A DICHOTOMY THEOREM FOR GRAPHS INDUCED BY COMMUTING FAMILIES OF BOREL INJECTIONS

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ABSTRACT. We prove a dichotomy theorem for oriented graphs induced by certain families of commuting partial injections.

An embedding of a graph $\mathcal{G}$ on a Polish space X into a graph $\mathcal{H}$ on a Polish space Y is an injective Borel function $\pi : X \to Y$ such that

$$\forall x_1, x_2 \in X \ ((x_1, x_2) \in \mathcal{G} \Leftrightarrow (\pi(x_1), \pi(x_2)) \in \mathcal{H}).$$

So as to maintain consistency with the notation of Kechris-Solecki-Todorčević [1], we write $\mathcal{G} \sqsubseteq_c^{\leftrightarrow} \mathcal{H}$ to indicate the existence of a continuous embedding.

Given partial injections f and g on X, we use $f \circ g$ to denote the partial injection such that $\text{dom}(f \circ g) = \text{dom}(g) \cap g^{-1}(\text{dom}(f))$ and $[f \circ g](x) = f(g(x))$, for all $x \in \text{dom}(f \circ g)$. We say that a partial injection is *Borel* if its graph is Borel. (Our results here generalize to partial injections with $\mathbf{\Sigma^1_1}$ graphs; we make the stronger assumption so as to simplify some of the proofs.)

Suppose now that $\langle g_0, g_1, \ldots \rangle$ is a sequence of Borel partial injections. Let $g_\emptyset$ denote the empty partial injection, and for each $s \in 2^{n+1}$, set

$$g_s = g_0^{s(0)} \circ \cdots \circ g_n^{s(n)}.$$

Let $\text{supp}(s) = \{k < |s| : s(k) = 1\}$. We say that a sequence $\langle g_0, g_1, \ldots \rangle$ of commuting Borel partial injections is *prismatic* if for all $n \in \mathbb{N}$, $s, t \in 2^n$, and $k \in \mathbb{N}$ such that $\text{supp}(t) \neq \text{supp}(s) \amalg \{k\}$, the composition $g_t^{-1} \circ g_k \circ g_s$ is fixed-point free. We say that a directed graph $\mathcal{G}$ on a Polish space is an *oriented prism* if there is a prismatic sequence $\langle g_0, g_1, \ldots \rangle$ such that $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \text{graph}(g_n)$.

For each $n \in \mathbb{N}$, let $g_n^\square$ be the partial injection of $2^\mathbb{N}$ such that $\text{dom}(g_n^\square) = \{x \in X : x(n) = 0\}$ and $g_n^\square(s0x) = s1x$, for all $s \in 2^n$ and $x \in 2^\mathbb{N}$. It is clear that $\langle g_0^\square, g_1^\square, \ldots \rangle$ is a prismatic sequence whose induced equivalence relation is E_0, so that the directed graph $\mathcal{G}_\square^\rightarrow = \bigcup_{n \in \mathbb{N}} \text{graph}(g_n^\square)$ is an oriented prism whose symmetrization is a graphing of E_0. As $\mathcal{G}_0^\rightarrow \subseteq \mathcal{G}_\square^\rightarrow$, it follows that $\chi_B(\mathcal{G}_\square^\rightarrow) = \mathfrak{c}$.

Theorem 1. *Suppose that X is a Polish space and $\mathcal{G}$ is an oriented prism on X. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$.
2. $\mathcal{G}_\square^\rightarrow \sqsubseteq_c^{\leftrightarrow} \mathcal{G}$.

Proof. As (1) $\Rightarrow \neg$(2) is straightforward, we shall prove only $\neg$(1) $\Rightarrow$ (2). Fix a prismatic sequence $\langle g_0, g_1, \ldots \rangle$ such that $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \text{graph}(g_n)$, and let $E_\mathcal{G}$ denote the equivalence relation induced by the symmetrization of $\mathcal{G}$. It is sufficient to produce a continuous injection $\pi : 2^\mathbb{N} \to X$ such that:

(i) $\forall \alpha E_0 \beta \ ((\alpha, \beta) \in \mathcal{G}_\square^{\rightarrow} \Leftrightarrow (\pi(\alpha), \pi(\beta)) \in \mathcal{G})$.

(ii) $\forall \alpha, \beta \in 2^{\mathbb{N}} \ (\pi(\alpha) E_{\mathcal{G}} \pi(\beta) \Rightarrow \alpha E_0 \beta)$.

Towards this end, let $\mathcal{I}$ denote the σ-ideal generated by $\mathcal{G}$-discrete Borel sets, and let H_n denote the finite set of Borel partial injections of the form $g_{i_1}^{\pm 1} \circ \cdot \circ g_{i_m}^{\pm 1}$, where $i_1, \ldots, i_m, m \leq n$. By standard change of topology results, we can assume that X is a zero-dimensional Polish space and each g_n is a partial homeomorphism with clopen domain and range. We will find clopen sets $A_n \subseteq X$ and natural numbers k_n, from which we define $h_s : X \to X$, for $s \in 2^{<\mathbb{N}}$, by $h_\emptyset = \mathrm{id}$ and

$$h_s = g_{k_0}^{s(0)} \ldots g_{k_n}^{s(n)},$$

for $s \in 2^{n+1}$. We will ensure that, for all $n \in \mathbb{N}$, the following conditions hold:

(a) $A_n \notin \mathcal{I}$.

(b) $A_{n+1} \subseteq A_n \cap g_{k_n}^{-1}(A_n)$.

(c) $\forall s, t \in 2^n \, \forall h \in H_n \ (h \circ h_s(A_{n+1}) \cap h_t \circ g_{k_n}(A_{n+1}) = \emptyset)$.

(d) $\forall s \in 2^{n+1} \ (\mathrm{diam}(h_s(A_{n+1})) \leq 1/n)$.

We begin by setting $A_0 = X$. Suppose now that we have found $\langle A_i \rangle_{i \leq n}$ and $\langle k_i \rangle_{i<n}$, and for each $k \in \mathbb{N}$, define an open set $U_k \subseteq A_n$ by

$$U_k = \{x \in A_n \cap g_k^{-1}(A_n) : \forall s, t \in 2^n \, \forall h \in H_n \ (g_k(x) \neq h_t^{-1} \circ h \circ h_s(x))\}.$$

Lemma 2. *There exists $k \in \mathbb{N}$ such that $U_k \notin \mathcal{I}$.*

Proof. As $A_n \notin \mathcal{I}$, it is enough to show that the set $A = A_n \setminus \bigcup_{k \in \mathbb{N}} U_k$ is in $\mathcal{I}$. Towards this end, let $\mathcal{G}|A = \mathcal{G} \cap (A \times A)$, and note that if $(x, y) \in \mathcal{G}|A$, then there exists $s, t \in 2^n$ and $h \in H_n$ such that $y = h_t^{-1} \circ h \circ h_s(x)$. It follows that the symmetrization of $\mathcal{G}|A$ is of bounded vertex degree. Proposition 4.6 of Kechris-Solecki-Todorčević [1] then ensures that $\chi_B(\mathcal{G}|A) < \aleph_0$, thus $A \in \mathcal{I}$. □

By Lemma 2, there exists $k \in \mathbb{N}$ such that $U_k \notin \mathcal{I}$. Set $k_n = k$. As each g_n is a partial homeomorphism with clopen domain and range, we can write U_k as the union of countably many clopen sets U such that:

(c′) $\forall s, t \in 2^n \, \forall h \in H_n \ (h \circ h_s(U) \cap h_t \circ g_{k_n}(U) = \emptyset)$.

(d′) $\forall s \in 2^{n+1} \ (\mathrm{diam}(h_s(U)) \leq 1/n)$.

Fix such a U which is not in $\mathcal{I}$, and set $A_{n+1} = U$.

This completes the recursive construction. For each $s \in 2^n$, put $B_s = h_s(A_n)$. Conditions (b) and (d) ensure that, for each $\alpha \in 2^{\mathbb{N}}$, the sets $B_{\alpha|0}, B_{\alpha|1}, \ldots$ are decreasing, clopen, and of vanishing diameter, and therefore have singleton intersection. Define $\pi : 2^{\mathbb{N}} \to X$ by

$$\pi(\alpha) = \text{ the unique element of } \bigcap_{n \in \mathbb{N}} B_{\alpha|n}.$$

It follows from conditions (c) and (d) that π is a continuous injection, so it only remains to check conditions (i) and (ii). We note first the following lemma:

Lemma 3. *Suppose that $n \in \mathbb{N}$, $s \in 2^n$, and $\alpha \in 2^{\mathbb{N}}$. Then $\pi(s\alpha) = h_s \circ \pi(0^n\alpha)$.*

Proof. Simply observe that

$$\begin{aligned}
\{\pi(s\alpha)\} &= \bigcap_{i \geq n} B_{(s\alpha)|i} \\
&= \bigcap_{i \geq 0} h_s h_{0^n(\alpha|i)}(A_{i+n}) \\
&= h_s \left(\bigcap_{i \geq 0} h_{0^n(\alpha|i)}(A_{i+n}) \right) \\
&= h_s \left(\bigcap_{i \geq n} B_{(0^n\alpha)|i} \right) \\
&= \{h_s \circ \pi(0^n\alpha)\},
\end{aligned}$$

thus $\pi(s\alpha) = h_s \circ \pi(0^n\alpha)$. □

To see (i), suppose first that $(\alpha, \beta) \in \mathcal{G}_{\square}^{\rightarrow}$, fix $n \in \mathbb{N}$ maximal such that $\alpha(n) \neq \beta(n)$, set $s = \alpha|n = \beta|n$, and fix $\gamma \in 2^{\mathbb{N}}$ such that $\alpha = s0\gamma$ and $\beta = s1\gamma$. Then Lemma 3 and the fact that $\langle g_0, g_1, \ldots \rangle$ is prismatic ensure that

$$\begin{aligned}
\pi(\beta) &= \pi(s1\gamma) \\
&= h_s \circ g_{k_n} \circ \pi(0^{n+1}\gamma) \\
&= g_{k_n} \circ h_s \circ \pi(0^{n+1}\gamma) \\
&= g_{k_n} \circ \pi(\alpha),
\end{aligned}$$

thus $(\pi(\alpha), \pi(\beta)) \in \mathcal{G}$.

Suppose now that $\alpha E_0 \beta$ and $(\pi(\alpha), \pi(\beta)) \in \mathcal{G}$. Fix $n \in \mathbb{N}$, $s, t \in 2^{n+1}$, and $\gamma \in 2^{\mathbb{N}}$ such that $\alpha = s\gamma$ and $\beta = t\gamma$. Then Lemma 3 ensures that $\pi(\alpha) = h_s \circ \pi(0^n\gamma)$ and $\pi(\beta) = h_t \circ \pi(0^n\gamma)$, so there exists $k \in \mathbb{N}$ such that $g_k \circ h_s \circ \pi(0^n\gamma) = h_t \circ \pi(0^n\gamma)$. Let $m = \max_{i \leq n} k_i$, and for each $u \in 2^{n+1}$, let u' be the element of 2^{m+1} such that $\mathrm{supp}(u') = \{k_i : i \in \mathrm{supp}(u)\}$, so that $h_u = g_{u'}$. Then $g_k \circ g_{s'} \circ \pi(0^n\gamma) = g_{t'} \circ \pi(0^n\gamma)$, and since $\langle g_0, g_1, \ldots \rangle$ is prismatic, it follows that $\mathrm{supp}(t') = \mathrm{supp}(s') \amalg \{k\}$, thus $k = k_i$, for some $i \leq n$, and $\mathrm{supp}(t) = \mathrm{supp}(s) \amalg \{k_i\}$, so $g_i^{\square}(\alpha) = g_i^{\square}(s\gamma) = t\gamma = \beta$, so $(\alpha, \beta) \in \mathcal{G}_{\square}^{\rightarrow}$.

To see (ii), it is enough to check that if $\alpha, \beta \in 2^{\mathbb{N}}$ and $\alpha(n) \neq \beta(n)$, then there is no $h \in H_n$ such that $h \circ \pi(\alpha) = \pi(\beta)$. Suppose, towards a contradiction, that there is such an $h \in H_n$. As H_n is symmetric, we can assume that $\alpha(n) = 0$ and $\beta(n) = 1$. Set $s = \alpha|n$ and $t = \beta|n$, and fix $\gamma, \delta \in 2^{\mathbb{N}}$ such that $\alpha = s0\gamma$ and $\beta = t1\delta$. Lemma 3 ensures that $\pi(\alpha) = h_s \circ \pi(0^{n+1}\gamma)$ and $\pi(\beta) = h_t \circ g_{k_n} \circ \pi(0^{n+1}\delta)$. As $\pi(0^{n+1}\gamma), \pi(0^{n+1}\delta) \in A_{n+1}$, it follows that $\pi(\beta) \in h \circ h_s(A_{n+1}) \cap h_t \circ g_{k_n}(A_{n+1})$, which contradicts condition (c). □

References

[1] A. Kechris, S. Solecki, and S. Todorčević. Borel chromatic numbers. *Adv. Math.*, **141 (1)**, (1999), 1–44