INVARIANT UNIFORMIZATIONS AND QUASI-TRANSVERSALS

BENJAMIN D. MILLER

ABSTRACT. We establish a dichotomy characterizing the class of \((E \times \Delta(Y))\)-invariant Borel sets \(R \subseteq X \times Y\), whose vertical sections are countable, that admit \((E \times \Delta(Y))\)-invariant Borel uniformizations, where \(X\) and \(Y\) are Polish spaces and \(E\) is a Borel equivalence relation on \(X\). We achieve this by establishing a dichotomy characterizing the class of Borel equivalence relations \(F \subseteq E\), where \(F\) has countable index below \(E\) and satisfies an additional technical definability condition, for which there is a Borel set intersecting each \(E\)-class in a non-empty finite union of \(F\)-classes.

INTRODUCTION

Endow \(\mathbb{N}\) with the discrete topology, and \(\mathbb{N}^\mathbb{N}\) with the corresponding product topology. A topological space is analytic if it is a continuous image of a closed subset of \(\mathbb{N}^\mathbb{N}\), and Polish if it is separable and admits a compatible complete metric. A subset of a topological space is Borel if it is in the smallest \(\sigma\)-algebra containing the open sets, and co-analytic if its complement is analytic. Every Polish space is analytic (see, for example, [Kec95, Theorem 7.9]), and Souslin’s theorem ensures that a subset of an analytic Hausdorff space is Borel if and only if it is analytic and co-analytic (see, for example, [Kec95, 14.11]).

A homomorphism from a binary relation \(R\) on a set \(X\) to a binary relation \(S\) on a set \(Y\) is a function \(\phi\) : \(X \to Y\) for which \((\phi \times \phi)(R) \subseteq S\), a reduction of \(R\) to \(S\) is a homomorphism from \(R\) to \(S\) that is also a homomorphism from \(\sim R\) to \(\sim S\), and an embedding of \(R\) into \(S\) is an injective reduction of \(R\) to \(S\). More generally, an embedding of a sequence \((R_i)_{i \in I}\) of binary relations on a set \(X\) into a sequence \((S_i)_{i \in I}\) of binary relations on a set \(Y\) is a function \(\phi\) : \(X \to Y\) that is an embedding of \(R_i\) into \(S_i\) for all \(i \in I\).

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1While the results in [Kec95] are stated for Polish spaces, the proofs of those to which we refer go through just as easily in the generality discussed here.
The diagonal on $X$ is given by $\Delta(X) = \{(x, y) \in X \times X \mid x = y\}$. Define $I(X) = X \times X$, and let $\mathbb{E}_0$ denote the equivalence relation on $2^\mathbb{N}$ given by $c \mathbb{E}_0 d \iff \exists n \in \mathbb{N} \forall m \geq n \ c(m) = d(m)$.

The product of binary relations $R$ on $X$ and $S$ on $Y$ is the binary relation given by $(x, y) (R \times S) (x', y') \iff (x \ R \ x' \text{ and } y \ S \ y')$. The vertical sections of a set $R \subseteq X \times Y$ are the sets of the form $R_x = \{y \in Y \mid (x, y) \in R\}$, where $x \in X$. A partial uniformization of a set $R \subseteq X \times Y$ over an equivalence relation $F$ on $Y$ is a set $U \subseteq R$ such that $F \upharpoonright U_x = I(U_x)$ for all $x \in X$.

Given an equivalence relation $E$ on a set $X$, the $E$-saturation of a set $Y \subseteq X$ is given by $|Y|_E = \{x \in X \mid \exists y \in Y \ x \ E \ y\}$, and a set $Y \subseteq X$ is $E$-complete if $X = |Y|_E$. A quasi-transversal of $E$ over a subequivalence relation $F$ is an $E$-complete set $Y \subseteq X$ for which there exists $k \in \mathbb{N}$ such that every $(E \upharpoonright Y)$-class is contained in a union of at most $k$ $F$-classes. The following fact is a generalization of the Glimm–Effros dichotomy for countable Borel equivalence relations:

**Theorem 1.** Suppose that $X$ is an analytic Hausdorff space, $E$ is a Borel equivalence relation on $X$, $F$ is a countable-index Borel subequivalence relation of $E$, and the projection onto the left coordinate of every $(\Delta(X) \times F)$-invariant Borel partial uniformization of $E$ over $F$ is Borel. Then exactly one of the following holds:

1. There is a partition $(B_n)_{n \in \mathbb{N}}$ of $X$ into $E$-invariant Borel sets such that for all $n \in \mathbb{N}$, there is an $F$-invariant Borel quasi-transversal of $E \upharpoonright B_n$ over $F \upharpoonright B_n$.

2. There is a continuous embedding $\pi: 2^\mathbb{N} \times \mathbb{N} \hookrightarrow X$ of $(\mathbb{E}_0 \times I(\mathbb{N}), \Delta(2^\mathbb{N}) \times \Delta(\mathbb{N}))$ into $(E, F)$ for which $[\pi(2^\mathbb{N} \times \mathbb{N})]_F$ is $E$-invariant.

Following the usual abuse of language, we say that a Borel equivalence relation is countable if all of its equivalence classes are countable. The special case of Theorem 1 where $E$ is countable originally arose in a conversation with Marks, and was used to eliminate the need for determinacy in an argument due to Thomas.

A uniformization of a set $R \subseteq X \times Y$ is a set $U \subseteq R$ such that $|U_x| = 1$ for all $x \in \text{proj}_X(R)$. A Borel equivalence relation $E$ on an analytic Hausdorff space $X$ is smooth if there is a Borel reduction $\pi: X \to 2^\mathbb{N}$ of $E$ to equality. Kechris has shown that the smooth Borel equivalence relations are precisely those with the property that every $(E \times \Delta(Y))$-invariant Borel set $R \subseteq X \times Y$ with countable vertical sections has an $(E \times \Delta(Y))$-invariant Borel uniformization (see [Kec201 Theorem 1.5]). He also asked the finer question as to the circumstances under which a given $(E \times \Delta(Y))$-invariant Borel set $R \subseteq X \times Y$ admits
such a uniformization. The following fact refines Kechris’s result and answers his question:

**Theorem 2.** Suppose that $X$ and $Y$ are Polish spaces, $E$ is a Borel equivalence relation on $X$, and $R \subseteq X \times Y$ is an $(E \times \Delta(Y))$-invariant Borel set whose vertical sections are countable. Then exactly one of the following holds:

1. There is an $(E \times \Delta(Y))$-invariant Borel uniformization of $R$.
2. There are a continuous embedding $\pi_X : 2\mathbb{N} \times 2\mathbb{N} \hookrightarrow X$ of $E_0 \times I(\mathbb{N})$ into $E$ and a continuous injection $\pi_Y : 2\mathbb{N} \times 2\mathbb{N} \hookrightarrow Y$ such that $R \cap (\pi_X(2\mathbb{N}) \times Y) = (\pi_X \times \pi_Y)(E_0 \times I(\mathbb{N}))$.

In §1, we establish a generalization of Theorem 1 in which $F$ need not be contained in $E$, while simultaneously strengthening it so as to ensure that, in condition (2), distinct points map to points that are inequivalent with respect to a given smooth countable Borel subequivalence relation of $E$ satisfying an additional technical property.

In §2, we establish a strengthening of Theorem 2 characterizing the circumstances under which $\text{proj}_X(R)$ is a countable union of $E$-invariant Borel sets on which $R$ admits an $((E \times F) \upharpoonright R)$-invariant Borel quasi-uniformization over a given countable Borel equivalence relation $F$. Here, a quasi-uniformization of a set $R \subseteq X \times Y$ over an equivalence relation $F$ on $Y$ is a set $U$ for which there exists $k \in \mathbb{Z}^+$ such that $U_x$ is contained in a non-empty union of at most $k$ $F$-classes for all $x \in \text{proj}_X(R)$.

1. Quasi-transversals

While the following two facts are consequences of their well-known analogs for $E_0$, we provide proofs for the reader’s convenience:

**Proposition 1.1.** Suppose that $B \subseteq 2\mathbb{N} \times \mathbb{N}$ is a non-meager set with the Baire property. Then there exists $(c, m) \in 2\mathbb{N} \times \mathbb{N}$ with the property that $B \cap ([c]_{E_0} \times \{m\})$ is infinite.

**Proof.** Fix $n \in \mathbb{N}$ and $s \in 2^{<\mathbb{N}}$ for which $B$ is comeager in $\mathcal{N}_s \times \{n\}$ (see, for example, [Kec95, Proposition 8.26]). It is sufficient to show that for all $k \in \mathbb{N}$, there are comeagerly-many $c \in \mathcal{N}_s$ with the property that $B \cap ([c]_{E_0} \times \mathbb{N}) \cap (\mathcal{N}_s \times \{n\})$ has at least element $k$ elements.

For each permutation $\sigma$ of $2^k$, let $\phi_\sigma$ be the corresponding homeomorphism of $\mathcal{N}_s \times \{n\}$, given by $\phi_\sigma(s \cdot t \cdot c)(0) = s \cdot \sigma(t) \cdot c$ for all $c \in 2^k$ and $t \in 2^k$. Then there are comeagerly-many $c \in \mathcal{N}_s$ with the property that $\phi_\sigma(c, n) \in B$ for all permutations $\sigma$ of $2^k$ (see, for example, [Kec95, Exercise 8.45]), and clearly $B \cap ([c]_{E_0} \times \mathbb{N}) \cap (\mathcal{N}_s \times \{n\})$ has at least $2^k$ elements for every such $c$. 

\[ \square \]
Proposition 1.2. Suppose that $E$ and $F$ are equivalence relations on $2^\mathbb{N} \times \mathbb{N}$ with the Baire property, every $E$-class is a countable union of $(E \cap F)$-classes, and $F \cap (\mathbb{E}_0 \times \Delta(\mathbb{N})) = \Delta(2^\mathbb{N}) \times \Delta(\mathbb{N})$. Then $E$ and $F$ are meager.

Proof. Suppose, towards a contradiction, that $F$ is not meager. As $F$ has the Baire property, the Kuratowski-Ulam theorem (see, for example, [Kec95, Theorem 8.41]) yields an $F$-class $C$ with the Baire property that is not meager. But $(\mathbb{E}_0 \times \Delta(\mathbb{N})) \uparrow C \nsubseteq \Delta(2^\mathbb{N}) \times \Delta(\mathbb{N})$ by Proposition 1.1, the desired contradiction. It follows that $F$ is meager.

The Kuratowski-Ulam theorem now ensures that every $F$-class is meager, in which case every $(E \cap F)$-class is meager, so every $E$-class is meager, thus $E$ is meager. \hfill \Box

An invariant embedding of an equivalence relation $E$ on $X$ into an equivalence relation $F$ on $Y$ is an embedding $\phi: X \hookrightarrow Y$ of $E$ into $F$ for which $\phi(X)$ is $F$-invariant.

Proposition 1.3. Suppose that $U \subseteq 2^\mathbb{N} \times \mathbb{N}$ is a non-empty open set. Then there is a continuous invariant embedding $\pi: 2^\mathbb{N} \times \mathbb{N} \hookrightarrow U$ of $\mathbb{E}_0 \times I(\mathbb{N})$ into $(\mathbb{E}_0 \times I(\mathbb{N})) \uparrow U$.

Proof. Fix $S \subseteq (\bigcup_{n \in \mathbb{N}} 2^{2n}) \times \mathbb{N}$ such that $\{ \mathcal{N}_s \times \{n\} \mid (s, n) \in S \}$ partitions $U$, as well as an injective enumeration $((s_k, n_k), t_k)_{k \in \mathbb{N}}$ of $S \times \{ c \in 2^\mathbb{N} \mid \exists n \in \mathbb{N} \forall m \geq n \, c(m) = 0 \}$, and define $\pi: 2^\mathbb{N} \times \mathbb{N} \to U$ by

$$\pi(c, k)(0)(i) = \begin{cases} s_k(i) & \text{if } i < |s_k|, \\ c((i - 1)/2) & \text{if } i \geq |s_k| \text{ is odd,} \\ t_k((i - 2|s_k|)/2) & \text{if } i \geq 2|s_k| \text{ is even, and} \\ c((i - |s_k|)/2) & \text{otherwise}, \end{cases}$$

and $\pi(c, k)(1) = n_k$. \hfill \Box

A homomorphism from a sequence $(R_i)_{i \in I}$ of binary relations on a set $X$ to a sequence $(S_i)_{i \in I}$ of binary relations on a set $Y$ is a function $\phi: X \to Y$ that is a homomorphism from $R_i$ to $S_i$ for all $i \in I$.

Proposition 1.4. Suppose that $R$ is a meager binary relation on $2^\mathbb{N} \times \mathbb{N}$. Then there is a continuous injective homomorphism $\phi: 2^\mathbb{N} \times \mathbb{N} \hookrightarrow 2^\mathbb{N} \times \mathbb{N}$ from $(\mathbb{E}_0 \times I(\mathbb{N}), \neg(I_0 \times I(\mathbb{N})))$ to $(\mathbb{E}_0 \times I(\mathbb{N}), \neg R)$ such that $\forall c \in 2^\mathbb{N} \phi([c]_{\mathbb{E}_0 \times \mathbb{N}})$ is an $(\mathbb{E}_0 \times I(\mathbb{N}))$-class.

Proof. Set $d_0 = r_0 = 1$ and $\ell_0 = 0$, and fix a decreasing sequence $(U_n)_{n \in \mathbb{N}}$ of dense open symmetric subsets of $(2^\mathbb{N} \times \mathbb{N}) \times (2^\mathbb{N} \times \mathbb{N})$ whose intersection is disjoint from $R$, as well as $\phi_0: 2^\mathbb{N} \times d_0 \leftrightarrow 2^{\ell_0} \times r_0$. 


Lemma 1.5. Suppose that \( n \in \mathbb{N}, d_n, \ell_n, r_n \in \mathbb{N} \), and \( \phi_n : 2^n \times d_n \leftrightarrow 2^{\ell_n} \times r_n \) is a bijection. Then there exist \( d_{n+1} > d_n, \ell_{n+1} > \ell_n, r_{n+1} > r_n \),
and a bijection \( \phi_{n+1} : 2^{n+1} \times d_{n+1} \leftrightarrow 2^{\ell_{n+1}} \times r_{n+1} \) such that:

1. \( \forall i < 2 \forall (t, m) \in 2^n \times d_n \ (\phi_n(t, m)(0) \subseteq \phi_{n+1}(t \setminus (i), m)(0) \) and \( \phi_n(t, m)(1) = \phi_{n+1}(t \setminus (i), m)(1) \)).
2. \( \forall i, j < 2 \forall (t, m) \in (2^n \times 2^n) \times (d_n \times d_n) \)
   \( (i = j \iff \forall \ell \in [\ell_n, \ell_{n+1}) \phi_{n+1}(t(0) \setminus (i), m(0))(0)(\ell) = \phi_{n+1}(t(1) \setminus (i), m(1))(0)(\ell)) \).
3. \( \forall (t, m) \in (2^n \times 2^n) \times (d_n \times d_n) \)
   \( \prod_{i \leq 2} \mathcal{N}_{\phi_{n+1}(t(i) \setminus (i), m(i))(0)} \times \{ \phi_{n+1}(t(i) \setminus (i), m(i))(1) \} \subseteq U_n \).

Proof. Fix an enumeration \( (t_k, m_k)_{k \leq 2^n d_n^2} \) of \( (2^n \times 2^n) \times (d_n \times d_n) \), as well as any pair \( u_0 \in 2^{< \mathbb{N}} \times 2^{< \mathbb{N}} \) such that \( \forall i < 2 \ u_0(i) \not\subseteq u_0(1 - i) \).

Given \( k < 4^n d_n^2 \) and \( u_k \in 2^{< \mathbb{N}} \times 2^{< \mathbb{N}} \), fix \( u_{k+1} \in 2^{< \mathbb{N}} \times 2^{< \mathbb{N}} \) such that:

- \( \forall i < 2 \ u_k(i) \not\subseteq u_{k+1}(i) \).
- \( \prod_{i < 2} \mathcal{N}_{\phi_{n}(t_k(i), m_k(i))(0)} \times \{ \phi_{n}(t_k(i), m_k(i))(1) \} \subseteq U_n \).

Fix \( \ell_n \geq \ell_n \) and \( u \in 2^{\ell_{n+1} - \ell_n} \times 2^{\ell_{n+1} - \ell_n} \) such that \( u_0 \cap u \) for all \( i < 2 \). Let \( d_{n+1} = 2^{\ell_{n+1} - \ell_n} d_n \) and \( r_{n+1} = 2 r_n \).

Then \( 2^{n+1} d_{n+1} = 2^{\ell_{n+1} - \ell_n} d_n, r_{n+1} = 2 r_n \) in which case there is a bijection \( \phi_{n+1} : 2^{n+1} \times d_{n+1} \leftrightarrow 2^{\ell_{n+1}} \times r_{n+1} \) with the property that \( \phi_{n+1}(t \setminus (i), m)(0) = \phi_n(t, m)(0) \cup u(i) \) and \( \phi_{n+1}(t \setminus (i), m)(1) = \phi_n(t, m)(1) \) for all \( (t, m) \in 2^n \times d_n \).

As \( \phi_n(t, m) \subseteq \phi_{n+1}(t \setminus (i), m) \) for all \( i < 2, n \in \mathbb{N} \), and \( (t, m) \in 2^n \times d_n \), we obtain a continuous function \( \phi : 2^N \times N \rightarrow 2^N \times N \) by setting \( \phi(c, m) = \bigcup_{n \geq m} \phi_n(c \upharpoonright n, m) \) for all \( c \in 2^N \) and \( m \in \mathbb{N} \).

To see that \( \phi \) is a homomorphism from \( \mathbb{E}_0 \times I(\mathbb{N}) \) to \( \mathbb{E}_0 \times I(\mathbb{N}) \), observe that if \( c \in \mathbb{E}_0 \times I(\mathbb{N}) \), then there exists \( n \geq \max_{i < 2} c(i)(1) \) with the property that \( \forall m \geq n \ c(0)(m) = c(1)(m) \), in which case \( \forall m \geq n \ \phi(c(0))(0)(m) = \phi(c(1))(0)(m) \).

To see that \( \phi \) is a homomorphism from \( \sim (\mathbb{E}_0 \times I(\mathbb{N})) \) to \( \sim R \), note that if \( c \in \sim (\mathbb{E}_0 \times I(\mathbb{N})) \), then there are infinitely many \( n \geq \max_{i < 2} c(i)(1) \) with the property that \( \phi(c(i))(i \leq 2) \in \prod_{i < 2} \mathcal{N}_{\phi_{n+1}(c(i)(0)|(n+1), c(i)(1))(0)} \times \{ \phi_{n+1}(c(i)(0) \geq n + 1, c(i)(1))(1) \} \subseteq U_n \), so \( \phi(c(i))(i \leq 2) \sim R \).

It remains to note that if \( (c, m) \in 2^N \times N \), then \( \phi([(c, m)]_{\mathbb{E}_0 \times I(\mathbb{N})}) = \bigcup_{n \geq m} \phi([c \upharpoonright m \times d_n]) = \bigcup_{n \geq m} \phi([c, m][c, m]|_{r_n \times I(r_n)} = [\phi(c, m)]_{\mathbb{E}_0 \times I(\mathbb{N})} \).

Given \( n \in \mathbb{N} \), an equivalence relation \( F \) on \( 2^n \times (n + 1) \), let \( F^* \) denote the corresponding equivalence relation on \( 2^n \times (n + 1) \) given by \( (c, \ell) F^*(d, m) \iff (c \upharpoonright n, \ell) F (d \upharpoonright n, m) \) and \( \forall k \geq n \ c(k) = d(k) \).

A one-step extension of \( F \) is an equivalence relation \( F' \) on \( 2^{n+1} \times (n + 2) \)
Proposition 1.6. Suppose that there is a clopen transversal one-step extension of $F$ has the further property that for all $i < 2$ and $(s, \ell), (t, m) \in 2^n \times (n + 1)$, and such an extension is splitting if it has the further property that $\neg(s \sim (i), \ell) F' (t \sim (1 - i), m)$ for all $i < 2$. A sequence $(F_n)_{n \in \mathbb{N}}$ is suitable if $F_0$ is the unique equivalence relation on $2^0 \times 1$, and $F_{n+1}$ is a splitting one-step extension of $F_n$ for all $n \in \mathbb{N}$.

Proof. Fix the unique transversal $S_0$ of $F_0$, and given a transversal $S_n$ of $F_n$, fix a transversal $S_{n+1} \supseteq \{(t \sim (i), m) \mid i < 2 \text{ and } (t, m) \in S_n\}$ of $F_{n+1}$. Set $S^* = \{(t \sim c, m) \mid c \in 2^n \text{ and } (t, m) \in S\}$ for all $n \in \mathbb{N}$ and $S \subseteq 2^n \times (n + 1)$, and define $U = \bigcup_{n \in \mathbb{N}} S^*_n$.

We can now establish our primary technical result.

Theorem 1.7. Suppose that $X$ is an analytic Hausdorff space, $E$ is a Borel equivalence relation on $X$, $F$ is a countable-index Borel subequivalence relation of $E$ for which the projection onto the left coordinate of every $(\Delta(X) \times F)$-invariant Borel partial uniformization of $E$ over $F$ is Borel, and $F_\perp$ is a Borel subequivalence relation of $E$ for which the $E$-saturation of every $F_\perp$-invariant Borel partial quasi-transversal of $E$ over $F_\perp$ is Borel. Then at least one of the following holds:

1. There is a partition $(B_n)_{n \in \mathbb{N}}$ of $X$ into $E$-invariant Borel sets such that for all $n \in \mathbb{N}$, at least one of the following holds:
   a. There is an $F$-invariant $(E \upharpoonright B_n)$-complete Borel partial quasi-transversal $A_n \subseteq B_n$ of $F$ over $F \cap F_\perp$.
   b. There is an $F_\perp$-invariant Borel quasi-transversal $A_n \subseteq B_n$ of $E \upharpoonright B_n$ over $F \upharpoonright F_\perp$ for some $F_\perp \in \{F, F_\perp\}$.
2. There exist a suitable sequence $(F_n)_{n \in \mathbb{N}}$ and a continuous homomorphism $\pi: 2^\mathbb{N} \times \mathbb{N} \to X$ from $(F^* \setminus (\Delta(2^\mathbb{N}) \times \Delta(\mathbb{N}))), (\mathbb{E}_0 \times I(\mathbb{N})) \setminus F^*$ to $(F \setminus F_\perp, E \setminus (F \cup F_\perp))$ with the property that $\forall c \in 2^\mathbb{N}, [\pi([c]_{\mathbb{E}_0} \times \mathbb{N})]_F$ is an $E$-class, where $F^* = \bigcup_{n \in \mathbb{N}} F^*_n$.

Proof. By [Mil20, Theorem 2.7], there are $(\Delta(X) \times F)$-invariant Borel partial uniformizations $R_n$ of $E$ over $F$ for which $E = \bigcup_{n \in \mathbb{N}} R_n$.

Lemma 1.8. Every $(\Delta(X) \times F)$-invariant Borel partial uniformization $R$ of $E$ over $F$ is contained in a $(\Delta(X) \times F)$-invariant Borel uniformization $S$ of $E$ over $F$.

Proof. Set $S_0 = R$, recursively define $S_{n+1} = (R_n \setminus (\text{proj}_0(S_n) \times Y)) \cup S_n$ for all $n \in \mathbb{N}$, and observe that the set $S = \bigcup_{n \in \mathbb{N}} S_n$ is as desired.
We can clearly assume that $R_0 = F$, and by Lemma 1.8, we can assume that each $R_n$ is a $(\Delta(X) \times F)$-invariant Borel uniformization of $E$ over $F$.

We can also assume that $F \setminus F_\perp \neq \emptyset$, since otherwise $X$ is a transversal of $F$ over $F \cap F_\perp$.

Finally, we can assume that $E \setminus (F \cup F_\perp) \neq \emptyset$. To see this, suppose otherwise, and define $A = \{x \in X \mid [x]_E \not\subseteq [x]_F\}$. Note that if $x \in A$, then there exists $y \in [x]_E \setminus [x]_F$, in which case $[y]_F \cup [y]_{F_\perp} \subseteq [x]_{F_\perp}$, so $[x]_E = [x]_{F_\perp}$, thus $A$ is a partial transversal of $E$ over $F_\perp$. By [Mii20, Proposition 2.1], there is an $F_\perp$-invariant Borel partial transversal $B \subseteq X$ of $E$ over $F_\perp$ containing $A$. But then $\sim |B|_E$ is an $E$-invariant Borel partial transversal of $E$ over $F$.

It now follows that there are continuous surjections $\phi_X : \mathbb{N}^n \to X$, $\phi_{F,F_\perp} : \mathbb{N}^n \to F \setminus F_\perp$, $\phi_{E \setminus (F \cup F_\perp)} : \mathbb{N}^n \to E \setminus (F \cup F_\perp)$, and $\phi_{R_n} : \mathbb{N}^n \to R_n$ for all $n \in \mathbb{N}$. Define $\phi_{E \setminus F_\perp} : \mathbb{N}^n \times 2 \to E \setminus F_\perp$ by

$$\phi_{E \setminus F_\perp}(b,i) = \begin{cases} \phi_{F \setminus F_\perp}(b) & \text{if } i = 1, \\ \phi_{E \setminus (F \cup F_\perp)}(b) & \text{otherwise.} \end{cases}$$

We will recursively define a decreasing sequence $(B^n)_{a < \omega_1}$ of $E$-invariant Borel subsets of $X$, off of which condition (1) holds. We begin by setting $B^0 = X$. For all limit ordinals $\lambda < \omega_1$, we set $B^\lambda = \bigcap_{\alpha < \lambda} B^\alpha$. To describe the construction at successor ordinals, we require several preliminaries.

An approximation is a sextuple $a = (n^a, D^a, F^a, \psi_X^a, \psi_R^a, \psi_{E \setminus F_\perp}^a)$ with the property that $n^a \in \mathbb{N}$, $D^a$ is a lexicographically downward-closed subset of $(n^a+1) \times 2^{n^a}$ containing $n^a \times 2^{n^a}$, $F^a$ is an equivalence relation on $D^a$, $\psi_{\ast}^a : D^a \to \mathbb{N}^{n^a}$ for all $\ast \in \{X, R\}$, and $\psi_{E \setminus F_\perp}^a : D^a \times D^a \to \mathbb{N}^{n^a}$.

If $a$ is an approximation for which $D^a \neq (n^a+1) \times 2^{n^a}$, then a one-step extension of $a$ is an approximation $b$ such that:

- $n^b = n^a$.
- $D^a = D^b \setminus \{ \max_{\text{lex}} D^b \}$.
- $F^a = F^b \upharpoonright D^a$.
- $\forall \ast \in \{X, R\} \psi_{\ast}^a = \psi_{\ast}^b \upharpoonright D^a$.
- $\psi_{E \setminus F_\perp}^a = \psi_{E \setminus F_\perp}^b \upharpoonright (D^a \times D^a)$.

If $a$ is an approximation for which $D^a = (n^a+1) \times 2^{n^a}$, then a one-step extension of $a$ is an approximation $b$ such that:

- $n^b = n^a + 1$.
- $D^b = n^b \times 2^{n^b}$. 
\[ \forall i < 2\forall(m, s), (n, t) \in D^a \]
\[ (m, s) F^a (n, t) \iff (m, s \land (i)) F^b (n, t \land (1 - i)) \land (m, s \land (i)) F^b (n, t \land (1 - i)). \]
\[ \forall i < 2\forall(m, s), (n, t) \in D^a \]
\[ \psi^b_{E \setminus F}((m, s), (n, t)) \subseteq \psi^b_{E \setminus F}((m, s \land (i)), (n, t \land (i))). \]

A configuration is a sextuple \( \gamma = (n^\gamma, D^\gamma, F^\gamma, \psi_X^\gamma, \psi_R^\gamma, \psi_{E \setminus F}^\gamma) \) with the property that \( n^\gamma \in \mathbb{N}, D^\gamma \) is a lexicographically downward-closed subset of \( (n^\gamma + 1) \times 2^{n^\gamma} \) containing \( n^\gamma \times 2^{n^\gamma} \), \( F^\gamma \) is an equivalence relation on \( D^\gamma \), \( \psi^\gamma : D^\gamma \rightarrow \mathbb{N}^\mathbb{N} \) for all \( * \in \{ X, R \} \), \( \psi_{E \setminus F}^\gamma : D^\gamma \times D^\gamma \rightarrow \mathbb{N}^\mathbb{N} \), \( \phi_R \circ \psi_R^\gamma(n, t) = ((\phi_X \circ \psi_X^\gamma)(0, t), (\phi_R \circ \psi_R^\gamma)(0, t)) \) for all \( (n, t) \in D^\gamma \), and \( \phi_{E \setminus F} \circ (\psi_{E \setminus F}^\gamma \times 1_{F^\gamma})(n, t) \) for all \( (m, s), (n, t) \in D^\gamma \). We say that \( \gamma \) is compatible with an \( E \)-invariant set \( X' \subseteq X \) if \( \phi_X \circ \psi_X^\gamma(D^\gamma) \subseteq X' \), and compatible with an approximation \( a \) if:

- \( (n^a, D^a, F^a) = (n^\gamma, D^\gamma, F^\gamma) \).
- \( \forall \gamma \in \{ X, R \} \forall(n, t) \in D^a \psi_a^\gamma(n, t) \subseteq \psi^\gamma(n, t) \).
- \( \forall(m, s), (n, t) \in D^a \psi_{E \setminus F}^a((m, s), (n, t)) \subseteq \psi_{E \setminus F}^\gamma((m, s), (n, t)). \)

We say that an approximation \( a \) is \( X' \)-terminal if no configuration is compatible with both \( X' \) and a one-step extension of \( a \).

For each configuration \( \gamma \) such that \( D^\gamma \neq (n^\gamma + 1) \times 2^{n^\gamma} \), let \( t^\gamma \) be the lexicographically minimal element of \( (n^\gamma + 1) \times 2^{n^\gamma} \) not in \( D^\gamma \), and set \( C^\gamma = (R^\gamma)_{(\phi_X \circ \psi_X^\gamma)(0, t^\gamma)} \). For each approximation \( a \) with the property that \( D^a \neq (n^a + 1) \times 2^{n^a} \) and each set \( X' \subseteq X \), define \( A'(a, X') = \cup\{ C^\gamma \mid \gamma \text{ is compatible with } a \text{ and } X' \} \).

**Lemma 1.9.** Suppose that \( X' \subseteq X \) is E-invariant and \( a \) is an \( X' \)-terminal approximation for which \( D^a \neq (n^a + 1) \times 2^{n^a} \). Then \( A'(a, X') \) is a partial quasi-transversal of \( F \) over \( F \cap F_L \).

**Proof.** Suppose, towards a contradiction, that there is a configuration \( \gamma \), compatible with \( a \) and \( X' \), with the property that \( C^\gamma \) contains strictly more than \( |D^\gamma| \) \( (F \cap F_L) \)-classes, in which case there exists \( y \in C^\gamma \setminus \{(\phi_X \circ \psi_X^\gamma)(D^\gamma)\}_{F \cap F_L} \). Define \( n^\delta = n^a \), as well as \( D^\delta = D^a \cup \{(n^a, t^a)\} \), and fix an extension \( \psi_X^\delta \) of \( \psi_X^\gamma \) to \( D^\delta \) for which \( (\phi_X \circ \psi_X^\delta)(n^a, t^a) = y \). Let \( F^\delta \) be the equivalence relation on \( D^\delta \) given by \( F^\delta \mid D^\gamma = F^\gamma \mid D^\gamma \) and \( (n, t) F^\delta (n^a, t^a) \iff (\phi_X \circ \psi_X^\gamma)(n, t) F (\phi_X \circ \psi_X^\delta)(n^a, t^a) \) for all \( (n, t) \in D^\delta \), fix an extension \( \psi_R^\delta \) of \( \psi_R^\gamma \) to \( D^\delta \) for which \( (\phi_R \circ \psi_R^\delta)(n^a, t^a) = y \), and fix an extension \( \psi_{E \setminus F}^\delta \) of \( \psi_{E \setminus F}^\gamma \) to \( D^\delta \times D^\delta \) such that \( (\phi_{E \setminus F} \circ (\psi_{E \setminus F}^\delta \times 1_{F^\delta}))(n, s), (n, t)) = ((\phi_X \circ \psi_X^\gamma)(n, s), (\phi_X \circ \psi_X^\gamma)(n, t)) \) for all \( (m, s), (n, t) \in D^\delta \) such that
\((n^a, t^a) \in \{(m, s), (n, t)\}\). Then \(\delta\) is compatible with a one-step extension of \(a\), contradicting the fact that \(a\) is \(X'\)-terminal.

Set \(X = X \times \{F, F_\perp\}\) and \(E = E \times I(\{F, F_\perp\})\), and define \(\overline{F}\) on \(X\) by \((x, F_*) \overline{F} (x', F'_*) \iff (F_* = F'_* \text{ and } x \in F_* \text{ or } x' \in F'_*)\). For each configuration \(\gamma\), set \(A^\gamma = (\phi_X \circ \phi_X)(D^\gamma)\), and for each approximation \(a\) with the property that \(D^a = (n^a + 1) \times 2^n\) and each \(E\)-invariant set \(X' \subseteq X\), define \(\mathcal{A}(a, X') = \{A^\gamma \mid \gamma\ \text{is compatible with } a \text{ and } X'\}\) and \(\overline{\mathcal{A}}(a, X') = \{A^\gamma \times \{F, F_\perp\} \mid \gamma\ \text{is compatible with } a \text{ and } X'\}\). We say that a family \(\overline{\mathcal{A}}\) of subsets of \(X\) is \(\overline{F}\)-intersecting if all distinct sets in \(\overline{\mathcal{A}}\) have disjoint \(\overline{F}\)-saturations, and \(\overline{E}\)-locally \(\overline{F}\)-intersecting if for every \(\overline{E}\)-class \(C\), the family \(\overline{\mathcal{A}}\mid C = \{A \in \overline{\mathcal{A}} \mid A \subseteq C\}\) is \(\overline{F}\)-intersecting.

**Lemma 1.10.** Suppose that \(X' \subseteq X\) and \(a\) is an \(X'\)-terminal approximation for which \(D^a = (n^a + 1) \times 2^n\). Then \(\overline{\mathcal{A}}(a, X')\) is \(\overline{E}\)-locally \(\overline{F}\)-intersecting.

**Proof.** Suppose, towards a contradiction, that there are configurations \(\gamma_0\) and \(\gamma_1\), both compatible with \(a\) and \(X'\), such that \(A^{\gamma_0}\) and \(A^{\gamma_1}\) are contained in the same \(E\)-class, but have disjoint \(F\)-saturations and disjoint \(F_\perp\)-saturations. Define functions \(\psi_{E,F_\perp}^\delta : D^\delta \to \mathbb{N}^\delta \) by \(\psi_{E,F_\perp}^\delta((m, s), (n, t) \in (i)) = \psi_{E,F_\perp}^\delta((m, s), (n, t) \in (1 - i))\) for all \(i \in \{X, R\}\), \(i \neq 2\), and \((m, s), (n, t) \in D^\delta\), let \(F_\perp^\delta\) be the equivalence relation on \(D^\delta\) given by \((m, s) F_\perp^\delta (n, t) \iff (\phi_X \circ \psi_{E,F_\perp}^\delta)(m, s) F_\perp (\phi_X \circ \psi_{E,F_\perp}^\delta)(n, t)\) for all \((m, s), (n, t) \in D^\delta\), and fix \(\psi_{E,F_\perp}^\delta : D^\delta \times D^\delta \to \mathbb{N}^\delta\) such that \(\psi_{E,F_\perp}^\delta((m, s), (n, t)) = \psi_{E,F_\perp}^\delta((m, s), (n, t))\) for all \(i \in \{X, R\}\), \(i \neq 2\), and \((m, s), (n, t) \in D^\delta\). Then \(\delta\) is compatible with a one-step extension of \(a\), contradicting the fact that \(a\) is \(X'\)-terminal.

Suppose that \(a\) is \(B^a\)-terminal. If \(D^a \neq (n^a + 1) \times 2^n\), then Lemma 1.9 and [Mil20, Proposition 2.1] yield an \(F\)-invariant Borel partial quasi-transversal \(A(a, B^a)\) of \(F\) over \(F \cap F_\perp\) containing \(A'(a, B^a)\), in which case we define \(B(a, B^a) = [A(a, B^a)]_E\). A set \(Y \subseteq X\) punctures a family \(\mathcal{A}\) of subsets of \(X\) if \(A \cap Y \neq \emptyset\) for all \(A \in \mathcal{A}\). If \(D^a = (n^a + 1) \times 2^n\), then Lemma 1.10 and [Mil20, Proposition 4.1] yield an \(\overline{F}\)-invariant Borel partial quasi-transversal \(\overline{A}(a, B^a)\) of \(\overline{E}\) over \(\overline{F}\) puncturing \(\overline{\mathcal{A}}(a, B^a)\), and it follows that the set \(\overline{A}_F(a, B^a) = \{x \in X \mid (x, F_*) \in \overline{A}(a, B^a)\}\) is an \(F_\ast\)-invariant Borel partial transversal of \(\overline{E}\) over \(\overline{F}_\ast\) for all \(F_\ast \in \{F, F_\perp\}\). The set \(\bigcup_{F_\ast \in \{F, F_\perp\}} \overline{A}_F(a, B^a)\) punctures \(\overline{A}(a, B^a)\), in which case we define \(B(a, B^a) = \bigcup_{F_\ast \in \{F, F_\perp\}} [A_F(a, B^a)]_E\).
Let $B^{\alpha+1}$ be the set obtained from $B^\alpha$ by subtracting the union of the sets of the form $B(a, B^\alpha)$, where $a$ varies over all $B^\alpha$-terminal approximations.

**Lemma 1.11.** Suppose that $\alpha < \omega_1$ and $a$ is a non-$B^{\alpha+1}$-terminal approximation. Then $a$ has a non-$B^\alpha$-terminal one-step extension.

**Proof.** Fix a one-step extension $b$ of $a$ for which there is a configuration $\gamma$ compatible with $b$ and $B^{\alpha+1}$. Then $(\phi_X \circ \phi_X^{-1})(D^\gamma) \subseteq B^{\alpha+1}$, so $b$ is not $B^\alpha$-terminal.

Fix $\alpha < \omega_1$ such that the families of $B^\alpha$- and $B^{\alpha+1}$-terminal approximations coincide, and let $a_0$ be the approximation given by $n^{a_0} = 0$ and $D^{a_0} = 1 \times 2^0$. As $\overline{\mathcal{A}}(a_0, X') = \{(x, F_*) \mid F_* \in \{F, F_{\perp}\}\mid x \in X'\}$ for all $E$-invariant sets $X' \subseteq X$, we can assume that $a_0$ is not $B^\alpha$-terminal, since otherwise $B^{\alpha+1} = \emptyset$, so condition (1) holds.

By recursively applying Lemma 1.11, we obtain non-$B^\alpha$-terminal one-step extensions $a_{n+1}$ of $a_n$ for all $n \in \mathbb{N}$. Let $(a_n)_{n \in \mathbb{N}}$ be the unique subsequence such that $D^{a_n} = (n+1) \times 2^n$ for all $n \in \mathbb{N}$. Define $F_n = F_{a_n}$ for all $n \in \mathbb{N}$, $\psi_n : 2^n \times \mathbb{N} \to \mathbb{N}$ by $\psi_n(c, m) = \bigcup_{n \geq m} \psi^a_n(m, c(0) \mid n)$ for all $* \in \{X, R\}$, and $\psi_{E \setminus F_{\perp}} : \mathbb{E}_0 \times I(\mathbb{N}) \to \mathbb{N}$ by $\psi_{E \setminus F_{\perp}}((b, \ell), (c, m)) = \bigcup_{n \geq \min(b, \ell), (c, m)} \psi_{E \setminus F_{\perp}}(\psi^a_n((\ell, b \mid n), (m, c \mid n)))$, where $n((b, \ell), (c, m))$ is the least natural number $n \geq \max\{\ell, m\}$ such that $\forall k \geq n$ $b(k) = c(k)$. We will show that the function $\pi = \phi_X \circ \psi_X$ is as desired.

To see that $\forall c \in 2^N \mid [\pi([c]_{\mathbb{E}_0} \times \mathbb{N})]_F$ is an $E$-class, we will show that if $c \in 2^N$ and $m \in \mathbb{N}$, then $(\phi_{R_{\mathbb{E}}} \circ \phi_R)(c, m) = (\pi(c, 0), \pi(c, m))$. As $X \times X$ is a Hausdorff space, it is sufficient to show that if $U$ is an open neighborhood of $(\pi(c, 0), \pi(c, m))$ and $V$ is an open neighborhood of $(\phi_{R_{\mathbb{E}}} \circ \phi_R)(c, m)$, then $U \cap V \neq \emptyset$. Towards this end, fix $n \geq m$ such that $\phi_X(\mathcal{N}_{\psi_X^a|_{(0,c|m)}} \times \phi_X(\mathcal{N}_{\psi_X^a|_{(m,c|m)}}) \subseteq U$ and $\phi_{R_{\mathbb{E}}} \circ \phi_R)(\mathcal{N}_{\psi_X^a|_{(0,c|m)}} \times \phi_X(\mathcal{N}_{\psi_X^a|_{(m,c|m)}}) \subseteq V$. As $a_n$ is not $B^\alpha$-terminal, there is a configuration $\gamma$ compatible with $a_n$, in which case $(\phi_X \circ \psi_X^\gamma(0, c \mid n), (\phi_X \circ \psi_X^\gamma)(m, c \mid n)) \in U$ and $(\phi_{R_{\mathbb{E}}} \circ \phi_R^\gamma)(m, c \mid n) \in V$, thus $U \cap V \neq \emptyset$.

It now only remains to establish that $\pi$ is a homomorphism from $(F^* \setminus (\Delta(2^\mathbb{N}) \times \Delta(\mathbb{N}))), (\mathbb{E}_0 \times I(\mathbb{N}))(F^* \setminus F_{\perp}, (E \setminus (F \cup F_{\perp}))).$ We will show the stronger fact that if $b \mathbb{E}_0 c$ and $\ell, m \in \mathbb{N}$, then $(\phi_{E \setminus F_{\perp}} \circ \psi_{E \setminus F_{\perp}} \times \mathbf{1}_{F^*})(b, \ell), (c, m)) = (\pi(b, \ell), (c, m))$. As $X \times X$ is a Hausdorff space, it is sufficient to show that if $U$ is an open neighborhood of $(\pi(b, \ell), (c, m))$ and $V$ is an open neighborhood of $(\phi_{E \setminus F_{\perp}} \circ \psi_{E \setminus F_{\perp}} \times \mathbf{1}_{F^*})(b, \ell), (c, m))$, then $U \cap V \neq \emptyset$. Towards this end, set $n = n((b, \ell), (c, m))$, and note that $\phi_X(\mathcal{N}_{\psi_X^a|_{(0,b|m)}} \times \phi_X(\mathcal{N}_{\psi_X^a|_{(m,c|m)}}) \subseteq U$ and $\phi_{E \setminus F_{\perp}}(\mathcal{N}_{\psi_X^a|_{(0,b|m)}} \times \mathbf{1}_{F^*})(b, \ell), (c, m))) \subseteq V$. As $a_n$ is not $B^\alpha$-terminal, there exists a configuration $\gamma$ compatible with $a_n$,.
in which case \((\phi_X \circ \psi^\gamma_X)(\ell, b \upharpoonright n), (\phi_X \circ \psi^\gamma_X)(m, c \upharpoonright n)) \in U\) and \(\phi_E(\psi^\gamma_{E\setminus F})(((\ell, b \upharpoonright n), (m, c \upharpoonright n)), 1_{F^*}((b, \ell), (c, m))) \in V\), and it follows that \(U \cap V \neq \emptyset\).

Remark 1.12. The apparent use of choice beyond \(\mathsf{DC}\) in the above argument can be eliminated by first running the analog of the argument without \([\text{Mil}20]\) Proposition 2.1 and replacing the use of \([\text{Mil}20]\) Propositions 4.1] with the use of its weakening without any definability constraints on the partial quasi-transversal puncturing the family (which can be proven in the same manner, but without using \([\text{Mil}20]\) Proposition 2.1]), in order to obtain an upper bound \(\alpha' < \omega_1\) on the least ordinal \(\alpha < \omega_1\) for which the sets of \(B^\alpha\)- and \(B^{\alpha+1}\)-terminal approximations coincide.

The composition of relations \(R\) on \(X \times Y\) and \(S\) on \(Y \times Z\) is given by \(R \circ S = \{(x, z) \in X \times Z \mid \exists y \in Y \; x R y \; R z\}\).
Suppose that \( E = (E \cap F) \circ F_\perp \). Then exactly one of the following holds:

1. There exist a partition \( (B_n)_{n \in \mathbb{N}} \) of \( X \) into \( E \)-invariant Borel sets such that for all \( n \in \mathbb{N} \), there is an \( (E \cap F) \)-invariant Borel quasi-transversal \( A_n \subseteq B_n \) of \( E \upharpoonright B_n \) over \( (E \cap F) \upharpoonright B_n \).

2. There is a continuous embedding \( \pi : 2^\mathbb{N} \times \mathbb{N} \to X \) of \((E_0 \times I(\mathbb{N}), \Delta(2^\mathbb{N}) \times \Delta(\mathbb{N}))\) into \((E, F \cup F_\perp)\) for which \([\pi(2^\mathbb{N} \times \mathbb{N})]_{E \cap F}\) is \( E \)-invariant.

Proof. To see that conditions (1) and (2) are mutually exclusive, note that if both hold, then there exists \( n \in \mathbb{N} \) for which \( \pi^{-1}(A_n) \) is not meager, thus \( \pi^{-1}(A_n) \) is not meager, contradicting Proposition 1.1.

Note that if \( A \subseteq X \) is an \( E \)-invariant Borel set for which there is an \( F_\perp \)-invariant Borel quasi-transversal of \( E \upharpoonright A \) over \( F_\perp \upharpoonright A \), then the smoothness of \( F_\perp \) and \textit{[HKL90, Theorem 1.1]} ensures that \( E \upharpoonright A \) is smooth. Moreover, if \( B \subseteq X \) is an \( E \)-invariant Borel set for which there is an \( (E \upharpoonright B) \)-complete \( (E \cap F) \)-invariant Borel partial quasi-transversal of \( E \cap F \) over \( E \cap F \cap F_\perp \), then \( B \) is a Borel partial quasi-transversal of \( E \) over \( F_\perp \), so \( E \upharpoonright B \) is smooth.

By \textit{[Mil20, Theorem 2.6]} and Theorem 1.7, we can therefore assume that there is a suitable sequence \((F_n)_{n \in \mathbb{N}}\) and a continuous homomorphism \( \phi : 2^\mathbb{N} \times \mathbb{N} \to X \) from \((F^* \setminus (\Delta(2^\mathbb{N}) \times \Delta(\mathbb{N})), (E_0 \times I(\mathbb{N})) \setminus F^*)\) to \(((E \cap F) \setminus F_\perp, E \setminus (F \cup F_\perp))\) such that \( \forall c \in 2^\mathbb{N} \ [\phi([c]_{E_0 \times \mathbb{N}})]_{E \cap F} \) is an \( E \)-class, where \( F^* = \bigcup_{n \in \mathbb{N}} F_n^* \). As Proposition 1.6 yields a clopen transversal \( U \subseteq 2^\mathbb{N} \times \mathbb{N} \) of \( F^* \), Proposition 1.3 gives rise to a continuous invariant embedding \( \chi : 2^\mathbb{N} \times \mathbb{N} \to U \) of \( E_0 \times I(\mathbb{N}) \) into \((E_0 \times I(\mathbb{N})) \setminus U\), in which case \( \phi \circ \chi \) is a continuous homomorphism from \((E_0 \times I(\mathbb{N})) \setminus (\Delta(2^\mathbb{N}) \times \Delta(\mathbb{N}))\) to \( E \setminus (F \cup F_\perp)\) with the property that \( \forall c \in 2^\mathbb{N} \ [([\phi \circ \chi]([c]_{E_0 \times \mathbb{N}})]_{E \cap F} \) is an \( E \)-class. As Proposition 1.1 ensures that the preimages \( E' \) and \( F' \) of \( E \) and \( F \) under \((\phi \circ \chi) \times (\phi \circ \chi) \) are meager, Proposition 1.4 yields a continuous injective homomorphism \( \psi : 2^\mathbb{N} \times \mathbb{N} \to 2^\mathbb{N} \times \mathbb{N} \) from \((E_0 \times I(\mathbb{N}), \sim(E_0 \times I(\mathbb{N})))\) to \((E_0 \times I(\mathbb{N}), \sim(E' \cup F'))\) with the property that \( \forall c \in 2^\mathbb{N} \ [\psi([c]_{E_0 \times \mathbb{N}})]_{E \cap F} \) is an \((E_0 \times I(\mathbb{N}))\)-class. Define \( \pi = \phi \circ \chi \circ \psi \).

2. Uniformizations

As a corollary of Theorem 1.13, we obtain the following:
Theorem 2.1. Suppose that $X$ and $Y$ are Polish spaces, $E$ is a Borel equivalence relation on $X$, $F$ is a countable Borel equivalence relation on $Y$, and $R \subseteq X \times Y$ is an $(E \times \Delta(Y))$-invariant Borel set whose vertical sections are contained in countable unions of $F$-classes. Then exactly one of the following holds:

1. There is a partition $(B_n)_{n \in \mathbb{N}}$ of $\text{proj}_X(R)$ into $E$-invariant Borel sets such that for all $n \in \mathbb{N}$, there is an $((E \times F) \restriction R)$-invariant Borel quasi-uniformization of $R \cap (B_n \times Y)$.

2. There are continuous embeddings $\pi_X : 2^\mathbb{N} \times \mathbb{N} \hookrightarrow X$ of $\mathbb{E}_0 \times I(\mathbb{N})$ into $E$ and $\pi_Y : 2^\mathbb{N} \times \mathbb{N} \hookrightarrow Y$ of $\Delta(2^\mathbb{N}) \times \Delta(\mathbb{N})$ into $F$ such that $R \cap (\pi_X(2^\mathbb{N}) \times Y) = [\{\pi_X \times \pi_Y \}([\mathbb{E}_0 \times I(\mathbb{N})])_{(\Delta(X) \times F)\restriction_R}$.

Proof. To see that conditions (1) and (2) are mutually exclusive, note that if both hold, then there is an $(\mathbb{E}_0 \times I(\mathbb{N}))$-complete Borel set whose intersection with each $(\mathbb{E}_0 \times I(\mathbb{N}))$-class is finite, contradicting Proposition \ref{prop:finite-class}.

Suppose now that condition (1) fails. Then Theorem \ref{thm:projection} yields a continuous embedding $\pi : 2^\mathbb{N} \hookrightarrow R$ of $(\mathbb{E}_0 \times I(\mathbb{N}), \Delta(2^\mathbb{N}) \times \Delta(\mathbb{N}))$ into $(E \times I(Y), (I(X) \times F) \cup (\Delta(X) \times I(Y)))$ for which $[\pi(2^\mathbb{N} \times \mathbb{N})]_{(E \times F)\restriction_R}$ is $((E \times I(Y)) \restriction R)$-invariant. Set $\pi_X = \text{proj}_X \circ \pi$ and $\pi_Y = \text{proj}_Y \circ \pi$. \(\Box\)

As a corollary, we obtain the following generalization of Theorem 2.

Theorem 2.2. Suppose that $X$ and $Y$ are Polish spaces, $E$ is a Borel equivalence relation on $X$, $F$ is a smooth countable Borel equivalence relation on $Y$, and $R \subseteq X \times Y$ is an $(E \times \Delta(Y))$-invariant Borel set whose vertical sections are contained in countable unions of $F$-classes. Then exactly one of the following holds:

1. There is an $((E \times F) \restriction R)$-invariant Borel uniformization of $R$ over $F$.

2. There are continuous embeddings $\pi_X : 2^\mathbb{N} \times \mathbb{N} \hookrightarrow X$ of $\mathbb{E}_0 \times I(\mathbb{N})$ into $E$ and $\pi_Y : 2^\mathbb{N} \times \mathbb{N} \hookrightarrow Y$ of $\Delta(2^\mathbb{N}) \times \Delta(\mathbb{N})$ into $F$ such that $R \cap (\pi_X(2^\mathbb{N}) \times Y) = [\{\pi_X \times \pi_Y \}([\mathbb{E}_0 \times I(\mathbb{N})])_{(\Delta(X) \times F)\restriction_R}$.

Proof. By Theorem \ref{thm:main} it is sufficient to show that if every vertical section of $R$ is contained in a union of finitely-many $F$-classes, then there is a Borel uniformization of $R$. But this is a straightforward consequence of the original Lusin-Novikov uniformization theorem. \(\Box\)

References

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B.D. MILLER


Benjamin D. Miller, Universität Wien, Augasse 2–6, 1090 Wien, Austria

E-mail address: benjamin.miller@univie.ac.at

URL: [http://www.logic.univie.ac.at/benjamin.miller](http://www.logic.univie.ac.at/benjamin.miller)