THE RATIO ERGODIC THEOREM

BENJAMIN D. MILLER AND ANUSH TSERUNYAN

ABSTRACT. We use a strengthening of the marker lemma to prove Dowker's ratio ergodic theorem.

INTRODUCTION

A function $\rho \colon R \to (0, \infty)$ on a transitive binary relation is a *cocycle* if $\rho(x, z) = \rho(x, y)\rho(y, z)$ for all $x \mathrel{R} y \mathrel{R} z$. The *orbit relation* induced by a function $T \colon X \to X$ is the transitive binary relation on X given by $x \mathrel{R_T^X} y \iff \exists n \in \mathbb{N}\ T^n(x) = y$, and a set $Y \subseteq X$ is T-*wandering* if $T^{-m}(B) \cap T^{-n}(B) = \emptyset$ for all distinct $m, n \in \mathbb{N}$. A Borel measure μ on a Borel space X is T-*quasi-invariant* if $\mu \sim T_* \mu$, and ρ-*invariant* if

$$\forall f \in L^1(\mu) \forall n \in \mathbb{N}\ \int f \ d\mu = \int \ (f \circ T^n)(x)\rho(x, T^n(x)) \ d\mu(x),$$

where $L^1(\mu)$ denotes the set of all Borel functions $f \colon X \to \mathbb{R}$ for which $\int |f| \ d\mu < \infty$. If μ is σ-finite and T-quasi-invariant, then one obtains a Borel cocycle $\rho \colon R_T^X \to (0, \infty)$ with respect to which μ is ρ-invariant by considering Radon-Nikodym derivatives of the form $d\mu/d(T^n)_*\mu$, where $n \in \mathbb{Z}^+$. Given T and ρ, we associate with each function $f \colon X \to \mathbb{R}$ the functions $S_n(f, \rho, T) \colon X \to \mathbb{R}$ given by $S_n(f, \rho, T)(x) = \sum_{i<n}(f \circ T^i)(x)\rho(x, T^i(x))$ for all $n \in \mathbb{Z}^+$, and with each function $g \colon X \to (0, \infty)$ the functions $R_n(f, g, \rho, T) \colon X \to \mathbb{R}$ given by $R_n(f, g, \rho, T)(x) = S_n(f, \rho, T)(x)/S_n(g, \rho, T)(x)$ for all $n \in \mathbb{Z}^+$, as well as the partial function $R_\infty(f, g, \rho, T) \colon X \rightharpoonup [-\infty, \infty]$ given by $R_\infty(f, g, \rho, T)(x) = \lim_{n \to \infty} R_n(f, g, \rho, T)(x)$.

Here we combine the strategy underyling [Tse18] with a strengthening of the marker lemma to obtain a simple proof of the following:

Theorem 1 (Dowker [Dow50, Theorem II]). *Suppose that X is a Borel space, $T \colon X \to X$ is a Borel function, $\rho \colon R_T^X \to (0, \infty)$ is a Borel cocycle, μ is a ρ-invariant σ-finite Borel measure on X that concentrates off of T-wandering Borel sets, $f \in L^1(\mu)$, and $g \colon X \to (0, \infty)$*

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is a μ-integrable Borel function such that $S_n(g,\rho,T)(x) \to \infty$ for all $x \in X$. Then $\int f \, d\mu = \int R_\infty(f,g,\rho,T)g \, d\mu$.

1. Preliminaries

A function $T\colon X \to X$ is *aperiodic* if there do not exist $n \in \mathbb{Z}^+$ and $x \in X$ for which $T^n(x) = x$, and a set $Y \subseteq X$ is *T-invariant* if $Y = T^{-1}(Y)$. Given a binary relation R on a set X, a sequence $(X_n)_{n\in\mathbb{N}}$ *separates R-related points* if for all $x \; R \; y$, either $x = y$ or there exists $n \in \mathbb{N}$ for which $x \in X_n$ and $y \notin X_n$. We say that a Borel function $T\colon X \to X$ on a Borel space is *separable* if there is a sequence $(B_n)_{n\in\mathbb{N}}$ of Borel subsets of X that separates R_T^X-related points. A Borel measure μ on a standard Borel space X is *ccc* if every family of pairwise-disjoint μ-positive Borel subsets of X is countable. Every Borel function is aperiodic and separable, off of a set where it induces a periodic function on the corresponding measure algebra:

Proposition 1.1. *Suppose that X is a Borel space, $T\colon X \to X$ is a Borel function, and μ is a ccc Borel measure that concentrates off of T-wandering Borel sets. Then there are T-invariant Borel sets $B_n \subseteq X$ such that:*

1. *$\forall n \in \mathbb{N} \forall B \subseteq B_n$ Borel $\mu(B \bigtriangleup T^{-n}(B)) = 0$.*
2. *$T \upharpoonright \sim\bigcup_{n\in\mathbb{N}} B_n$ is aperiodic and separable.*

Proof. For all $n \in \mathbb{N}$, fix a maximal family $\mathcal{B}_n$ of μ-positive T-invariant pairwise-disjoint Borel sets $B \subseteq X$ such that $\mu(A \bigtriangleup T^{-n}(A)) = 0$ for all Borel sets $A \subseteq B$. As any such set is necessarily countable, the corresponding set $B_n = \bigcup \mathcal{B}_n$ is Borel. To see that the restriction of T to the set $C = \sim\bigcup_{n\in\mathbb{N}} B_n$ is aperiodic and separable off of a μ-null T-invariant Borel set, it is sufficient to show that for all $n \in \mathbb{Z}^+$ and μ-positive T-invariant Borel sets $B \subseteq C$, there is a μ-positive T-invariant Borel set $A \subseteq B$ for which there is a sequence $(A_i)_{i\in\mathbb{N}}$ of Borel subsets of B such that $A \subseteq \bigcup_{i\in\mathbb{N}} A_i \bigtriangleup T^{-n}(A_i)$, as a measure exhaustion argument then yields the desired result. Towards this end, fix a Borel set $A_0 \subseteq B$ for which the set $D = A_0 \bigtriangleup T^{-n}(A_0)$ is μ-positive. As $T^{-i}(D) \setminus \bigcup_{j>i} T^{-j}(D)$ is T-wandering for all $i \in \mathbb{N}$, the T-invariant set $A = \bigcap_{i\in\mathbb{N}} \bigcup_{j\geq i} T^{-j}(D)$ is μ-positive, and the sets $A_i = T^{-i}(A_0)$, where $i \in \mathbb{N}$, are as desired. ⊠

For all $x \in X$, the x^{th} *vertical section* of a set $R \subseteq X \times Y$ is given by $R_x = \{y \in Y \mid x \; R \; y\}$. Given a function $T\colon X \to X$, we say that a set $Y \subseteq X$ is *T-complete* if it intersects every vertical section of R_T^X. We say that a decreasing sequence $(X_n)_{n\in\mathbb{N}}$ is *vanishing* if

$\bigcap_{n \in \mathbb{N}} X_n = \emptyset$. We next note that the proof of the usual marker lemma for Borel automorphisms goes through for Borel functions:

Proposition 1.2 (Slaman–Steel [SS88, Lemma 1 of §3])**.** *Suppose that X is a Borel space and $T \colon X \to X$ is an aperiodic separable Borel function. Then there is a decreasing vanishing sequence $(B_n)_{n \in \mathbb{N}}$ of T-complete Borel subsets of X.*

Proof. Fix a sequence $(A_n)_{n \in \mathbb{N}}$ of Borel subsets of X that separates R_T^X-related points. For all $s \in 2^{<\mathbb{N}}$, set $A_s = \bigcap_{s(n)=0} A_n \cap \bigcap_{s(n)=1} \sim A_n$. For all $n \in \mathbb{N}$, let $s_n(x)$ be the lexicographically least sequence $s \in 2^n$ for which $x \in \bigcap_{i \in \mathbb{N}} \bigcup_{j \geq i} T^{-j}(A_s)$, and observe that the intersection of the set $B'_n = \bigcup_{s \in 2^n} A_s \cap s_n^{-1}(\{s\})$ with every vertical section of R_T^X is infinite. Moreover, the set $B' = \bigcap_{n \in \mathbb{N}} B'_n$ is T-wandering, for if $x \in B'$ and $i \in \mathbb{Z}^+$, then there exists $n \in \mathbb{N}$ for which $x \in A_n$ and $T^i(x) \notin A_n$, so the fact that $x \in B'_{n+1}$ ensures that $T^i(x) \notin B'_{n+1}$, thus $T^i(x) \notin B'$. It follows that the sets $B_n = B'_n \setminus B'$, where $n \in \mathbb{N}$, are as desired. ⊠

Given a function $T \colon X \to X$, define $T^{-\leq n}(Y) = \bigcup_{i \leq n} T^{-i}(Y)$ for all $n \in \mathbb{N}$. We say that a set $Y \subseteq X$ is *T-bounded* if there exists $n \in \mathbb{N}$ for which $X = T^{-\leq n}(Y)$.

Proposition 1.3. *Suppose that X is a Borel space and $T \colon X \to X$ is a Borel function for which there is a decreasing vanishing sequence $(A_n)_{n \in \mathbb{N}}$ of T-complete Borel sets. Then there is a decreasing vanishing sequence $(B_n)_{n \in \mathbb{N}}$ of T-bounded Borel sets.*

Proof. We can clearly assume that $A_0 = X$. For all $n \in \mathbb{N}$, define $B_n = A_n \cup \bigcup_{m<n} A_m \setminus T^{-\leq n}(A_{m+1})$.

To see that $B_{n+1} \subseteq B_n$, note that $A_{n+1}, A_n \setminus T^{-\leq n+1}(A_{n+1}) \subseteq A_n$ and $A_m \setminus T^{-\leq n+1}(A_{m+1}) \subseteq A_m \setminus T^{-\leq n}(A_{m+1})$ for all $m < n$. To see that $\bigcap_{n \in \mathbb{N}} B_n = \emptyset$, note that if $x \in A_m \setminus A_{m+1}$, then there exists $n > m$ for which $x \in T^{-\leq n}(A_{m+1})$, in which case $x \notin B_n$. To see that $X = T^{-\leq n^2}(B_n)$, note that if $m < n$, then $A_m \subseteq B_n \cup T^{-\leq n}(A_{m+1})$, so the obvious induction ensures that $A_m \subseteq T^{-\leq n(n-m)}(B_n)$. ⊠

Given a set $Y \subseteq X$, we use $\mathbb{1}_Y$ to denote the characteristic function of Y. Given a function $T \colon X \to X$ and a T-complete set $Y \subseteq X$, we define $n_Y \colon X \to \mathbb{Z}^+$ by $n_Y(x) = \min\{n \in \mathbb{Z}^+ \mid T^n(x) \in Y\}$. The analog of the following fact for the trivial cocycle and T-complete sets is well known (see [Kac47]):

Proposition 1.4. *Suppose that X is a Borel space, $T \colon X \to X$ is a Borel function, $\rho \colon R_T^X \to (0, \infty)$ is a Borel cocycle, μ is a ρ-invariant Borel measure on X, $B \subseteq X$ is a T-bounded Borel set, and $f \in L^1(\mu)$. Then $\int f \, d\mu = \int_B S_{n_B(x)}(f, \rho, T)(x) \, d\mu(x)$.*

Proof. For all $n \in \mathbb{N}$, set $B_n = \bigcup_{1 \leq i \leq n} T^{-i}(B)$, and observe that

$$\begin{aligned}\int_{\sim(B \cup B_n)} &(f \circ T^n)(x)\rho(x, T^n(x))\ d\mu(x) \\ &= \int_{\sim T^{-1}(B \cup B_n)} (f \circ T^{n+1})(x)\rho(T(x), T^{n+1}(x))\rho(x, T(x))\ d\mu(x) \\ &= \int_{\sim(B \cup B_{n+1})} (f \circ T^{n+1})(x)\rho(x, T^{n+1}(x))\ d\mu(x) \\ &\quad + \int_{B \setminus B_{n+1}} (f \circ T^{n+1})(x)\rho(x, T^{n+1}(x))\ d\mu(x).\end{aligned}$$

As $\int f\ d\mu = \int_{\sim(B \cup B_0)} f\ d\mu + \int_B f\ d\mu$, a straightforward inductive argument ensures that $\int f\ d\mu = \int_{\sim(B \cup B_n)} (f \circ T^n)(x)\rho(x, T^n(x))\ d\mu(x) + \sum_{i \leq n} \int_{B \setminus B_i} (f \circ T^i)(x)\rho(x, T^i(x))\ d\mu(x)$. As B is T-bounded, there exists $n \in \mathbb{N}$ for which $X = B \cup B_n$, so it only remains to note that

$$\begin{aligned}\int f\ d\mu &= \sum_{i \leq n} \int_{B \setminus B_i} (f \circ T^i)(x)\rho(x, T^i(x))\ d\mu(x) \\ &= \int_B \sum_{i \leq n} \mathbb{1}_{\sim B_i}(x)(f \circ T^i)(x)\rho(x, T^i(x))\ d\mu(x),\end{aligned}$$

and that for all $x \in B$, the latter integrand is $S_{n_B(x)}(f, \rho, T)(x)$. ⊠

2. Limits

Set $\mathbb{1} = \mathbb{1}_X$. The following well-known observation reduces Theorem 1 to the special case where $g = \mathbb{1}$ and μ is finite:

Proposition 2.1. *Suppose that X is a Borel space, $T\colon X \to X$ is a Borel function, $\rho\colon R_T^X \to (0, \infty)$ is a Borel cocycle, μ is a ρ-invariant Borel measure on X, $f \in L^1(\mu)$, $g\colon X \to (0, \infty)$ is a Borel function such that $S_n(g, \rho, T)(x) \to \infty$ for all $x \in X$, ν is the Borel measure on X given by $\nu(B) = \int_B g\ d\mu$, and $\sigma\colon R_T^X \to (0, \infty)$ is the cocycle given by $\sigma(x, T^n(x)) = \rho(x, T^n(x))(g \circ T^n)(x)/g(x)$. Then:*

1. *ν is σ-invariant.*
2. *$\forall x \in X\ S_n(\mathbb{1}, \sigma, T)(x) \to \infty$.*
3. *$\int f\ d\mu = \int f/g\ d\nu$.*
4. *$\int R_\infty(f/g, \mathbb{1}, \sigma, T)\ d\nu = \int R_\infty(f, g, \rho, T)g\ d\mu$.*

Proof. Note first that if $B \subseteq X$ is a Borel set and $n \in \mathbb{N}$, then

$$\begin{aligned}\nu(B) &= \int_B g\ d\mu \\ &= \int_{T^{-n}(B)} (g \circ T^n)(x)\rho(x, T^n(X))\ d\mu(x) \\ &= \int_{T^{-n}(B)} g(x)\rho(x, T^n(x))(g \circ T^n)(x)/g(x)\ d\mu(x) \\ &= \int_{T^{-n}(B)} \rho(x, T^n(x))(g \circ T^n)(x)/g(x)\ d\nu(x) \\ &= \int_{T^{-n}(B)} \sigma(x, T^n(x))\ d\nu(x).\end{aligned}$$

Observe next that $S_n(\mathbb{1}, \sigma, T)(x) = S_n(g, \rho, T)(x)g(x)$ for all $n \in \mathbb{N}$, thus $S_n(\mathbb{1}, \sigma, T) \to \infty$, for all $x \in X$. By the definition of ν, it only remains to observe that if $n \in \mathbb{N}$ and $x \in X$, then

$$\begin{aligned} &R_n(f/g, \mathbb{1}, \sigma, T)(x) \\ &\quad = \textstyle\sum_{i<n} \sigma(x, T^i(x))((f/g) \circ T^i)(x) / \sum_{i<n} \sigma(x, T^i(x)) \\ &\quad = \textstyle\sum_{i<n} (f \circ T^i)(x)\rho(x, T^i(x)) / \sum_{i<n} (g \circ T^i)(x)\rho(x, T^i(x)) \\ &\quad = R_n(f, g, \rho, T)(x), \end{aligned}$$

so $R_\infty(f/g, \mathbb{1}, \sigma, T)(x) = R_\infty(f, g, \rho, T)(x)$. ⊠

In light of Proposition 1.1, the following observation further reduces what remains of Theorem 1 to the aperiodic separable case:

Proposition 2.2. *Suppose that X is a Borel space, $T \colon X \to X$ is a Borel function, $\rho \colon R_T^X \to (0, \infty)$ is a Borel cocycle, μ is a ρ-invariant Borel measure on X for which there exists $n \in \mathbb{Z}^+$ with the property that $\mu(B \bigtriangleup T^{-n}(B)) = 0$ for all Borel sets $B \subseteq X$, and $f \in L^1(\mu)$. Then $\int f \, d\mu = \int R_\infty(f, \mathbb{1}, \rho, T) \, d\mu$.*

Proof. As $\int_B d\mu = \int_{T^{-n}(B)} \rho(x, T^n(x)) \, d\mu(x) = \int_B \rho(x, T^n(x)) \, d\mu(x)$ for all Borel sets $B \subseteq X$, it follows that $\rho(x, T^n(x)) = 1$ μ-almost everywhere. And if $(U_i)_{i \in \mathbb{N}}$ is a sequence of Borel subsets of $\mathbb{R}$ that separates points, then $\bigcup_{n \in \mathbb{N}} f^{-1}(U_i) \bigtriangleup T^{-n}(f^{-1}(U_i))$ is a μ-null Borel set off of which $f = f \circ T^n$. It only remains to note that

$$\begin{aligned} \textstyle\int f \, d\mu &= \textstyle\int \sum_{i<n} f(x)\rho(x, T^i(x)) / S_n(\mathbb{1}, \rho, T)(x) \, d\mu(x) \\ &= \textstyle\sum_{i<n} \int (f \circ T^n)(x)\rho(x, T^i(x)) / S_n(\mathbb{1}, \rho, T)(T^n(x)) \, d\mu(x) \\ &= \textstyle\sum_{i<n} \int (f \circ T^{n-i})(x) / S_n(\mathbb{1}, \rho, T)(T^{n-i}(x)) \, d\mu(x) \\ &= \textstyle\sum_{i<n} \int (f \circ T^{n-i})(x)\rho(x, T^{n-i}(x)) / S_n(\mathbb{1}, \rho, T)(x) \, d\mu(x) \\ &= \textstyle\int R_n(f, \mathbb{1}, \rho, T) \, d\mu, \end{aligned}$$

and $R_n(f, \mathbb{1}, \rho, T) = R_\infty(f, \mathbb{1}, \rho, T)$ μ-almost everywhere. ⊠

Define functions $\underline{R}(f, g, \rho, T)(x) = \liminf_{n \to \infty} R_n(f, g, \rho, T)(x)$ and $\overline{R}(f, g, \rho, T)(x) = \limsup_{n \to \infty} R_n(f, g, \rho, T)(x)$.

Proposition 2.3. *Suppose that X is a Borel space, $T \colon X \to X$ is an aperiodic separable Borel function, $\rho \colon R_T^X \to (0, \infty)$ is a Borel cocycle, μ is a ρ-invariant Borel measure on X, $f, h \in L^1(\mu)$, h is T-invariant, and $\overline{R}(f, \mathbb{1}, \rho, T)(x) \geq h(x)$ for all $x \in X$. Then $\int f \, d\mu \geq \int h \, d\mu$.*

Proof. It is enough to show that if $\epsilon > 0$ and $\overline{R}(f, \mathbb{1}, \rho, T)(x) > h(x)$ for all $x \in X$, then $\int f \, d\mu \geq \int h \, d\mu - \epsilon$. Let ν be the Borel measure given by $\nu(B) = \int_B |f| + |h| \, d\mu$. By Propositions 1.2 and 1.3, there is a T-bounded Borel set $A \subseteq X$ for which the set $B = \{x \in X \mid \sup_{n \leq n_A(x)} R_n(f, \mathbb{1}, \rho, T)(x) < h(x)\}$ has ν-measure at most ϵ. For all $x \in A$, set $n_0(x) = 0$, and given $i \in \mathbb{N}$ and $n_i(x) < n_A(x)$, let $n_{i+1}(x)$ be the least natural number n such that $n_i(x) < n \leq n_A(x)$ and $R_{n-n_i}(f, \mathbb{1}, \rho, T)(T^{n_i}(x)) \geq h(x)$, or $n_i(x) + 1$ if no such n exists. Let $i(x)$ be the least natural number i for which $n_i = n_A(x)$, set $I(x) = \{i < i(x) \mid R_{n-n_i}(f, \mathbb{1}, \rho, T)(T^{n_i}(x)) \geq h(x)\}$ and $N(x) = \bigcup_{i \in I(x)} [n_i(x), n_{i+1}(x))$, and define $C(x) = \{T^n(x) \mid n \in N(x)\}$ and $D(x) = \{T^n(x) \mid n < n_A(x)\} \setminus C(x)$. Proposition 1.4 yields that

$$\begin{aligned}
&\int f \ d\mu \\
&\quad = \int_A S_{n_A(x)}(f, \rho, T)(x) \ d\mu(x) \\
&\quad = \int_A S_{n_A(x)}(f \mathbb{1}_{C(x)}, \rho, T)(x) + S_{n_A(x)}(f \mathbb{1}_{D(x)}, \rho, T)(x) \ d\mu(x) \\
&\quad \geq \int_A S_{n_A(x)}(f \mathbb{1}_{C(x)}, \rho, T)(x) \ d\mu(x) - \int_A S_{n_A(x)}(|f| \mathbb{1}_B, \rho, T)(x) \ d\mu(x) \\
&\quad = \int_A S_{n_A(x)}(f \mathbb{1}_{C(x)}, \rho, T)(x) \ d\mu(x) - \int_B |f| \ d\mu
\end{aligned}$$

and

$$\begin{aligned}
&\int_A S_{n_A(x)}(f \mathbb{1}_{C(x)}, \rho, T)(x) \ d\mu(x) \\
&\quad = \int_A \sum_{i<i(x)} S_{n_{i+1}(x)-n_i(x)}(f, \rho, T)(T^{n_i(x)}(x)) \rho(x, T^{n_i(x)}(x)) \ d\mu(x) \\
&\quad \geq \int_A \sum_{i<i(x)} S_{n_{i+1}(x)-n_i(x)}(h, \rho, T)(T^{n_i(x)}(x)) \rho(x, T^{n_i(x)}(x)) \ d\mu(x) \\
&\quad = \int_A S_{n_A(x)}(h, \rho, T)(x) - S_{n_A(x)}(h \mathbb{1}_{D(x)}, \rho, T)(x) \ d\mu(x) \\
&\quad \geq \int_A S_{n_A(x)}(h, \rho, T)(x) \ d\mu(x) - \int_A S_{n_A(x)}(|h| \mathbb{1}_B, \rho, T)(x) \ d\mu(x) \\
&\quad = \int h \ d\mu - \int_B |h| \ d\mu,
\end{aligned}$$

so $\int f \ d\mu \geq \int h \ d\mu - \int_B |f| + |h| \ d\mu \geq \int h \ d\mu - \epsilon$. ⊠

For the remaining special case of Theorem 1, it is enough to show the weakening whose conclusion is that $\int \overline{R}(f, \mathbb{1}, \rho, T) \ d\mu \leq \int f \ d\mu$ (since it yields that $\int f \ d\mu \leq \int \underline{R}(f, \mathbb{1}, \rho, T) \ d\mu$, when f is replaced with $-f$). Towards this end, note that $\overline{R}(f, \mathbb{1}, \rho, T)$ is T-invariant, since $S_n(\mathbb{1}, \rho, T)(x) \to \infty$ for all $x \in X$. Then the set $B = \overline{R}(f, \mathbb{1}, \rho, T)^{-1}(\mathbb{R})$ is μ-conull and $\overline{R}(f, \mathbb{1}, \rho, T)\mathbb{1}_B \in L^1(\mu)$, since otherwise there exists $r > 0$ for which $\int |f| \ d\mu < \int \min\{|\overline{R}(f, \mathbb{1}, \rho, T)|, r\} \ d\mu$, contradicting the fact that Proposition 2.3 at $|f|$ and $\min\{|\overline{R}(f, \mathbb{1}, \rho, T)|, r\}$ yields that $\int |f| \ d\mu \geq \int \min\{|\overline{R}(f, \mathbb{1}, \rho, T)|, r\} \ d\mu$. But then Proposition 2.3 at f and $\overline{R}(f, \mathbb{1}, \rho, T)\mathbb{1}_B$ yields that $\int f \ d\mu \geq \int \overline{R}(f, \mathbb{1}, \rho, T) \ d\mu$.

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Benjamin D. Miller, Universität Wien, Department of Mathematics, Kolingasse 14–16, 1090 Wien, Austria

E-mail address: `benjamin.miller@univie.ac.at`

URL: `http://www.logic.univie.ac.at/benjamin.miller`

Anush Tserunyan, University of Illinois at Urbana-Champaign, Department of Mathematics, 1409 West Green Street, Urbana, IL 61801, USA

E-mail address: `anush@ullinois.edu`

URL: `https://faculty.math.illinois.edu/~anush/`