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Cutting up graphs revisited – a short proof of Stallings' structure theorem

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Abstract. This is a short proof of the existence of finite sets of edges in graphs with more than one end, such that after removing them we obtain two components which are nested with all their isomorphic images. This was first done in "Cutting up graphs" [4]. Together with a certain tree construction and some elementary Bass–Serre theory this yields a combinatorial proof of Stallings' theorem on the structure of finitely generated groups with more than one end.

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Keywords. ???.

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1 Stallings' theorem

Let VX be a set and EX a set of two-element subsets of VX. Then $X = (VX, EX)$ is called a *graph* with set of *vertices* VX and set of *edges* EX. A *path* is a finite or infinite sequence (x_i) of vertices such that $\{x_i, x_{i+1}\} \in EX$, for all i. A set C of vertices is called *connected* if any pair of vertices in C can be connected by a path which is contained in C. *Components* of a set of vertices are maximal connected subsets.

Sets of vertices A, B are *separated* by a set of vertices S if any $x \in A$ and $y \in B$ lie in distinct components of $VX \setminus S$. Sets of vertices A, B are *separated* by a set of edges F if any $x \in A$ and $y \in B$ lie in distinct components of the graph $X - F = (VX, EX \setminus F)$, respectively. A vertex x is said to be separated from a set of vertices A (or a vertex) if $\{x\}$ is separated from A. A *ray* is a one-way infinite path $(x_i)_{i \geq 0}$. A *tail* of a ray is an infinite subpath of a ray. Two rays are said to be separated by a set (of vertices or edges) if this set separates some tails of the rays. We call two rays *equivalent* if they cannot be separated by a finite set of edges. The corresponding equivalence classes are called the *ends* of X. In non-locally finite graphs there are different notions of ends. Usually they are defined by separation by removing finite sets of vertices and not edges. For locally finite graphs, all definitions of ends coincide and they correspond to Freudenthal's end compactification of locally compact Hausdorff space [7, 8].

Let a group G be generated by S. The *Cayley-graph* $X = \mathrm{Cay}(G, S)$ is defined by $VX = G$ and vertices x, y are adjacent if $x^{-1}y \in S$. The number of ends of a Cayley graph $\mathrm{Cay}(G, S)$ does not depend on the choice of S, as long as S is a finite generating set. The *number of ends of a finitely generated group* is defined as the number of ends of $\mathrm{Cay}(G, S)$. The group G *splits over the subgroup* A if G is a free product $H *_A J$ with amalgamation over A or G is an HNN-extension $H*^A$.

Theorem 1.1 (Stallings' theorem [17, 18]). *A finitely generated group has more than one end if and only if it splits over some finite subgroup.*

Consider a group G acting on a graph X with more than one end. In the following sections we will decompose the graph X into sets of vertices (so-called "blocks") which correspond to the vertices of a tree T. These blocks are separated by finite sets of edges in X, and each of these sets of edges will correspond to an edge of T. The group G will act with one or two orbits on T.

A group G acts on its Cayley graphs X by left multiplication. The stabilizer of an edge of the action of G on T is finite, because it is the stabilizer of a finite set of edges in the Cayley graph. The following result from Bass–Serre theory will complete the proof of Stallings' theorem.

Theorem 1.2 ([1, 15, 16]). *Let G act without inversion of edges on a tree that has no vertices of degree one and suppose G acts transitively on the set of (undirected) edges. If G acts transitively on the tree then G is an HNN-extension of the stabilizer of a vertex over the pointwise stabilizer of an edge. If there are two orbits on the vertices of the tree then G is the free product of the stabilizers of two adjacent vertices with amalgamation over the pointwise stabilizer of an edge.*

2 Thin cuts

For $C, D \subset VX$ let $\delta(C, D)$ denote the set of edges with one vertex in C and one vertex in D. We write $C^{\mathrm{c}} = VX \setminus C$ and call $\delta C = \delta(C, C^{\mathrm{c}})$ the *edge boundary* of C. A *k-separator* is a k-element edge boundary of a set of vertices C, where C and C^{c} are connected. For a set of edges F define $X - F$ as the graph $(VX, EX \setminus F)$.

Lemma 2.1 (Proposition 4.1 in [19]). *Let e be an edge of a connected graph X and let k be an integer. There are only finitely many k-separators which contain e.*

Proof. We prove the statement by induction on k. The case $k = 1$ is obvious.

Suppose the statement holds for all connected graphs for some integer $k \geq 1$. We show the statement in X for $(k+1)$-separators containing $e = \{x, y\}$. If there is no such $(k+1)$-separator then there is nothing to prove. Otherwise $X - \{e\}$ is connected, because $k + 1 \geq 2$. Hence there is a path π from x to y in $X - \{e\}$. Every $(k+1)$-separator in X which contains e also contains an edge e' of π. By the induction hypothesis there are only finitely many k-separators in $X - \{e\}$ which contain e'. Now the statement follows, because π is finite and different $(k+1)$-separators in X which contain e and e' correspond to different k-separators in $X - \{e\}$ which contain e'. □

A *cut* is a set of vertices C with finite edge boundary such that C and C^{c} are both connected and contain a ray. If there is a cut then let κ be the minimal cardinality of all boundaries of cuts. Cuts C with $|\delta C| = \kappa$ are called *thin*. In connected graphs with more than one end there is a thin cut.

Lemma 2.2. *Let C and D be thin cuts. If $C \cap D$ and $C^{\mathrm{c}} \cap D^{\mathrm{c}}$ are cuts then they are thin cuts.*

In [9, Theorem 2] and [10, Proposition 2.1] Jung and Watkins prove a similar result.

Proof. According to Figure 1 we set

$$\begin{aligned}
a &= |\delta(C \cap D, C^{\mathrm{c}} \cap D)|,\\
b &= |\delta(C \cap D, C \cap D^{\mathrm{c}})|,\\
c &= |\delta(C \cap D^{\mathrm{c}}, C^{\mathrm{c}} \cap D^{\mathrm{c}})|,\\
d &= |\delta(C^{\mathrm{c}} \cap D, C^{\mathrm{c}} \cap D^{\mathrm{c}})|,\\
e &= |\delta(C \cap D, C^{\mathrm{c}} \cap D^{\mathrm{c}})|,\\
f &= |\delta(C \cap D^{\mathrm{c}}, C^{\mathrm{c}} \cap D)|.
\end{aligned}$$

Then

$$\kappa = |\delta C| = a + e + f + c = |\delta D| = b + e + f + d \quad \text{and hence}$$
$$2\kappa = a + b + c + d + 2e + 2f. \tag{1}$$

The sets $C \cap D$ and $C^{\mathrm{c}} \cap D^{\mathrm{c}}$ contain an end and so

$$|\delta(C \cap D)| = a + e + b \geq \kappa \quad \text{and} \quad |\delta(C^{\mathrm{c}} \cap D^{\mathrm{c}})| = c + e + d \geq \kappa.$$

Hence $a + b + c + d + 2e \geq 2\kappa$ and, by (1), $a + b + c + d + 2e = 2\kappa$ and $f = 0$. Finally, $a + e + b = c + e + d = \kappa$ and $|\delta(C \cap D)| = |\delta(C^{\mathrm{c}} \cap D^{\mathrm{c}})| = \kappa$. □

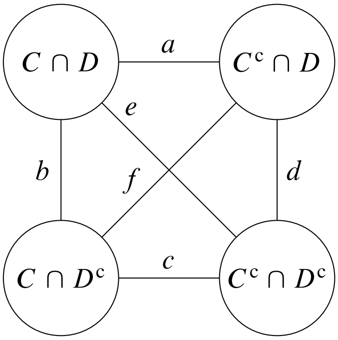

Figure 1. Cuts and their corners.

3 Main theorem

Sets of vertices C and D are *nested* if one of following intersections is empty $C \cap D, C \cap D^c, C^c \cap D, C^c \cap D^c$. These intersections are called *corners* of C and D. According to Figure 1, we say that $C \cap D$ is *opposite* to $C^c \cap D^c$, and $C^c \cap D$ is *opposite* to $C \cap D^c$. Considering sets of vertices C, D as indicator functions, we say that C is *constant* on D if either $D \subset C$ or $C \cap D = \emptyset$. Note that C is nested with D if and only if C is constant on D or on D^c.

Lemma 3.1. *If E is not nested with two opposite corners of C and D then E is neither nested with C nor with D.*

Let C and D be cuts which are not nested. If a set E is nested with C and D then E is nested with all corners of C and D.

Proof. If E is not nested with two opposite corners of C and D, then E is not constant on each of these corners. Hence E is not constant on any of the sets C, C^c, D, D^c and the statement follows.

Since E is nested with C, it is constant on two adjacent corners of C and D. Since E is nested with D it is constant on another two adjacent corners of C and D. So E is constant on at least three corners. All corners are not empty, because C and D are not nested, and two adjacent corners are connected, because C, D are cuts. Hence E is constant on the union of three corners. Thus for any corner A, the set E is constant on A or on A^c. □

Let C be a cut and let $M(C)$ be the set of thin cuts which are not nested with C. Set $m(C) = |M(C)|$. It follows from Lemma 2.1 that $m(C)$ is finite.

Lemma 3.2. *Let C and D be thin cuts which are not nested and suppose $C \cap D$ and $C^c \cap D^c$ are cuts, then*

$$m(C \cap D) + m(C^c \cap D^c) < m(C) + m(D).$$

Proof. It follows from Lemma 2.2 that $C \cap D$ and $C^c \cap D^c$ are thin.

Let E be a thin cut. If E is in $M(C^c \cap D^c) \cap M(C \cap D)$ then, by Lemma 3.1, E is in $M(C)$ and in $M(D)$. Hence if E is counted twice on the left of the above inequality then it is also counted twice on the right.

If E is in $M(C \cap D) \setminus M(C^c \cap D^c)$ or in $M(C^c \cap D^c) \setminus M(C \cap D)$, that is E is counted once on the left, then, again by Lemma 3.1, E is in $M(C)$ or in $M(D)$. Hence E is counted at least once on the right side of the inequality. We have now proved that $m(C \cap D) + m(C^c \cap D^c) \leq m(C) + m(D)$. Since $C \in M(D)$ and $D \in M(C)$, the cuts C and D are counted on the right side, but not on the left side. Thus the inequality is a strict inequality. □

Let $\mathcal{C}$ be the set of all thin cuts. Set $m = \min\{m(C) \mid C \in \mathcal{C}\}$. This minimum exists, because the values $m(C)$ are all finite. A thin cut C with $m(C) = m$ is called *optimally nested*.

Theorem 3.3. *Optimally nested cuts are all nested with each other.*

Proof. Suppose there are optimally nested thin cuts E and F which are not nested. Then $m \geq 1$. There is a pair of opposite corners, each of which contains an end. By relabeling C as C^c if necessary, we may assume that these corners are $C \cap D$ and $C^c \cap D^c$. By Lemma 2.2, $C \cap D$ and $C^c \cap D^c$ are both thin cuts. Now Lemma 3.2 says that

$$m(C \cap D) + m(C^c \cap D^c) < m(C) + m(D) = 2m.$$

Thus one of the summands on the left side is less than m, contradicting the minimality of m. □

4 Blocks and trees from nested systems

We write βC to denote the set vertices which are contained in an edge of δC. That is, $\beta C = \bigcup \delta C$. Let $\mathcal{C}$ be a set of sets of vertices. A nonempty set of vertices B is called *$\mathcal{C}$-inseparable* if no pair of vertices in B can be separated by βC, for any $C \in \mathcal{C}$. In other words, for all $C \in \mathcal{C}$ either $B \subset C \cup \beta C$ or $B \subset C^c \cup \beta C$. Maximal $\mathcal{C}$-inseparable sets are called the *$\mathcal{C}$-blocks*. Note that edges are $\mathcal{C}$-inseparable and distinct blocks are not necessarily disjoint. For a block B, let $\mathcal{C}(B)$ denote the set of all C in $\mathcal{C}$ which are minimal with respect to the inclusion $B \subset C \cup \beta C$. That is, C is in $\mathcal{C}(B)$ if $B \subset D \cup \beta D \subset C \cup \beta C$, for $C, D \in \mathcal{C}$, implies $C = D$.

We call $\mathcal{C}$ nested if any two sets in $\mathcal{C}$ are nested.

Lemma 4.1. *Let $\mathcal{C}$ be a nested set of sets of vertices and $C \in \mathcal{C}$. No pair of vertices in βC is separated by βD, for any $D \in \mathcal{C}$. Let $\mathcal{C}$ be a nested set of thin cuts. For each $C \in \mathcal{C}$ there is precisely one $\mathcal{C}$-block B_C such that $C \in \mathcal{C}(B_C)$. If $D \in \mathcal{C}(B_C)$ then $\beta D \subsetneq B_C$. Moreover,*

$$\bigcup_{D \in \mathcal{C}(B_C)} \beta D \subset B_C = \bigcap_{D \in \mathcal{C}(B_C)} D \cup \beta D. \tag{2}$$

Proof. Suppose $x, y \in \beta C$ are separated by βD. After possibly replacing C with C^{c}, D with D^{c} and x with y, we have $C \cap D = \emptyset$ and $x \in D$. Since $x \in \beta C$, x is adjacent to some vertex in C which is contained in D^{c}. Hence $x \in \beta D$, contradicting the assumption that βD separates x from another vertex.

Suppose there are different blocks B, B' such that $C \in \mathcal{C}(B) \cap \mathcal{C}(B')$. Then there are vertices $x \in B' \setminus B$ and $y \in B \setminus B'$. They are separated by βD, for some $D \in \mathcal{C}$. Any path $\pi \subset C$ from x to y intersects D and D^{c}. Hence $C \cap D \neq \emptyset$ and $C \cap D^{\mathrm{c}} \neq \emptyset$. So either $C^{\mathrm{c}} \cap D = \emptyset$ or $C^{\mathrm{c}} \cap D^{\mathrm{c}} = \emptyset$, equivalently $D \subset C$ or $D^{\mathrm{c}} \subset C$. One of the sets $D \cup \beta D$, $D^{\mathrm{c}} \cup \beta D$ contains B, the other B'. If $D \subset C$ and $B \subset D \cup \beta D$ then $B \subset D \cup \beta D \subsetneq C \cup \beta C$ in contradiction to $C \in \mathcal{C}(B)$. The case $D^{\mathrm{c}} \subset C$ leads to a contradiction in the same way.

The intersection in (2) is maximal inseparable, it contains βC and it is contained in $C \cup \beta C$. Hence this is the unique block B_C such that $C \in \mathcal{C}(B_C)$.

If $D, E \in \mathcal{C}(B_C)$ then $E^{\mathrm{c}} \subset D$ which implies $E^{\mathrm{c}} \cup \beta E \subset D \cup \beta D$ and

$$\beta E \subset \bigcap_{D \in \mathcal{C}(B_C)} D \cup \beta D = B_C.$$

This establishes the inclusion in (2). If $\mathcal{C}(B_C) = \{C\}$ then $B_C = C \cup \beta C$ and βC is a proper subset of B_C. Otherwise $\mathcal{C}(B_C)$ contains a cut D, $D \neq C$. Then $\beta C \cup \beta D \subset B_C$ and again βD is a proper subset of B_C. □

Let $\mathcal{C}$ be a nested set of thin cuts such that if $C \in \mathcal{C}$ then also $C^{\mathrm{c}} \in \mathcal{C}$. We define a graph $T = T(\mathcal{C})$. Let VT be the set of $\mathcal{C}$-blocks and $ET = \{\{B_C, B_{C^{\mathrm{c}}}\} \mid C \in \mathcal{C}\}$.

Theorem 4.2. *Let $\mathcal{C}$ be a nested thin system of cuts. Then $T(\mathcal{C})$ is a tree.*

Proof. Let $\{B_C, B_{C^c}\}$ be an edge of T. Then βC separates $C \setminus \beta C$ from $C^c \setminus \beta C$ and hence any path in T from a component in $C \setminus \beta C$ to a component in $C^c \setminus \beta C$ has to pass through the edge $\{B_C, B_{C^c}\}$. Thus $T - \{\{B_C, B_{C^c}\}\}$ is disconnected and T is a forest.

Let π be a path in X between blocks $v, w \in VT$. Lemma 2.1 implies that there are only finitely many sets βC, $C \in \mathcal{C}$, which contain some edge in π. Hence there is a finite path in T connecting the blocks v and w and T is a tree. □

Proof of Stallings' theorem 1.1. Let G be a group with more than one end which is generated by some finite set S. Let C be an optimally nested cut in $\mathrm{Cay}(G, S)$ and set

$$\mathcal{C} = \{g(C),\ g(C^c) \mid g \in G\}.$$

Then G acts transitively on the edges of $T(\mathcal{C})$. If there are inversions of edges (i.e. there is a $g \in G$ such that $g(C) = C^c$) then we simply replace every edge of T by a path of length 2 and the action will still be transitive on the edges of the resulting tree. Now Theorem 1.2 applies and we obtain Stallings' theorem. □

5 Comparison to other papers

In [5] Dunwoody and the author have proved the existence of cuts which are based on the principle of removing finite sets of vertices instead of edges. The proof in the present paper is based on ideas from this paper. The results in [5] imply a generalization of Stallings' theorem from finitely generated to arbitrarily generated groups. Namely, a group splits over a finite subgroup if and only if it has a Cayley graph with more than one vertex end. That is, a Cayley graph with two rays which can be separated by removing finitely many vertices. Another application of [5] is the generalization of Tutte's tree decomposition of 2-connected graphs to k-connected graphs for any integer k.

The way of proving Stallings' theorem in the present paper is in principle not new and was also mentioned as application in [4]. What is new are the arguments that give a short proof of the existence of cuts which we need to build the tree, in particular Section 3. The short proof of Thomassen and Woess of Lemma 2.1 also simplified the original proof in [4].

In previous papers and monographs (for instance [2, 3, 11–14, 19]), vertices of the structure tree where not defined as inseparable blocks (sets of vertices) but as equivalence classes of cuts. Cuts $C, D \in \mathcal{C}$ are called *equivalent* if either (a) $C = D$ or if (b) $C^c \subset D$ and $C^c \subset E \subset D$ implies $C^c = E$ or $E = D$. To prove transitivity of this relation is a bit technical. Using blocks instead of equivalence classes of cuts does not shorten the construction significantly, but we think that this approach is more accessible to the reader.

It is common to consider so-called structure maps $\varphi : VX \to VT$ and, for locally finite graphs, $\Phi : \Omega X \to VT \cup \Omega T$. A vertex $x \in VX$ is mapped to the vertex (equivalence class) $v \in VT$ if x is contained in all cuts of this equivalence class. Here it may happen that $\varphi^{-1}(v) = \emptyset$. Hence there may be vertices of the tree which do not correspond to sets of vertices in the graph. The block which corresponds to the vertex v of the tree was called *region of* v in [11].

In a recent pre-print, Evangelidou and Papasoglu show that the cuts we are considering in the present paper can be encoded by a cactus, see [6]. In particular, the cuts which are not nested have a certain circular structure. They also consider equivalence classes of cuts. Instead of the definition above they define cuts as equivalent if they separate the same sets of ends.

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