ISOMORPHISM VIA FULL GROUPS

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Abstract. At the request of Medynets, we give a measure-theoretic characterization of the circumstances under which Borel subsets A, B of a Polish space X can be mapped to one another via an element of the full group of a countable Borel equivalence relation on X.

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. The *full group* of E is the group $[E]$ of Borel automorphisms $f : X \to X$ such that $\mathrm{graph}(f) \subseteq E$. The *full semigroup* of E is the semigroup $[\![E]\!]$ of Borel isomorphisms $f : A \to B$, where $A, B \subseteq X$ are Borel, such that $\mathrm{graph}(f) \subseteq E$. We write $A \sim B$ to indicate that there exists $f \in [\![E]\!]$ such that $f(A) = B$.

Theorem 1. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $A, B \subseteq X$ are Borel. Then the following are equivalent:*

1. *$A \sim B$.*
2. *The following conditions are satisfied:*
   (a) *$[A]_E = [B]_E$.*
   (b) *Every $(E|A)$-invariant finite measure on A extends to an $(E|(A \cup B))$-invariant finite measure on $A \cup B$ such that $\mu(A) = \mu(B)$.*
   (c) *Every $(E|B)$-invariant finite measure on B extends to an $(E|(A \cup B))$-invariant finite measure on $A \cup B$ such that $\mu(A) = \mu(B)$.*

Proof. As the proof of (1) $\Rightarrow$ (2) is straightforward, we prove only (2) $\Rightarrow$ (1). By Feldman-Moore [2], there is a countable group $\Gamma = \{\gamma_n\}_{n \in \mathbb{N}}$ of Borel automorphisms of X with $E = E_\Gamma^X$. Define recursively $A_n \subseteq A$ and $B_n \subseteq B$ by

$$A_n = \Big(A \setminus \bigcup_{m<n} A_m\Big) \cap \gamma_n^{-1}\Big(B \setminus \bigcup_{m<n} B_m\Big)$$

and

$$B_n = \gamma_n\Big(A \setminus \bigcup_{m<n} A_m\Big) \cap \Big(B \setminus \bigcup_{m<n} B_m\Big).$$

Put $A_\infty = \bigcup_{n \in \mathbb{N}} A_n$ and $B_\infty = \bigcup_{n \in \mathbb{N}} B_n$. As $\langle A_n \rangle_{n \in \mathbb{N}}$ and $\langle B_n \rangle_{n \in \mathbb{N}}$ partition A_∞ and B_∞, respectively, there is a Borel isomorphism $g : A_\infty \to B_\infty$ in $[\![E]\!]$ such that $\forall n \in \mathbb{N}\ (g|A_n = \gamma_n|A_n)$.

Lemma 2. $\forall x \in X$ $(A \cap [x]_E = A_\infty \cap [x]_E$ *or* $B \cap [x]_E = B_\infty \cap [x]_E)$.

Proof. Suppose, towards a contradiction, that there exists $x \in X$ such that both $(A \setminus A_\infty) \cap [x]_E$ and $(B \setminus B_\infty) \cap [x]_E$ are non-empty. Fix $x_A \in (A \setminus A_\infty) \cap [x]_E$ and $x_B \in (B \setminus B_\infty) \cap [x]_E$, and find $n \in \mathbb{N}$ such that $\gamma_n \cdot x_A = x_B$. As $\bigcup_{m<n} A_m \subseteq A_\infty$ and $\bigcup_{m<n} B_m \subseteq B_\infty$, it follows that $x_A \in A_n \subseteq A_\infty$, the desired contradiction. □

It follows from Lemma 2 that the sets $X_A = [A \setminus A_\infty]_E$ and $X_B = [B \setminus B_\infty]_E$ are disjoint. Set $Y = X \setminus (X_A \cup X_B)$, and observe that $f_Y = g|(A \cap Y)$ is a Borel isomorphism of $A \cap Y$ with $B \cap Y$ in $[\![E|Y]\!]$.

It remains to find Borel isomorphisms $f_A \in [\![E|X_A]\!]$ and $f_B \in [\![E|X_B]\!]$ of $A \cap X_A$ with $B \cap X_A$ and $A \cap X_B$ with $B \cap X_B$, respectively. We will describe only the construction of f_A, as the construction of f_B is essentially similar.

Following standard convention, we say that E is *compressible* if there is a Borel set $C \sim X$ such that $X \setminus C$ is an E-complete section. More generally, we say that a set $D \subseteq X$ is *compressible* if $E|D$ is compressible. We will require the following remarkable theorem of Nadkarni [3]:

Theorem 3 (Nadkarni). *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then exactly one of the following holds:*

(i) There is an E-invariant probability measure on X.

(ii) E is compressible.

We verify next that the remaining sets under consideration are compressible:

Lemma 4. *$A \cap X_A$ and $B \cap X_A$ are compressible.*

Proof. To see that $B \cap X_A$ is compressible suppose, towards a contradiction, that it is not. By Theorem 3, there is an $E|(B \cap X_A)$-invariant probability measure μ on $B \cap X_A$. We can extend this to an $(E|B)$-invariant probability measure on B by insisting that $\mu(B \setminus X_A) = 0$. It then follows from condition (2c) that μ extends to an $(E|(A \cup B))$-invariant finite measure ν on $A \cup B$ such that $\nu(A) = \nu(B) = 1$. It follows from invariance that ν is supported on X_A. As the set $A \setminus g^{-1}(B)$ intersects every equivalence class of $E|X_A$, another appeal to invariance gives that $\nu(A \setminus g^{-1}(B)) > 0$, thus $\nu(A) > \nu(g^{-1}(B))$, and one final appeal to invariance implies that $\nu(A) > \nu(B)$, the desired contradiction.

It follows that $g^{-1}(B \cap X_A)$ is also compressible, thus so too is $A \cap X_A$. □

A Borel set $C \subseteq X$ is *countably paradoxical* if it can be partitioned into Borel sets $C_0, C_1, \ldots \subseteq C$ such that $\forall i, j \in \mathbb{N}\ (C_i \sim C_j)$. We will need the following fact from Becker-Kechris [1]:

Proposition 5 (Becker-Kechris). *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then X is compressible $\Leftrightarrow$ X is countably paradoxical.*

Using this, we can now establish the following general fact:

Lemma 6. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $C \subseteq X$ is a compressible Borel E-complete section. Then $C \sim X$.*

Proof. By a straightforward Schröder-Bernstein argument, it is enough to find $f \in [\![E]\!]$ such that $f(X) \subseteq C$. By Theorem 5, there is a partition $C_0, C_1, \ldots \subseteq C$ of C into Borel sets as well as bijections $f_n \in [\![E]\!]$ of C_0 with C_n, for each $n \in \mathbb{N}$. By Feldman-Moore [2], there is a countable group $\Gamma = \{\gamma_n\}_{n \in \mathbb{N}}$ of Borel automorphisms of X with $E = E_\Gamma^X$. For each $x \in X$, let $n(x)$ be the least natural number such that $\gamma_{n(x)} \cdot x \in C_0$, and observe that the function $f(x) = f_{n(x)}(\gamma_{n(x)} \cdot x)$ is an element of $[\![E]\!]$ such that $f(X) \subseteq C$. □

By Lemmas 4 and 6, there are Borel isomorphisms $g_A, g_B \in [\![E]\!]$ of $A \cap X_A$ with X_A and $B \cap X_A$ with X_A, respectively, and it follows that the function $g_B^{-1} \circ g_A$ is the desired element of $[\![E]\!]$ which sends $A \cap X_A$ to $B \cap X_A$. □

As an immediate corollary, we now have the following:

Theorem 7. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $A, B \subseteq X$ are Borel, and set $A^c = X \setminus A$ and $B^c = X \setminus B$. The following are equivalent:*

1. *There exists $f \in [E]$ such that $f(A) = B$.*
2. *The following conditions are satisfied:*
   (a) $[A]_E = [B]_E$.
   (b) *Every $(E|A)$-invariant finite measure on A extends to an $(E|(A \cup B))$-invariant finite measure on $A \cup B$ such that $\mu(A) = \mu(B)$.*
   (c) *Every $(E|B)$-invariant finite measure on B extends to an $(E|(A \cup B))$-invariant finite measure on $A \cup B$ such that $\mu(A) = \mu(B)$.*
   (d) $[A^c]_E = [B^c]_E$.
   (e) *Every $(E|A^c)$-invariant finite measure on A^c extends to an $(E|(A^c \cup B^c))$-invariant finite measure on $A^c \cup B^c$ such that $\mu(A^c) = \mu(B^c)$.*
   (f) *Every $(E|B^c)$-invariant finite measure on B^c extends to an $(E|(A^c \cup B^c))$-invariant finite measure on $A^c \cup B^c$ such that $\mu(A^c) = \mu(B^c)$.*

References

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