# THE DUAL SCHRÖDER-BERNSTEIN THEOREM AND BOREL EQUIVALENCE RELATIONS

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Abstract. At the request of Andrés Caicedo, we describe how ideas from the study of Borel equivalence relations can be used to establish the consistency of the failure of the (weak) dual Schröder-Bernstein theorem from Con($\mathtt{ZF}$).

The *Schröder-Bernstein theorem* ($\mathtt{SBT}$) is the statement that for all sets $X$ and $Y$, if there are injections from each of $X$ and $Y$ into the other, then there is a bijection from $X$ to $Y$. Of course, it is well known that $\mathtt{ZF} \Rightarrow \mathtt{SBT}$. The *dual Schröder-Bernstein theorem* ($\mathtt{DSBT}$) is the statement that for all sets $X$ and $Y$, if there are surjections from each of $X$ and $Y$ onto the other, then there is a bijection from $X$ to $Y$. Clearly $\mathtt{ZFC} \Rightarrow \mathtt{DSBT}$, and it is known that $\mathtt{ZF} + \mathtt{DC} \not\Rightarrow \mathtt{DSBT}$. The *weak dual Schröder-Bernstein theorem* ($\mathtt{WDSBT}$) is the statement that for all sets $X$ and $Y$, if there are surjections from each of $X$ and $Y$ onto the other, then there is an injection from at least one of $X$ or $Y$ into the other.

Let $\mathtt{BP}$ abbreviate the statement that all subsets of $2^\omega$ have the Baire property. Let $\mathtt{CU}$ abbreviate the statement that for all sets $R \subseteq 2^\omega \times 2^\omega$ with full projections, there is a comeager set $C \subseteq 2^\omega$ and a function $f : C \to 2^\omega$ whose graph is contained in $R$. Shelah has shown that $\mathrm{Con}(\mathtt{ZF}) \Rightarrow \mathrm{Con}(\mathtt{ZF} + \mathtt{DC} + \mathtt{BP} + \mathtt{CU})$. We will show that $\mathtt{ZF} + \mathtt{BP} \Rightarrow \neg\mathtt{DSBT}$ and $\mathtt{ZF} + \mathtt{BP} + \mathtt{CU} \Rightarrow \neg\mathtt{WDSBT}$.

We begin with an observation that suggests Borel equivalence relations on Polish spaces as a natural place to look for counterexamples to $\mathtt{DSBT}$.

**Proposition 1.** *Suppose that $X$ is a Polish space and $E$ is a Borel equivalence relation on $X$ with uncountably many equivalence classes. Then there is a surjective Borel homomorphism from $E$ to $\Delta(2^\omega)$. In particular, if $Y$ is a Polish space and $F$ is a Borel equivalence relation on $Y$, then there is a surjective Borel homomorphism from $E$ to $F$, so there is a surjection from $X/E$ onto $Y/F$.*

*Proof.* By Silver's theorem, there is a continuous embedding $\phi : 2^\omega \to X$ of $\Delta(2^\omega)$ into $E$. Then the set

$$[\phi(2^\omega)]_E = \{x \in X : \exists y \in 2^\omega \ (xE\phi(y))\} = \{x \in X : \exists! y \in 2^\omega \ (xE\phi(y))\}$$

is Borel, as is the function $\psi : [\phi(2^\omega)]_E \to 2^\omega$ given by $\psi(x) = y \Leftrightarrow xE\phi(y)$. Fix $y_0 \in 2^\omega$, and define $\pi : X \to 2^\omega$ by

$$\pi(x) = \begin{cases} \psi(x) & \text{if } x \in [\phi(2^\omega)]_E, \\ y_0 & \text{otherwise.} \end{cases}$$

Then $\pi$ is a surjective Borel homomorphism from $E$ to $\Delta(2^\omega)$. The desired surjective Borel homomorphism from $E$ to $F$ can be obtained by composing $\pi$ with any Borel

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surjection from $2^\omega$ onto $Y$, and the resulting map then factors over the quotients to give the desired surjection from $X/E$ onto $Y/F$. ⊠

Define $E_0$ on $2^\omega$ by $xE_0y \Leftrightarrow \exists n < \omega \forall m \geq n\ (x(m) = y(m))$. It is well known that there is no Baire measurable reduction of $E_0$ to $\Delta(2^\omega)$.

**Theorem 2** ($\mathtt{ZF} + \mathtt{BP}$)**.** *There are surjections from each of $2^\omega$ and $2^\omega/E_0$ onto the other, but there is no injection from $2^\omega/E_0$ into $2^\omega$. In particular, it follows that the dual Schröder-Bernstein theorem is false.*

*Proof.* Proposition 1 implies that there are surjections from each of $2^\omega$ and $2^\omega/E_0$ onto the other (this can also be easily established directly).

To see that there is no injection from $2^\omega/E_0$ into $2^\omega$, simply observe that otherwise there is a reduction of $E_0$ to $\Delta(2^\omega)$, so BP ensures that there is a Baire measurable reduction of $E_0$ to $\Delta(2^\omega)$, a contradiction. ⊠

On the other hand, it is easy to see that there is an injection from $2^\omega$ into $2^\omega/E_0$. Moreover, results of Harrington-Kechris-Louveau and Hjorth-Kechris imply that if $X$ and $Y$ are Polish spaces and $E$ and $F$ are Borel equivalence relations on $X$ and $Y$, at least one of which is countable, then after reversing the roles of $(X, E)$ and $(Y, F)$ if necessary, there is a dense, $G_\delta$ set $C \subseteq X$ for which there is a continuous reduction of $E \restriction C$ to $F$, thus there is an injection from $C/E$ into $Y/F$. So if we wish to obtain an analogous counterexample to WDSBT, we must leave the realm of countable Borel equivalence relations. (There are similar counterexamples in the countable case if we replace Baire category with Lebesgue measure, but the proof seems to require sophisticated ergodic-theoretic arguments, and the consistency strength of the corresponding set of axioms goes beyond that of ZF.)

It will be convenient to think of $E_0^\omega$ as an equivalence relation on $2^{\omega\times\omega}$.

**Proposition 3.** *Suppose that $Y$ is a Polish space, $F$ is an $F_\sigma$ equivalence relation on $Y$, $\pi \colon 2^{\omega\times\omega} \to Y$ is Baire measurable, and $C \subseteq 2^{\omega\times\omega}$ is comeager. Then $\pi \restriction C$ is not a reduction of $E_0^\omega \restriction C$ to $F$.*

*Proof.* We can assume, without loss of generality, that $(x, y) \notin E_0^\omega \Rightarrow (\pi(x), \pi(y)) \notin F$, for all $x, y \in C$. By refining $C$ if necessary, we can also assume that $C$ is $G_\delta$ and $\pi \restriction C$ is continuous. Fix dense, open sets $U_n \subseteq 2^{\omega\times\omega}$ such that $C = \bigcap_{n<\omega} U_n$, as well as open sets $V_n \subseteq Y \times Y$ with $(Y \times Y) \setminus F = \bigcap_{n<\omega} V_n$. For $s, t \in 2^{\leq\omega\times\omega}$, set $\delta(s, t) = \{(i, j) \in \mathrm{dom}(s) \cap \mathrm{dom}(t) \colon s(i, j) \neq t(i, j)\}$.

Set $s_0 = t_0 = \emptyset$. Given $s_n, t_n \in 2^{<\omega\times\omega}$, fix $(u_n, v_n) \sqsupseteq (s_n, t_n)$ with $n \times n \subseteq \mathrm{dom}(u_n) = \mathrm{dom}(v_n)$, $\delta(s_n, t_n) = \delta(u_n, v_n)$, and $\mathcal{N}_{u_n} \cup \mathcal{N}_{v_n} \subseteq U_n$. Let $\phi$ denote the homeomorphism from $\mathcal{N}_{u_n}$ to $\mathcal{N}_{v_n}$ given by

$$[\phi(x)](i, j) = \begin{cases} v_n(i, j) & \text{if } (i, j) \in \mathrm{dom}(u_n), \\ x(i, j) & \text{if } (i, j) \notin \mathrm{dom}(u_n) \text{ and } i < n, \\ 1 - x(i, j) & \text{otherwise.} \end{cases}$$

Then $(C \cap \mathcal{N}_{u_n}) \cap \phi^{-1}(C \cap \mathcal{N}_{v_n})$ is comeager, so there exists $x_n \in (C \cap \mathcal{N}_{u_n}) \cap \phi^{-1}(C \cap \mathcal{N}_{v_n})$. Set $y_n = \phi(x_n)$. Then $\delta(s_n, t_n) \cap (n \times \omega) = \delta(u_n, v_n) \cap (n \times \omega) = \delta(x_n, y_n) \cap (n \times \omega)$ and $(x_n, y_n) \notin E_0^\omega$, so $(\pi(x_n), \pi(y_n)) \notin F$, thus there is an initial segment $(s_{n+1}, t_{n+1}) \sqsupseteq (u_n, v_n)$ of $(x_n, y_n)$ with $(\pi(x), \pi(y)) \in V_n$, for all $x \in C \cap \mathcal{N}_{s_{n+1}}$ and $y \in C \cap \mathcal{N}_{t_{n+1}}$.

Set $x = \lim_{n\to\infty} s_n$ and $y = \lim_{n\to\infty} t_n$, and note that $x, y \in C$, $xE_0^\omega y$, and $(\pi(x), \pi(y)) \notin F$, thus $\pi$ is not a reduction of $E_0^\omega \restriction C$ to $F$. ⊠

Given a topological space $X$ and a point $x \in X$, let $\tau_X(x)$ denote the family of all open neighborhoods of $x$.

**Proposition 4** (Hjorth)**.** *Suppose that $G$ and $H$ are Polish groups, $X$ is a Polish $G$-space, $Y$ is a Polish $H$-space, $\pi \colon X \to Y$ is Baire measurable, $C \subseteq X$ is comeager, $\pi \restriction C$ is a homomorphism from $E_G \restriction C$ to $E_H$, and $V$ is an open neighborhood of $1_H$. Then $\forall^* x \in X \exists U \in \tau_G(1_G) \forall^* g \in U$ $(x, g \cdot x \in C$ and $\pi(g \cdot x) \in V \cdot \pi(x))$.*

*Proof.* Fix an open neighborhood $W$ of $1_H$ such that $WW^{-1} \subseteq V^{-1}$, as well as a countable, dense set $D \subseteq H$. For each $h \in D$ and $x \in C$, define $G_{h,x} \subseteq G$ by

$$G_{h,x} = \{g \in G \colon g \cdot x \in C \text{ and } \pi(g \cdot x) \in Wh \cdot \pi(x)\}.$$

**Lemma 5.** *Suppose that $g_0, g_1 \in G_{h,x}$. Then $\pi(g_0 \cdot x) \in V^{-1} \cdot \pi(g_1 \cdot x)$.*

*Proof.* For each $i < 2$, fix $w_i \in W$ with $\pi(g_i \cdot x) = w_i h \cdot \pi(x)$. Then $\pi(g_0 \cdot x) = w_0 h (w_1 h)^{-1} \cdot \pi(g_1 \cdot x) = w_0 w_1^{-1} \cdot \pi(g_1 \cdot x) \in WW^{-1} \cdot \pi(g_1 \cdot x) \subseteq V^{-1} \cdot \pi(g_1 \cdot x)$. ⊠

Observe now that $\forall g \in G \forall^* x \in X$ $(g \cdot x \in C)$, so the Kuratowski-Ulam theorem implies that $\forall^* x \in X \forall^* g \in G$ $(g \cdot x \in C)$. It follows that for comeagerly many $x \in X$, the set $\bigcup_{h \in D} G_{h,x}$ is comeager, so there are open sets $U_{h,x} \subseteq G$ such that $G_{h,x}$ is comeager in $U_{h,x}$ and $\bigcup_{h \in D} U_{h,x}$ is comeager, thus

$$\begin{aligned}&\forall^* g_0 \in U_{h,x} \forall^* g_1 \in U_{h,x} g_0^{-1} \\ &\quad (g_0 \cdot x, g_1 \cdot (g_0 \cdot x) \in C \text{ and } \pi(g_0 \cdot x) \in V^{-1} \cdot \pi(g_1 \cdot (g_0 \cdot x))).\end{aligned}$$

Since $g_0 \in U_{h,x} \Rightarrow 1_G \in U_{h,x} g_0^{-1}$, it follows that

$$\begin{aligned}&\forall^* x \in X \forall^* g_0 \in G \exists U \in \tau_G(1_G) \forall^* g_1 \in U \\ &\quad (g_0 \cdot x, g_1 \cdot (g_0 \cdot x) \in C \text{ and } \pi(g_0 \cdot x) \in V^{-1} \cdot \pi(g_1 \cdot (g_0 \cdot x))).\end{aligned}$$

The Kuratowski-Ulam theorem then implies that

$$\begin{aligned}&\forall^* g_0 \in G \forall^* x \in X \exists U \in \tau_G(1_G) \forall^* g_1 \in U \\ &\quad (g_0 \cdot x, g_1 \cdot (g_0 \cdot x) \in C \text{ and } \pi(g_0 \cdot x) \in V^{-1} \cdot \pi(g_1 \cdot (g_0 \cdot x))).\end{aligned}$$

Fix $g_0 \in G$ such that the set

$$\begin{aligned}B = \{&x \in X \colon \exists U \in \tau_G(1_G) \forall^* g_1 \in U \\ &(g_0 \cdot x, g_1 \cdot (g_0 \cdot x) \in C \text{ and } \pi(g_0 \cdot x) \in V^{-1} \cdot \pi(g_1 \cdot (g_0 \cdot x)))\}\end{aligned}$$

is comeager. Then so too is the set

$$g_0(B) = \{x \in X \colon \exists U \in \tau_G(1_G) \forall^* g \in U \ (x, g \cdot x \in C \text{ and } \pi(x) \in V^{-1} \cdot \pi(g \cdot x))\},$$

thus $\forall^* x \in X \exists U \in \tau_G(1_G) \forall^* g \in U$ $(x, g \cdot x \in C$ and $\pi(g \cdot x) \in V \cdot \pi(x))$. ⊠

Define $E_1$ on $(2^\omega)^\omega$ by $x E_1 y \Leftrightarrow \exists n < \omega \forall m \geq n$ $(x(m) = y(m))$. It will be convenient to think of $E_1$ as an equivalence relation on $2^{\omega \times \omega}$.

**Theorem 6** (Kechris-Louveau)**.** *Suppose that $H$ is a Polish group, $Y$ is a Polish $H$-space, $\pi \colon 2^{\omega \times \omega} \to Y$ is Baire measurable, and $C \subseteq 2^{\omega \times \omega}$ is comeager. Then $\pi \restriction C$ is not a reduction of $E_1 \restriction C$ to $E_H$.*

*Proof (Hjorth).* Let the Polish group $G = ((\mathbb{Z}/2\mathbb{Z})^\omega)^\omega$ act on $2^{\omega \times \omega}$ by

$$[g \cdot x](i,j) = \begin{cases} x(i,j) & \text{if } [g(i)](j) \equiv 0 \pmod 2, \\ 1 - x(i,j) & \text{otherwise.} \end{cases}$$

Set $G_n = \{g \in G \colon \forall i \geq n \forall j < \omega \ ([g(i)](j) \equiv 0 \pmod 2)\}$. Then $E_1 = \bigcup_{n<\omega} E_{G_n}$.

We can assume, without loss of generality, that $(x, y) \in E_1 \Rightarrow (\pi(x), \pi(y)) \in E_H$, for all $x, y \in C$. By refining $C$ if necessary, we can also assume that $C$ is $G_\delta$ and $\pi \restriction C$ is continuous. Fix dense, open sets $U_n \subseteq 2^{\omega\times\omega}$ such that $C = \bigcap_{n<\omega} U_n$. Proposition 4 ensures that the set $C_0$ of $x \in C$ for which

$$\forall n < \omega \forall V \in \tau_H(1_H) \exists U \in \tau_{G_n}(1_{G_n}) \forall^* g \in U \ (g \cdot x \in C \text{ and } \pi(g \cdot x) \in V \cdot \pi(x))$$

is comeager. The Kuratowski-Ulam theorem implies that each of the sets defined recursively by $C_{k+1} = \bigcap_{n<\omega} \{x \in C_k \colon \forall^* g \in G_n \ (g \cdot x \in C_k)\}$ is comeager, thus so too is the set $C_\infty = \bigcap_{k<\omega} C_k$. Note that $\forall n < \omega \forall x \in C_\infty \forall^* g \in G_n \ (g \cdot x \in C_\infty)$.

Fix $x \in C_\infty$ and a compatible, complete metric $d_H$ on $H$. We will recursively construct $s_n \in 2^{<\omega\times\omega}$, $g_n \in G$, and $h_n \in H$ such that for all $n < \omega$, the following conditions are satisfied:

(1) $g_n \cdots g_0 \cdot x \in C_\infty \cap \mathcal{N}_{s_n}$.
(2) $\pi(g_n \cdots g_0 \cdot x) = h_n \cdots h_0 \cdot \pi(x)$.
(3) $d_H(h_n \cdots h_0, h_{n+1} \cdots h_0) < 1/2^n$.
(4) $s_n \sqsubseteq s_{n+1}$.
(5) $n \times n \subseteq \mathrm{dom}(s_{n+1})$.
(6) $\mathcal{N}_{s_{n+1}} \subseteq U_n$.
(7) $\delta(s_{n+1}, x) \cap (\{n\} \times \omega) \neq \emptyset$.

We begin by setting $s_0 = \emptyset$, $g_0 = 1_G$, and $h_0 = 1_H$. Suppose now that we have already found $s_m$, $g_m$, and $h_m$, for $m \leq n$. Define $V \subseteq H$ by

$$V = \{h \in H \colon d_H(h_n \cdots h_0, h h_n \cdots h_0) < 1/2^n\}.$$

Fix an open neighborhood $U$ of $1_{G_{n+1}}$ such that

$$\forall^* g \in U \ (g g_n \cdots g_0 \cdot x \in C \text{ and } \pi(g g_n \cdots g_0 \cdot x) \in V \cdot \pi(g_n \cdots g_0 \cdot x)),$$

fix $g_{n+1} \in U$ with $g_{n+1} \cdots g_0 \cdot x \in C_\infty \cap \mathcal{N}_{s_n}$ and $\delta(g_{n+1} \cdots g_0 \cdot x, x) \cap (\{n\} \times \omega) \neq \emptyset$, fix $h_{n+1} \in V$ with $\pi(g_{n+1} \cdots g_0 \cdot x) = h_{n+1} \cdots h_0 \cdot \pi(x)$, and fix an initial segment $s_{n+1} \sqsupseteq s_n$ of $g_{n+1} \cdots g_0 \cdot x$ with $n \times n \subseteq \mathrm{dom}(s_{n+1})$, $\mathcal{N}_{s_{n+1}} \subseteq U_n$, and $\delta(s_{n+1}, x) \cap (\{n\} \times \omega) \neq \emptyset$. This completes the recursive construction.

Set $y = \lim_{n\to\infty} s_n$ and $h = \lim_{n\to\infty} h_n \cdots h_0$. Then $y \in C$, $(x, y) \notin E_1$, and $\pi(y) = h \cdot \pi(x)$, thus $\pi \restriction C$ is not a reduction of $E_1 \restriction C$ to $E_H$. ⊠

**Theorem 7** (ZF + BP + CU)**.** *There are surjections from each of $2^{\omega\times\omega}/E_0^\omega$ and $2^{\omega\times\omega}/E_1$ onto the other, but there is no injection from either of $2^{\omega\times\omega}/E_0^\omega$ or $2^{\omega\times\omega}/E_1$ into the other. In particular, it follows that the weak dual Schröder-Bernstein theorem is false.*

*Proof.* Proposition 1 implies that there are surjections from each of $2^{\omega\times\omega}/E_0^\omega$ and $2^{\omega\times\omega}/E_1$ onto the other (this can also be easily established directly).

Suppose, towards a contradiction, that there is an injection from $2^{\omega\times\omega}/E_0^\omega$ into $2^{\omega\times\omega}/E_1$. Then CU yields a comeager set $C \subseteq 2^{\omega\times\omega}$ and a reduction $\pi \colon C \to 2^{\omega\times\omega}$ of $E_0^\omega \restriction C$ to $E_1$. As $E_1$ is $F_\sigma$ and BP implies that $\pi$ is Baire measurable, this contradicts Proposition 3.

Suppose now, towards a contradiction, that there is an injection from $2^{\omega \times \omega}/E_1$ into $2^{\omega \times \omega}/E_0^\omega$. Then CU yields a comeager set $C \subseteq 2^{\omega \times \omega}$ and a reduction $\pi \colon C \to 2^{\omega \times \omega}$ of $E_1 \upharpoonright C$ to $E_0^\omega$. As $E_0^\omega$ is generated by a continuous action of $(\mathbb{Z}/2\mathbb{Z})^\omega$ and BP implies that $\pi$ is Baire measurable, this contradicts Theorem 6. ⊠