$. Thus the orbit of an end in $\Omega X \setminus \{\mathcal{D}(g), \mathcal{D}(g^{-1})\}$ under g is infinite. Now find an element h such that $\mathcal{D}(h) \notin \{\mathcal{D}(g), \mathcal{D}(g^{-1})\}$ (here we use the assumption that $\mathcal{H}(G)$ has at least three elements). If $\mathcal{D}(h^{-1}) \notin \{\mathcal{D}(g), \mathcal{D}(g^{-1})\}$ then we see that the orbits under h of the ends $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ are both infinite. Thus we are left with the case where $\mathcal{D}(h^{-1}) \in \{\mathcal{D}(g), \mathcal{D}(g^{-1})\}$. For the sake of the argument let us assume $\mathcal{D}(h^{-1}) = \mathcal{D}(g^{-1})$. Note that the orbit of $\mathcal{D}(g)$ under h is infinite. The group G fixes no end of X so there is an element $f \in G$ such that $f(\mathcal{D}(g^{-1})) \neq \mathcal{D}(g^{-1})$. If $f(\mathcal{D}(g^{-1})) \neq \mathcal{D}(g)$ then the orbit of $f(\mathcal{D}(g^{-1}))$ under g is infinite and thus the orbit of $\mathcal{D}(g^{-1})$ under G is infinite. If $f(\mathcal{D}(g^{-1})) = \mathcal{D}(g)$ then we consider the orbit of $f(\mathcal{D}(g^{-1}))$ under h and arrive at the same conclusion. The proof that all the orbits of G on ΩX are infinite is now finished and the last statement in the corollary follows from Lemma 2. □

The next lemma has a technical flavour, but a simple proof. This and the following lemma will be used repeatedly.

Lemma 3 *Let X be a connected graph and G a group acting on X. Suppose that $\mathcal{H}(G)$ has at least three elements and G fixes no end of X. Given a hyperbolic element g in G it is possible to find a conjugate h of g such that $\mathcal{D}(g), \mathcal{D}(g^{-1}), \mathcal{D}(h)$ and $\mathcal{D}(h^{-1})$ are all distinct ends of X.*

Proof Let g be a hyperbolic element of G. Set $D = \{\mathcal{D}(g), \mathcal{D}(g^{-1})\}$. By Corollary 2 there is an element f in G such that $f(D) \cap D = \emptyset$. Set $h = fgf^{-1}$. Then $\mathcal{D}(h) = f(\mathcal{D}(g))$ and $\mathcal{D}(h^{-1}) = f(\mathcal{D}(g^{-1}))$, and h has the properties demanded. □

Note that $g^n(\mathcal{D}(h)) = \mathcal{D}(g^n h g^{-n}) \in \mathcal{H}(G)$. The orbit of $\mathcal{D}(h)$ under g is infinite, because a hyperbolic element fixes precisely two ends. Thus $\mathcal{H}(G)$ is also infinite.

Lemma 4 (Cf. [21, Lemma 3]) *Let X be a connected graph and G a group acting on X. Suppose G fixes no end of X and $\mathcal{H}(G)$ has at least three elements. If C is a connected set of vertices with $|NC| < \infty$ and some direction of G lives in C then there is a hyperbolic element $g \in G$ such that both $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ are in C. We can choose g such that g is conjugate to some fixed hyperbolic element of G.*

Proof Let g_0 be some fixed hyperbolic element in G. Suppose $f \in G$ is a hyperbolic automorphism of X such that $\mathcal{D}(f)$ lives in C. By Lemma 3, there must be a conjugate g_1 of g_0 such that $\mathcal{D}(g_1), \mathcal{D}(g_1^{-1}), \mathcal{D}(g_0)$ and $\mathcal{D}(g_0^{-1})$ are four distinct ends. Then either for $i = 0$ or for $i = 1$, we have $\mathcal{D}(f^{-1}) \notin \{\mathcal{D}(g_i), \mathcal{D}(g_i^{-1})\}$. By Lemma 2, we can find a number m such that $f^m(\mathcal{D}(g_i))$ and $f^m(\mathcal{D}(g_i^{-1}))$ both live in C. But $f^m(\mathcal{D}(g_i))$ is the direction of $f^m g_i f^{-m}$, and $f^m(\mathcal{D}(g_i^{-1}))$ is the direction of its inverse $f^m g_i^{-1} f^{-m}$. The element $f^m g_i f^{-m}$ is hyperbolic and a conjugate of g_0. □

The following lemma describes a method of constructing a hyperbolic automorphism such that its direction and the direction of its inverse belong to predeterminated components of $VX \setminus S$ for a finite set S of vertices.

Lemma 5 *Let X be a connected graph. Suppose S is a finite connected set of vertices of X and C is an infinite component of $VX \setminus S$. Suppose furthermore that f and g are hyperbolic automorphisms of X such that $\mathcal{D}(f)$ and $\mathcal{D}(f^{-1})$ are both in C, and $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ are in C^*, and neither $f(S)$ nor $g(S)$ intersects S. Then fg is hyperbolic and $\mathcal{D}(fg) \in \Omega C$ but $\mathcal{D}((fg)^{-1}) \in \Omega C^*$.*

Proof Since $g(S) \cap S = \emptyset$ and both $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ are in C^* and since $C \cup S$ is connected, we can apply Lemma 1. Hence $g(C \cup S) \subset C^*$ or $VX \setminus g(C \cup S) \subset C^*$. The latter implies $C \subset g(C \cup S)$. This means that g fixes an end which lives in C. But the only ends which are fixed by g are its directions, which are contained in C^*. Hence $g(C \cup S) \subset C^*$. Similarly, $f(C^*) \subseteq C$. Thus $fg(C \cup S)$ is a subset of C. Therefore fg is hyperbolic with $\mathcal{D}(fg) \in \Omega C$ but $\mathcal{D}((fg)^{-1}) \in \Omega C^*$. □

We can also proof an analogue for parabolic automorphisms.

Lemma 6 *Let X be a connected graph. Suppose that f and g are parabolic automorphisms of X such that $\mathcal{D}(f) \neq \mathcal{D}(g)$. Let S be a finite set of vertices of X separating $\mathcal{D}(f)$ and $\mathcal{D}(g)$. Let C be the component of $VX \setminus S$ containing $\mathcal{D}(f)$ and C' the component of $VX \setminus S$ containing $\mathcal{D}(g)$. Then there is a number m such that $f^m g^m$ is hyperbolic and $\mathcal{D}(f^m g^m) \in \Omega C$ but $\mathcal{D}((f^m g^m)^{-1}) \in \Omega C^*$.*

Proof By Theorem 2 there is a number m such that $f^m(S) \subseteq C$ and $g^m(S) \subseteq C'$. Then either $f^m(C) \subseteq C$ or $f^m(C^*) \subseteq C$. The first possibility cannot occur because then f would be a hyperbolic automorphism. Hence $f^m(C^*) \in C$ and in particular $f^m(C' \cup S) \in C$. Similarly we show that $g^m(C \cup S) \in C'$. Then note that $g^m(C \cup S) \subset C'$ and $f^m(C') \subseteq C$ imply that $f^m g^m(C \cup S) \subseteq C$. By Theorem 1 it follows that $f^m g^m$ is hyperbolic and that $\mathcal{D}(f^m g^m) \in \Omega C$ but $\mathcal{D}((f^m g^m)^{-1}) \in \Omega C^*$. □

Corollary 5 *Let X be a connected graph and G a group acting on X. If $\mathcal{P}(G)$ has more than one element then $\mathcal{H}(G)$ is nonempty, in which case $\mathcal{P}(G)$ and $\mathcal{H}(G)$ are infinite.*

Pavone proves a stronger result along the lines of Corollary 3. We say that $\mathcal{H}(G)$ is *bilaterally dense* if for each choice of disjoint non-empty open sets U and V in ΩX there is a hyperbolic automorphism $g \in G$ such that $\mathcal{D}(g) \in U$ and $\mathcal{D}(g^{-1}) \in V$. The following result can be proved by combining Lemmas 4 and 5. In the context of [21] it is only proved for locally finite graphs but it is true without the assumption of local finiteness.

Theorem 4 (Cf. [21, Theorem 5]) *Let X be a connected graph and G a group acting on X such that G fixes no end of X. If $\mathcal{H}(G)$ is dense in ΩX then $\mathcal{H}(G)$ is bilaterally dense in ΩX.*

Remark In his paper Pavone works with *contractive G-compactifications* (see [26] or [27, (20.B)]) of metric spaces. When one considers a locally finite connected graph with the graph metric and the action of a group G of automorphisms then the end compactification is an example of a contractive G-compactification.

Theorem 5 *Let G act on a connected graph X and suppose $\mathcal{D}(G)$ has more than two elements. Then $\mathcal{H}(G)$ is dense in $\mathcal{D}(G)$, and either $\mathcal{P}(G) = \emptyset$ or $\mathcal{P}(G)$ is also dense in $\mathcal{D}(G)$. The set of directions of G is perfect* (*i.e. has no isolated points*).

Proof If $\mathcal{P}(G) = \emptyset$ then $\mathcal{D}(G) = \mathcal{H}(G)$ and there is nothing to prove.

Suppose $\mathcal{P}(G) \neq \emptyset$. Let p be a parabolic automorphism of X. If ω is the direction of some hyperbolic automorphism h of X then the sequence $\{p^n(\omega)\}$ converges to $\mathcal{D}(p)$, by Theorem 2. The end $p^n(\omega)$ is the direction of $p^n h p^{-n}$ which is a hyperbolic automorphism of X. Hence $\mathcal{H}(G)$ is dense in $\mathcal{D}(G)$.

Let h be a hyperbolic element in G. Suppose p is a parabolic automorphism of X. Then the sequence $\{h^n\mathcal{D}(p)\}_{n\in\mathbb{N}} = \{\mathcal{D}(h^n p h^{-n})\}_{n\in\mathbb{N}}$ converges to $\mathcal{D}(h)$, by Theorem 2 and we can conclude that $\mathcal{P}(G)$ is dense in $\mathcal{D}(G)$.

From the above it is obvious that the set $\mathcal{D}(G)$ is perfect. □

Corollary 6 *Let G act on a locally finite connected graph X and suppose $\mathcal{D}(G)$ has more than two elements. Then the closure of $\mathcal{D}(G)$ is a Cantor set.*

Proof The end boundary of a graph is totally disconnected and the end boundary of a locally finite graph is compact. Hence $\overline{\mathcal{D}(G)}$ is compact. If X is locally finite, then VX is countable. The elements of the basis for the topology on the ends are of the form ΩC where C is a set of vertices such that NC is finite. For each finite set of vertices S there are only finitely many sets C such that $NC = S$. Hence the end topology on ΩX has a countable base. The end boundary of a locally finite graph satisfies all separation axioms. Every regular T_1 space with a countable base is metrizable. By definition, a Cantor set is a non-empty, perfect, compact, totally disconnected and metrizable space and these properties characterize it uniquely. □

Example 1 Let X be the Cayley graph of $G = \mathbb{Z} * \mathbb{Z}^2$ with respect to the generating set $S = \{\pm 1, \binom{\pm 1}{0}, \binom{0}{\pm 1}\}$. This means that $VX = G$ and vertices x and y are adjacent whenever $x^{-1}y \in S$. The group G acts on the Cayley graph X by left multiplication. The parabolic elements are the conjugates of elements of $\mathbb{Z}^2 \setminus \{\binom{0}{0}\}$. Their parabolic directions correspond to the left cosets of $\mathbb{Z}^2$ in the following way: Let $g \in G$. Then $p = gqg^{-1}$ is a parabolic element where $q \in \mathbb{Z}^2 \setminus \{\binom{0}{0}\}$ and $g\mathbb{Z}^2$ contains all but finitely many vertices of every ray in $\mathcal{D}(p)$. All other elements of G (except for the identity element of G) are hyperbolic. The parabolic directions and the hyperbolic directions are both dense in ΩX, and ΩX is a Cantor set.

2 The main result

Theorem 6 *Let G be a group acting on a connected graph X. Suppose that G contains a hyperbolic element and that no end of X is fixed by G. Then there is a free subgroup F of G such that $\mathcal{H}(F)$ is dense in $\mathcal{H}(G)$.*

Proof If $|\mathcal{H}(G)| = 2$ then we can take F to be the cyclic group generated by some hyperbolic element of G. If $|\mathcal{H}(G)| > 2$ then Lemma 3 applies and $\mathcal{H}(G)$ is infinite.

By assumption, G contains a hyperbolic element g_0. There is a finite connected set of vertices S which separates $\mathcal{D}(g_0)$ from $\mathcal{D}(g_0^{-1})$. Set $k = \mathrm{diam}(S)$.

Let F be a maximal free subgroup of G subject to the following conditions:

(i) The elements of $F \setminus \{1\}$ are hyperbolic automorphisms of X.
(ii) For every element $f \in F \setminus \{1\}$ and every vertex $v \in X$ the distance between v and $f(v)$ is larger than k.

The existence of F follows from Zorn's lemma. Because G contains some hyperbolic element, we can be sure that F is non-trivial.

Suppose now that $\mathcal{H}(F)$ is not dense in $\mathcal{H}(G)$. Then there is a connected finite set of vertices S_0 and a component D of $VX \setminus S_0$ such that there is a hyperbolic element h_0 of G whose direction lives in D but such that no direction of F lives in D. There is an integer n such that $h_0^n(S) \subset D$. Set $S' = h_0^n(S)$. One of the components of $VX \setminus S'$ containing $h(\mathcal{D}(g_0))$ or $h(\mathcal{D}(g_0^{-1}))$ is contained in D. Let us denote this component by C. Let g_1 be a hyperbolic element in G such that $\mathcal{D}(g_1)$ and $\mathcal{D}(g_1^{-1})$ both live in C (exists by Lemma 4). By replacing g_1 with g_1^m for some number m, if necessary, we may assume that $S' \cap g_1(S') = \emptyset$.

Let g_2 be an element of $F \setminus \{1\}$. Then both $\mathcal{D}(g_2)$ and $\mathcal{D}(g_2^{-1})$ live in C^*. By Lemma 5, we see that if $g = g_1 g_2$ then $\mathcal{D}(g) \in C$ but $\mathcal{D}(g^{-1}) \in C^*$. There is a number n such that $S_1 = g^n(S') \subsetneq C$. Set $C_1 = g^n(C)$. Choose a number m such that $d(S_1, g^m(S_1)) > k$. Define $S_2 = g^m(S_1)$ and $C_2 = g^m(C_1)$.

Because $\mathcal{D}(g) \in C_2$, we can apply Lemma 4 to find a hyperbolic element h in G such that $\mathcal{D}(h)$ and $\mathcal{D}(h^{-1})$ both live in C_2. By replacing h with h^m for some number m, if necessary, we may assume that $d(v, h(v)) > k$ for all vertices $v \in X$.

Let us sum up the properties of S_1, S_2, C_1, C_2, F and h as defined above:

- S_1 and S_2 are finite connected sets of vertices, each of diameter k and such that the distance between S_1 and S_2 is greater than k.
- C_i is an infinite component of $VX \setminus S_i$ and $C_2 \subsetneq C_1$. Also $S_2 \subsetneq C_1$.
- F is a free group acting on X such that the elements of F are hyperbolic automorphisms of X and the directions of elements in F all live in $VX \setminus C_1$.
- For every vertex $v \in X$ and every element $f \in F \setminus \{1\}$ we have $d(v, f(v)) > k$.
- h is a hyperbolic automorphism of X, the directions $\mathcal{D}(h)$ and $\mathcal{D}(h^{-1})$ both live in C_2 and furthermore for every vertex $v \in X$ and every integer $n \neq 0$ we have $d(v, h^n(v)) > k$.

Theorem 6 will now be established once the following has been proved, because the group F' generated by F together with h contradicts the maximality of F.

Main Claim *Assume that S_1, S_2, C_1, C_2, F and h are as above. Then $F' = \langle F, h \rangle$ is a free group, all elements in F' are hyperbolic and $d(v, g(v)) > k$ for every element $g \in F' \setminus \{1\}$ and every vertex $v \in X$.*

Proof of the main claim. For $i = 1, 2$ define

$$A_i = \bigcup_{f \in F \setminus \{1\}} f(C_i) \quad \text{and} \quad B_i = VX \Big\backslash \Big(\bigcup_{f \in F} f(C_i) \Big).$$

Then $VX = A_i \cup B_i \cup C_i$.

We start the proof by investigating the effect of elements of F' on A_i, B_i and C_i.

First note that B_1 and B_2 are both invariant under F.

If f is a non-identity element of F then, because $\mathrm{diam}(S_i) = k$, it follows that $f(S_i) \cap S_i = \emptyset$. The images of the connected set S_i under elements in F are thus disjoint. The boundary of C_i is contained in S_i. Hence there are, for a non-identity element $f \in F$, four possibilities (i) $f(C_i) \cap C_i = \emptyset$, (ii) $f(C_i) \subsetneq C_i$, (iii) $C_i \subsetneq f(C_i)$ and (iv) $C_i^* \subsetneq f(C_i)$. The possibility (ii) would imply that $\mathcal{D}(f)$ lives in C_i and (iii) would imply that $\mathcal{D}(f^{-1})$ lives in C_i. Both cases would contradict our assumptions. Since neither $\mathcal{D}(f)$ nor $\mathcal{D}(f^{-1})$ lives in C_i and $f(S_i) \cap S_i = \emptyset$, we have $f(S_i) \subsetneq VX \setminus C_i$. The fourth possibility would imply that $f(S_i) \subsetneq C_i$. Then we could find, using Theorem 2, an element f' in F such that $f'(S_i)$ is in some component of $VX \setminus (C_i \cup S_i)$. By Theorem 3 the direction of one of the elements f, f', $f^{-1}f'^{-1}$ would be contained in C_i. But these elements are in F so this would contradict our basic assumption. Hence possibility (iv) can also be excluded. We can now conclude that $f(C_i) \cap C_i = \emptyset$ for all $f \in F \setminus \{1\}$ and $A_i \cap C_i = \emptyset$.

Because the images of S_i under F are disjoint, we see that if $S_i \cap f(C_i) \neq \emptyset$ then $S_i \subseteq f(C_i)$. This would imply that either $C_i \subseteq f(C_i)$ or $C_i^* \subseteq f(C_i)$. Both these possibilities were excluded above. So we conclude that $S_i \cap f(C_i) = \emptyset$ for all $f \in F$ and therefore $S_i \subseteq B_i$. Also note that $h^n(VX \setminus C_i) \subsetneq C_i$ for all numbers $n \neq 0$ and $i = 1, 2$.

An arbitrary element g of F' can be written as $g = h^{m_l} f_l h^{m_{l-1}} f_{l-1} \cdots f_1 h^{m_1} f_1$, where $f_1, \ldots, f_l$ are elements of F and all, except possibly f_1, are not equal to the identity and $m_i \neq 0$ for $i = 1, \ldots, l-1$. The word $h^{m_l} f_l h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1} f_1$ will be called a *standard expression* for g. Note that *a priori* we can not claim that an element in G has a unique standard expression.

Claim 1 *If $g \in F' \setminus F$ then $g(B_1) \subseteq A_2 \cup C_2$.*

Take a vertex v in B_1. Let $h^{m_l} f_l h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1} f_1$ be a standard expression for g. We use induction over l. Because $g \notin F$, we have $m_1 \neq 0$.

If $l = 1$ then $g = h^{m_1} f_1$, where $m_1 \neq 0$. Since B_1 is invariant under F, we see that $f_1(v) \in f_1(B_1) = B_1$. Since $h^{m_1}(B_1) \subsetneq C_2$, one concludes that $g(v) = h^{m_1} f_1(v) \in C_2$.

Suppose now that we have shown that whenever $g \in F' \setminus F$ and g can be written as $h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1} f_1$ with $m_{l-1} \neq 0$ then $g(v) \in C_2$, but if $m_{l-1} = 0$ then $g(v) \in A_2$. Suppose $h^{m_l} f_l h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1} f_1$ is a standard expression for g and $l \geq 2$. Set $g' = h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1} f_1$. By the induction hypothesis, $g'(v) \in C_2$. If $m_l = 0$ then $g(v) = f_l g'(v) \in A_2$ and if $m_l \neq 0$ then $g(v) = h^{m_l} f_l g'(v) = h^{m_l}(f_l g'(v)) \in C_2$. Thus $g(B_1) \subseteq A_2 \cup C_2$.

Claim 2 *If $v \in B_1$ and $g \in F' \setminus \{1\}$ then $d(v, g(v)) > k$.*

If $g \in F$ then $d(v, g(v)) > k$ by assumption. Suppose $g \in F' \setminus F$. Then $g(v) \in A_2 \cup C_2$, say $g(v) \in f(C_2)$ for some $f \in F$. A path from v to $g(v)$ must contain vertices from both $f(S_1)$ and $f(S_2)$. Thus $d(v, g(v)) > k$ because $d(S_1, S_2) > k$.

Claim 3 *The group F' is a free group.*

Let $h^{m_l} f_l h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1} f_1$ be a non-trivial standard expression (non-trivial means that f_1 is not the identity or $m_1 \neq 0$) of some element g in F'. We have seen above (in the proof of Claim 1) that $d(v, g(v)) > k$ for every vertex v in B_1 if $m_1 \neq 0$. And if $m_1 = 0$ then $g = f_1$ and f_1 is not the identity element. Hence g is not the identity element. This shows that F' is the free product of F and $\langle h \rangle$ and since F is a free group it follows that F' is also a free group. Note that this implies that the standard expressions of elements of F' are unique.

Claim 4 *Each non-identity element of F' is a hyperbolic automorphism of X.*

Suppose $g \in F'$ has standard expression of the form $g = h^{m_l} f_l h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1} f_1$ with $f_i \neq 1$ and $m_i \neq 0$ for all i. First we look at the case when $l = 1$, that is $g = h^{m_1} f_1$. Because f_1 is a hyperbolic element with both directions in the complement of C_1 and because $f_1(S_1) \cap S_1 = \emptyset$, we see that $f_1(C_1 \cup S_1) \subsetneq VX \setminus C_1$. And h^{m_1} is a hyperbolic element with both directions inside C_1. Hence $h^{m_1}(VX \setminus C_1) \subsetneq C_1$. Therefore $g(C_1 \cup S_1) \subseteq C_1$ and g is hyperbolic. With induction on l we prove that if $g \in F'$ has standard expression $g = h^{m_l} f_l h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1} f_1$ with $f_i \neq 1$ and $m_i \neq 0$ for all i, then g is hyperbolic. The induction hypothesis is that if the length of a standard expression of this form is less than l then g is hyperbolic and $g(C_1 \cup S_1) \subseteq C_1$. Set $g' = h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1} f_1$ and $g'' = h^{m_l} f_l$. By the induction hypothesis, both g' and g'' are hyperbolic and $g'(C_1 \cup S_1) \subsetneq C_1$ and $g''(C_1 \cup S_1) \subsetneq C_1$. Now $g = g'' g'$ and hence $g(C_1 \cup S_1) = g''(g'(C_1 \cup S_1)) \subsetneq C_1$. We conclude that g is hyperbolic.

Next we tackle the general case. Again we use induction on l. Let g be an element in F' with standard expression $g = h^{m_l} f_l h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1} f_1$ (allowing $f_1 = 1$ and/or $m_l = 0$).

When $l = 1$ we have three possibilities, $g = f_1$, $g = h_1^m$ and $g = h^{m_1} f_1$. The first two are hyperbolic by the original assumptions about F and h, and the third one is treated above.

Next there is the case when g has standard expression $g = f_l h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1} f_1$ with $f_1 \neq 1$. Then $g' = f_l^{-1} g f_l$ has a shorter standard expression than g. By the induction hypothesis, g' is hyperbolic and since g and g' are conjugate we conclude that g is also hyperbolic.

Assume now that g has the standard expression $g = f_l h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1}$ with $m_1 \neq 0$ and $f_l \neq 1$. Now we look at $g' = h^{m_1} g h^{-m_1} = h^{m_1} f_l h^{m_{l-1}} f_{l-1} \cdots f_2$. By the argument above, g' is hyperbolic and hence so is g.

Finally we have to consider the case where g has standard expression of the form $h^{m_l} f_l h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1}$ with $m_l \neq 0$. Look at $g' = h^{m_1} g h^{-m_1} = h^{m_l + m_1} f_l h^{m_{l-1}} f_{l-1} \cdots f_2$. By the above we can conclude that g' and thus also g are hyperbolic.

Claim 5 *Suppose g is a hyperbolic automorphism of X. Then there is a double ray L which is invariant under g^n for some positive integer n such that if w is a vertex in L then $d(w, g^n(w)) \leq d(v, g(v))$ for all vertices v in X.*

Here we have to consider the details of the proof of Halin's [12, Theorem 7]. Find a vertex w_0 and a number n such that

$$d(w_0, g^n(w_0)) = \min\{d(v, g^i(v)) \mid v \in X \text{ and } i \geq 1\}.$$

Let P be a path of shortest possible length from w_0 to $g^n(w_0)$. Define $L = \cup_{i \in \mathbf{Z}} g^{in}(P)$. Halin shows that L is a double ray and that g^n acts as a non-trivial translation on L. If w is some vertex on the double ray then it is clear from the way that the double ray was constructed that $d(w, g^n(w)) = d(w_0, g^n(w_0))$ and the claim follows.

Claim 6 *If $v \in X$ and $g \in F'$ then $d(v, g(v)) > k$.*

Consider first the case when g has a standard expression of the type $h^{m_l} f_l h^{m_{l-1}} f_{l-1} \cdots f_2 h^{m_1} f_1$ with $f_i \neq 1$ and $m_i \neq 0$ for all i. In the proof of Claim 4 we saw that $g(C_1) \subsetneq C_1$. Hence $\mathcal{D}(g) \in C_1$ and $\mathcal{D}(g^{-1}) \notin C_1$. Let L and n be as described in Claim 5. The double ray L must intersect both C_1 and C_1^*. Therefore L contains some vertex w from S_1. If v is some vertex in X then $d(v, g(v)) \geq d(w, g^n(w))$ and since $w \in S_1 \subseteq B_1$ we see from Claim 2 that $d(w, g^n(w)) > k$.

For $g \in G$ define $d_g = \min\{d(v, g(v)) \mid v \in X\}$. If g and h are conjugate elements in G then $d_g = d_h$. The claim can now be proved by induction in the same way as Claim 4.

Now our Main Claim is proved and the proof of Theorem 6 is complete. □

Corollary 7 *The free group F can be chosen such that it is freely generated by mutually conjugate elements of G.*

Proof We choose a hyperbolic element g_0 from G. Let k be an integer such that $\mathcal{D}(g_0)$ and $\mathcal{D}(g_0^{-1})$ can be separated by a finite connected set of vertices of diameter at most k. If necessary we can replace g_0 with some power of g_0 such that $d(v, g_0^n(v)) > k$ for all vertices $v \in X$ and all non-zero integers n. In the proof of Theorem 6, we now add another constraint on the subgroup F, such that F is a maximal subgroup of G subject to the conditions:

(i) The elements of F are all hyperbolic automorphisms of X.
(ii) For every element $f \in F$ and every vertex $v \in X$ the distance between v and $f(v)$ is greater or equal k.
(iii) F is freely generated by elements which are conjugates of g_0 (in G).

Now we can follow the proof of Theorem 6 and the only addition is to choose h as a conjugate of g_0, which can be done by Lemma 4. Then the group $F' = \langle F, h\rangle$ is freely generated by conjugates of g_0, contradicting the maximality of F. □

In the proof of Theorem 6 we use the axiom of choice disguised in Zorn's lemma. It is possible to organize the proof differently and give it a more constructive flavour, by using induction (if needed, transfinite induction) to construct a free generating set for the free group F.

As a consequence of Corollary 3 and Theorem 6, we obtain the following theorem.

Theorem 7 *A group which acts transitively on a connected graph and fixes no end has a free subgroup whose directions are dense in the end boundary.*

Proof There is nothing to prove if the graph X has no ends. The case that the graph has exactly one end is impossible, because this end would be fixed by the group G. Hence there are at least two ends, Corollary 3 applies and $\mathcal{H}(G)$ is dense in ΩX. By Theorem 6, there is a free subgroup F of G such that $\mathcal{H}(F)$ is dense in $\mathcal{H}(G)$. Thus $\mathcal{H}(F)$ is dense in ΩX. □

3 Commentary

1. If we only consider the case of trees then the proof of the result that a group G acting with out fixed ends contains a free subgroup F such that $\mathcal{H}(F)$ is dense in $\mathcal{H}(G)$ is much easier. In this case we can choose the constant k appearing in the proof of Theorem 6 to be

0. When the graph X is locally finite then one can use Dunwoody's theory of structure trees (see the survey paper [19] and the book [7]) to reduce the question to the case of a group acting on trees.

2. The concept of $\mathbb{R}$-trees has been widely studied, see for instance the survey [4]. There are many parallels between the theory of groups acting on $\mathbb{R}$-trees and the theory of groups acting on ordinary combinatorial trees. One can define hyperbolic isometries and directions in a similar way as for combinatorial trees. The same ideas as used in this paper to prove Theorem 6 can be used to prove the following.

Theorem 8 *Let X be an $\mathbb{R}$-tree. Suppose G is a group acting on X by isometries and that G fixes no boundary point of X. Then G contains a free group F such that $\mathcal{H}(F)$ is dense in $\mathcal{H}(G)$.*

3. In [17], the first author defines and studies "metric ends". In [18] group actions on graphs with respect to the metric ends are studied. The theory is similar to the theory of vertex ends. Theorem 6 is also valid for metric ends and the same proof works.

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