COORDINATEWISE DECOMPOSITION OF
GROUP-VALUED BOREL FUNCTIONS

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Abstract. Answering a question of Klopotowski-Nadkarni-Sarbadhikari-Srivastava [6], we characterize the Borel sets \( S \subseteq X \times Y \) on which every Borel function \( f : S \to C \) is of the form \( uv|S \), where \( u : X \to C \) and \( v : Y \to C \) are Borel.

Suppose that \( S \subseteq X \times Y \) and \( \Gamma \) is a group. A coordinatewise decomposition of a function \( f : S \to \Gamma \) is a pair \((u, v)\), where \( u : X \to \Gamma \), \( v : Y \to \Gamma \), and \( \forall (x, y) \in S \ (f(x, y) = u(x)v(y)) \).

While our main goal here is to study coordinatewise decompositions in the descriptive set-theoretic context, we will first study the existence of coordinatewise decompositions without imposing any definability restrictions.

For the sake of notational convenience, we will assume that \( X \cap Y = \emptyset \). The graph associated with \( S \) is the graph on the set \( Z_S = X \cup Y \) given by \( G_S = S \cup S^{-1} \).

The following fact was proven essentially by Cowsik-Klopotowski-Nadkarni [1]:

**Proposition 1.** Suppose that \( X, Y \) are disjoint, \( S \subseteq X \times Y \), and \( \Gamma \) is a non-trivial group. Then the following are equivalent:

1. Every function \( f : S \to \Gamma \) admits a coordinatewise decomposition;
2. \( G_S \) is acyclic.

**Proof.** To see \( \neg(2) \Rightarrow \neg(1) \) note that, by reversing the roles of \( X \) and \( Y \) if necessary, we can assume that there is a proper cycle of the form \( x_0, y_0, x_1, y_1, \ldots, x_{n+1} = x_0 \) through \( G_S \). Fix \( \gamma_0 \in \Gamma \setminus \{1_\Gamma\} \), define \( f : S \to \Gamma \) by

\[
    f(x, y) = \begin{cases} 
        \gamma_0 & \text{if } (x, y) = (x_0, y_0), \\
        1_\Gamma & \text{otherwise},
    \end{cases}
\]

and suppose that \((u, v)\) is a coordinatewise decomposition of \( f \). Then

\[
    \gamma_0 = f(x_0, y_0)f(x_1, y_0)^{-1} \cdots f(x_n, y_n)f(x_{n+1}, y_n)^{-1}
    = (u(x_0)v(y_0))(u(x_1)v(y_0))^{-1} \cdots (u(x_n)v(y_n))(u(x_{n+1})v(y_n))^{-1}
    = u(x_0)u(x_1)^{-1} \cdots u(x_n)u(x_{n+1})^{-1}
    = u(x_0)u(x_{n+1})^{-1}
    = 1_\Gamma,
\]

which contradicts our choice of \( \gamma_0 \).
To see \((2) \Rightarrow (1)\), let \(E_S\) be the equivalence relation whose classes are the connected components of \(G_S\), fix a transversal \(B \subseteq Z_S\) of \(E_S\) (i.e., a set which intersects every \(E_S\)-class in exactly one point), and define \(B_n \subseteq Z\) by
\[
B_n = \{ z \in Z : d_S(z, B) = n \},
\]
where \(d_S\) denotes the graph metric associated with \(G_S\). For \(z \in B_{n+1}\), let \(g(z)\) denote the unique \(G\)-neighbor of \(z\) in \(B_n\), and define recursively \(u : X \to \Gamma, v : Y \to \Gamma\) by
\[
u(y) = \begin{cases} 1_\Gamma & \text{if } y \in B, \\ u(g(y))^{-1}f(g(y), y) & \text{otherwise.} \end{cases}
\]

As a corollary of the proof of Proposition 1, we obtain a sufficient condition for the existence of Borel coordinatewise decompositions:

**Corollary 2.** Suppose that \(X\) and \(Y\) are Polish spaces, \(S \subseteq X \times Y\) is Borel, \(G_S\) is acyclic, and \(E_S\) admits a Borel transversal. Then every standard Borel group-valued Borel function on \(S\) admits a Borel coordinatewise decomposition.

**Proof.** It is sufficient to check that if \(f : S \to \Gamma\) is a standard Borel group-valued Borel function, then the functions \(u\) and \(v\) constructed in the proof of Proposition 1 are Borel. Letting \(B_n \subseteq Z_S\) and \(g : Z_S \to Z_S\) be as constructed above, it follows from the fact that \(G_S\) is acyclic that
\[
z \in B_{n+1} \iff z \notin \bigcup_{i \leq n} B_i \text{ and } \exists w \in B_n \ ((z, w) \in \mathcal{G})
\]
\[
\iff z \notin \bigcup_{i \leq n} B_i \text{ and } \exists ! w \in B_n \ ((z, w) \in \mathcal{G}),
\]
and it follows from results of Souslin and Lusin (see, for example, Theorems 14.11 and 18.11 of Kechris [5]) that each of these sets is Borel. As
\[
\operatorname{graph}(g) = \bigcup_{n \in \mathbb{N}} G_S \cap (B_{n+1} \times B_n),
\]
it follows that \(g\) is Borel as well (see, for example, Theorem 14.12 of Kechris [5]), and this easily implies that \(u\) and \(v\) are Borel. \(\square\)

Our main theorem is that the sufficient condition given in Corollary 2 is also necessary to guarantee the existence of Borel coordinatewise decompositions:
Theorem 3. Suppose that $X, Y$ are disjoint Polish spaces, $S \subseteq X \times Y$ is Borel, and $\Gamma$ is a non-trivial standard Borel group. Then the following are equivalent:

1. Every Borel function $f : S \to \Gamma$ admits a Borel coordinatewise decomposition;

2. $\mathcal{G}_S$ is acyclic and $E_S$ admits a Borel transversal.

Proof. As $(2) \Rightarrow (1)$ follows from Corollary 2, we need only show that $(1) \Rightarrow (2)$.

As the map $f$ described in the proof of $\neg (2) \Rightarrow \neg (1)$ of Proposition 1 is clearly Borel, it follows that $\mathcal{G}_S$ is acyclic, thus $E_S$ is Borel (by Theorems 14.11 and 18.11 of Kechris [5]).

Fix a non-trivial countable subgroup $\Delta \leq \Gamma$, endow $\Delta$ with the discrete topology, and endow $\Delta^\mathbb{N}$ with the corresponding product topology. Define $E_{\Delta}^0$ on $\Delta^\mathbb{N}$ by

$$\alpha n \in \mathbb{N} \forall m > n (\alpha(m) = \beta(m)),$$

and define $F_{\Delta}^0 \subseteq E_{\Delta}^0$ on $\Delta^\mathbb{N}$ by

$$\alpha n \in \mathbb{N} (\alpha(0) \cdots \alpha(n) = \beta(0) \cdots \beta(n) \text{ and } \forall m > n (\alpha(m) = \beta(m))).$$

Let $\Delta$ act freely on $\Delta^\mathbb{N}$ by left multiplication on the 0th-coordinate, i.e.,

$$\delta \cdot \alpha = (\delta \alpha(0), \alpha(1), \alpha(2), \ldots).$$

Lemma 4. The action of $\Delta$ on $\Delta^\mathbb{N}$ induces a free action of $\Delta$ on $\Delta^\mathbb{N}/F_{\Delta}^0$.

Proof. It is enough to show that

$$\forall \delta \in \Delta \forall \alpha, \beta \in \Delta^\mathbb{N} (\alpha F_{\delta}^0 \beta \Rightarrow \delta \cdot \alpha F_{\delta}^0 \delta \cdot \beta).$$

Towards this end, suppose that $\delta \in \Delta$ and $(\alpha, \beta) \in F_{\delta}^0$, fix $n \in \mathbb{N}$ such that

$$\alpha(0) \cdots \alpha(n) = \beta(0) \cdots \beta(n) \text{ and } \forall m > n (\alpha(m) = \beta(m)),$$

and note that

$$\delta \alpha(0) \cdots \alpha(n) = \delta \beta(0) \cdots \beta(n) \text{ and } \forall m > n (\alpha(m) = \beta(m)),$$

thus $\delta \cdot \alpha F_{\delta}^0 \delta \cdot \beta$. 

Suppose now that $F \subseteq E$ are Borel equivalence relations on a Polish space $Z$. We say that a set $B \subseteq Z$ is $F$-invariant if $\forall z_1 \in B \forall z_2 \in Z (z_1 F z_2 \Rightarrow z_2 \in B)$, and $B \subseteq Z$ is an $E$-complete section if $\forall z_1 \in Z \exists z_2 \in B (z_1 E z_2)$. We say that $E$ is relatively ergodic over $F$ if there is no Borel way of choosing a non-empty proper subset of the $F$-classes within each $E$-class, i.e., if there is no $F$-invariant Borel set $B \subseteq Z$ such that both $B$ and $Z \setminus B$ are $E$-complete sections.

Lemma 5. $E_{\delta}^0$ is relatively ergodic over $F_{\delta}^0$.

Proof. Suppose, towards a contradiction, that $B \subseteq \Delta^\mathbb{N}$ is an $F_{\delta}^0$-invariant Borel set such that both $B$ and $\Delta^\mathbb{N} \setminus B$ are $E_{\delta}^0$-complete sections. As $B$ is an $E_{\delta}^0$-complete
section, it follows that $B$ is non-meager, thus there exists $s \in \Delta^{\leq N}$ such that $B$ is comeager in $\mathcal{N}_s$. Define $C \subseteq \Delta^N$ by

$$C = \Delta^N \setminus [\mathcal{N}_s \setminus B]_{E_0^\alpha},$$

and observe that $C$ is an $E_0^\Delta$-invariant comeager Borel set and $\mathcal{N}_s \cap C \subseteq B \cap C$. It only remains to show that $C \subseteq B$, which implies that $\Delta^N \setminus B$ is meager and therefore contradicts the fact that $\Delta^N \setminus B$ is an $E_0^\Delta$-complete section. Towards this end, put $n = |s|$, and given any $\alpha \in C$, define $\delta \in \Delta$ by

$$\delta = (s(0) \cdots s(n-1))^{-1}(\alpha(0) \cdots \alpha(n)).$$

As $\alpha E_0^\Delta s(0) \cdots s n + 1)\delta \alpha(n + 1)\alpha(n + 2) \ldots$, it follows that $\alpha \in B$. \hfill \Box

Suppose now that $E_1$ and $E_2$ are Borel equivalence relations on Polish spaces $Z_1$ and $Z_2$, respectively. A reduction of $E_1$ into $E_2$ is a function $\pi : Z_1 \to Z_2$ such that $\forall z, z' \in Z_1 (z E_1 z' \iff \pi(z) E_2 \pi(z'))$. An embedding is an injective reduction. Let $E_0$ denote the equivalence relation on $2^N$ which is given by

$$\alpha E_0 \beta \iff \exists n \in \mathbb{N} \forall m > n (\alpha(m) = \beta(m)).$$

While our next lemma follows from the much more general results of Dougherty-Jackson-Kechris [2], it is easy enough to prove directly:

**Lemma 6.** There is a Borel embedding $\pi_1 : \Delta^N \to 2^N$ of $E_0^\Delta$ into $E_0$.

**Proof.** Fix an enumeration $(k_n, \delta_n)$ of $\mathbb{N} \times \Delta$, and define $\pi_1 : \Delta^N \to 2^N$ by

$$[\pi_1(\alpha)](n) = \begin{cases} 1 & \text{if } \alpha(k_n) = \delta_n, \\ 0 & \text{otherwise}. \end{cases}$$

It is straightforward to check that $\pi_1$ is the desired embedding. \hfill \Box

Now suppose, towards a contradiction, that $E_S$ has no Borel transversal.

**Lemma 7.** There is a Borel embedding $\pi_2 : 2^N \to Z_S$ of $E_0$ into $E_S|X$.

**Proof.** An equivalence relation $E$ on a Polish space $Z$ is said to be smooth if there is a Borel reduction of $E$ into the trivial equivalence relation $\Delta(\mathbb{R}) = \{(x, x) : x \in \mathbb{R}\}$, or equivalently, if $E$ admits a Borel separating family, i.e., a family $B_0, B_1, \ldots$ of Borel subsets of $Z$ such that

$$\forall z_1, z_2 \in Z (z_1 E z_2 \iff \forall n \in \mathbb{N} (z_1 \in B_n \iff z_2 \in B_n)).$$

Suppose, towards a contradiction, that there is no Borel embedding of $E_0$ into $E_S|X$. As $E_S$ is Borel, so too is $E_S|X$. It follows from Theorem 1.1 of Harrington-Kechris-Louveau [3] that $E_S|X$ is smooth. Fix a Borel separating family $B_0, B_1, \ldots$ for $E_S|X$, and observe that the sets

$$A_n = B_n \cup \{y \in Y : \exists x \in B_n ((x, y) \in S)\}$$

form a $\Sigma_1^1$ separating family for $E_S|\{X \cup \text{proj}_Y[S]\}$, where $\text{proj}_Y : X \times Y \to Y$ denotes the projection function. It then follows from Theorem 1.1 of Harrington-Kechris-Louveau [3] that $E_S$ is smooth. As $G_S$ is acyclic, it follows from Hjorth [4] (see also Miller [7]) that $E_S$ admits a Borel transversal, which contradicts our assumption that it does not. \hfill \Box
For $x_1 E_S x_2$, we say that $z$ is $G_S$-between $x_1$ and $x_2$ if $z$ lies along the unique injective $G_S$-path from $x_1$ to $x_2$. Define $B \subseteq Z_S$ by

$$B = \{ z \in Z_S : \exists x_1, x_2 \in \text{rng}(\pi_2 \circ \pi_1) (z \text{ is } G_S\text{-between } x_1 \text{ and } x_2) \}.$$  

As $G_S$ is acyclic and $\text{rng}(\pi_2 \circ \pi_1)$ intersects every $E_S$-class in a countable set, it follows that $B$ is Borel. As $E_S \cap (B \times \text{rng}(\pi_2 \circ \pi_1))$ has countable sections, the Lusin-Novikov uniformization theorem (see, for example, §18 of Kechris [5]) ensures that it has a Borel uniformization $\pi_3 : B \to \text{rng}(\pi_2 \circ \pi_1)$. We can clearly assume that $\pi_3|\text{rng}(\pi_2 \circ \pi_1) = \text{id}$. Define $\pi : B \to \Delta^N$ by

$$\pi = (\pi_2 \circ \pi_1)^{-1} \circ \pi_3,$$

and finally, define $f : S \to \Delta$ by

$$f(x, y) = \begin{cases} 1 \text{ if } x \notin B \text{ or } y \notin B, \text{ and} \\ \delta \text{ if } x, y \in B \text{ and } \delta \cdot \pi(y)F_0^\Delta \pi(x). \end{cases}$$

Now suppose, towards a contradiction, that there is a Borel coordinatewise decomposition $(u, v)$ of $f$.

**Lemma 8.** Suppose that $x, x' \in B \cap X$ and $xE_Sx'$. Then:

1. $u(x)u(x')^{-1} \in \Delta$.
2. $u(x)u(x')^{-1} \cdot \pi(x')F_0^\Delta \pi(x)$.

**Proof.** Let $x_0, y_0, \ldots, x_n, y_n, x_{n+1}$ be the unique $G_S$-path from $x$ to $x'$. To see (1), observe that for all $i \leq n$,

$$u(x_i)u(x_{i+1})^{-1} = (u(x_i)v(y_i))(u(x_{i+1})v(y_i))^{-1} = f(x_i, y_i)f(x_{i+1}, y_i)^{-1},$$

thus $u(x_i)u(x_{i+1})^{-1} \in \Delta$. Noting that

$$u(x_0)u(x_{n+1})^{-1} = u(x_0)u(x_1)^{-1}u(x_1)u(x_2)^{-1} \cdots u(x_n)u(x_{n+1})^{-1},$$

it follows that $u(x)u(x')^{-1} \in \Delta$.

To see (2), recall that $\Delta$ acts freely on $\Delta^N/F_0^\Delta$, thus for all $i \leq n$,

$$u(x_i)u(x_{i+1})^{-1} \cdot [\pi(x_{i+1})]_{F_0^\Delta} = f(x_i, y_i)f(x_{i+1}, y_i)^{-1} \cdot [\pi(x_{i+1})]_{F_0^\Delta} = f(x_i, y_i) \cdot [\pi(y_i)]_{F_0^\Delta} = [\pi(x_i)]_{F_0^\Delta}.$$  

It then follows that

$$u(x_0)u(x_{n+1})^{-1} \cdot [\pi(x_{n+1})]_{F_0^\Delta} = u(x_0)u(x_1)^{-1} \cdots u(x_n)u(x_{n+1})^{-1} \cdot [\pi(x_{n+1})]_{F_0^\Delta} = u(x_0)u(x_1)^{-1} \cdots u(x_{n-1})u(x_n)^{-1} \cdot [\pi(x_n)]_{F_0^\Delta} \cdots = [\pi(x_0)]_{F_0^\Delta},$$

which completes the proof of the lemma. \qed
Define now \( w : \Delta^N \to \Gamma \) by \( w = u \circ \pi_2 \circ \pi_1 \). Fix a countable Borel separating family \( \Gamma_0, \Gamma_1, \ldots \) for \( \Gamma \), and define \( n : \Delta^N \to \Gamma \) by
\[
\begin{align*}
n(\alpha) &= \min \{ n \in \mathbb{N} : \exists \delta_1, \delta_2 \in \Delta \ (\delta_1 w(\alpha) \in \Gamma_n \text{ and } \delta_2 w(\alpha) \notin \Gamma_n) \}.
\end{align*}
\]
Lemma 8 ensures that if \( \alpha E_0^\Delta \beta \), then \( w(\alpha)w(\beta)^{-1} \in \Delta \), thus
\[
\begin{align*}
\Delta w(\alpha) &= \Delta w(\alpha)w(\beta)^{-1}w(\beta) \\
&= \Delta w(\beta),
\end{align*}
\]
and it follows that \( n(\alpha) = n(\beta) \). As \( \pi_3 \mid \text{rng}(\pi_2 \circ \pi_1) = \text{id} \), Lemma 8 ensures also that \( w(\alpha)w(\beta)^{-1} \circ F_0^\Delta \alpha \). It follows that if \( \alpha = \delta \cdot \beta \), then \( w(\alpha)w(\beta)^{-1} = \delta \), thus \( w(\alpha) = \delta w(\beta) \). Defining \( A \subseteq \Delta^N \) by
\[
A = \{ \alpha \in \Delta^N : w(\alpha) \in \Gamma_n(x) \},
\]
it follows that both \( A \) and \( \Delta^N \setminus A \) are \( E_0^\Delta \)-complete sections. As \( A \) is clearly \( F_0^\Delta \)-invariant, it follows that \( E_0^\Delta \) is not relatively ergodic over \( F_0^\Delta \), which contradicts Lemma 5, and therefore completes the proof of the theorem. \( \square \)

Klopotowski-Nadkarni-Sarbadhikari-Srivastava [6] have studied coordinatewise decomposition using another equivalence relation \( L \) which, modulo straightforward identifications, is the equivalence relation whose classes are the connected components of the dual graph \( \tilde{G}_S \) on \( S \), which is given by
\[
\tilde{G}_S = \{ ((x_1, y_1), (x_2, y_2)) \in S \times S : (x_1, y_1) \neq (x_2, y_2) \text{ and } (x_1 = x_2 \text{ or } y_1 = y_2) \}.
\]
The equivalence classes of \( L \) are called the \textit{linked components} of \( S \), and the linked components of \( S \) are said to be \textit{uniquely linked} if \( G_S \) is acyclic.

**Conjecture 9 (Klopotowski-Nadkarni-Sarbadhikari-Srivastava).** Suppose that \( X, Y \) are disjoint Polish spaces and \( S \subseteq X \times Y \) is Borel. Then the following are equivalent:

1. Every Borel function \( f : S \to \mathbb{C} \) admits a Borel coordinatewise decomposition;
2. The linked components of \( S \) are uniquely linked and \( L \) is smooth.

In light of Theorem 3 and the above remarks, the following observation implies that Conjecture 9 is indeed correct:

**Proposition 10.** Suppose that \( X \) and \( Y \) are disjoint Polish spaces, \( S \subseteq X \times Y \) is Borel, and \( G_S \) is acyclic. Then the following are equivalent:

1. \( E_S \) admits a Borel transversal;
2. \( L \) is smooth.

**Proof.** To see (1) \( \Rightarrow \) (2), suppose that \( E_S \) admits a Borel transversal \( B \subseteq Z_S \). Let \( \pi_1 : Z_S \to Z_S \) be the function which sends \( z \) to the unique element of \( B \cap \lfloor z \rfloor_{E_S} \), and let \( \pi_2 = \text{proj}_X |S \). Then \( \pi_1 \) is a Borel reduction of \( E_S \) into \( \Delta(Z_S) \) and \( \pi_2 \) is a Borel reduction of \( L \) into \( E_S \), thus \( \pi_1 \circ \pi_2 \) is a Borel reduction of \( L \) into \( \Delta(Z_S) \), so \( L \) is smooth.
To see (2) ⇒ (1), suppose that $L$ is smooth, and fix a Borel reduction $\pi_1 : S \to \mathbb{R}$ of $L$ into $\Delta(\mathbb{R})$. Put $Z = \text{proj}_X[S] \cup \text{proj}_Y[S]$. By the Jankov-von Neumann uniformization theorem (see, for example, §18 of Kechris [5]), there is a $\sigma(\Sigma_1^1)$-measurable reduction $\pi_2 : Z \to S$ of $E_S|Z$ into $L$, thus $\pi_1 \circ \pi_2$ is a $\sigma(\Sigma_1^1)$-measurable reduction of $E_S|Z$ into $\Delta(\mathbb{R})$. It then follows from Theorem 1.1 of Harrington-Kechris-Louveau [3] that $E_S$ is smooth. As $G_S$ is acyclic, it then follows from Hjorth [4] (see also Miller [7]) that $E_S$ admits a Borel transversal.

\[\square\]

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