ANALYTIC FAMILIES OF ALMOST DISJOINT SETS

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Abstract. We give a streamlined version of Törnquist's proof of Mathias's theorem that there are no infinite maximal analytic families of pairwise almost disjoint subsets of $\mathbb{N}$.

We say that sets A and B are *almost disjoint* if $|A \cap B| < \aleph_0$, and we write $A \subseteq^* B$ to indicate that $|A \setminus B| < \aleph_0$.

Let $[\mathbb{N}]^{\aleph_0}$ denote the set of all infinite subsets of $\mathbb{N}$, and for all countable sets $\mathscr{C} \subseteq [\mathbb{N}]^{\aleph_0}$, let $[\mathscr{C}]^{<\aleph_0}$ denote the set of all finite sets $\mathscr{F} \subseteq \mathscr{C}$.

Proposition 1. *Suppose that $\mathscr{A} \subseteq [\mathbb{N}]^{\aleph_0}$ and $\mathscr{C} \subseteq [\mathbb{N}]^{\aleph_0}$ is a countable set with the following properties:*

(1) $\forall A \in \mathscr{A} \exists \mathscr{F} \in [\mathscr{C}]^{<\aleph_0}\ A \subseteq^* \bigcup \mathscr{F}$.
(2) $\forall \mathscr{F} \in [\mathscr{C}]^{<\aleph_0}\ \mathbb{N} \not\subseteq^* \bigcup \mathscr{F}$.

Then some set in $[\mathbb{N}]^{\aleph_0}$ is almost disjoint from every set in $\mathscr{A}$.

Proof. If $\mathscr{C}$ is finite, then $\mathbb{N} \setminus \bigcup \mathscr{C}$ is as desired. Otherwise, fix an enumeration $(C_n)_{n \in \mathbb{N}}$ of $\mathscr{C}$, and for all $n \in \mathbb{N}$, set $\mathscr{F}_n = \{C_m \mid m < n\}$ and fix $k_n \geq n$ in $\mathbb{N} \setminus \bigcup \mathscr{F}_n$. Then $\{k_n \mid n \in \mathbb{N}\}$ is as desired. ⊠

Endow $[\mathbb{N}]^{\aleph_0}$ with the topology it inherits via its natural identification with the set of sequences in $2^{\mathbb{N}}$ with infinite support.

Proposition 2 (Törnquist)**.** *Suppose that $\pi \colon \mathbb{N}^{\mathbb{N}} \to [\mathbb{N}]^{\aleph_0}$ is continuous and T is a tree on $\mathbb{N}$ for which $\pi([T])$ is a set of pairwise almost disjoint sets and $|\pi([T])| \geq 2$. Then there exist $n \in \mathbb{N}$ and $s, t \in T$ such that*

$$\forall A \in \pi(\mathcal{N}_s \cap [T]) \forall B \in \pi(\mathcal{N}_t \cap [T])\ A \cap B \subseteq n.$$

Proof. Fix distinct $C, D \in \pi([T])$, as well as $j \in \mathbb{N}$ with $C \cap j \neq D \cap j$. Fix $c, d \in [T]$ with $C = \pi(c)$ and $D = \pi(d)$, as well as $i \in \mathbb{N}$ such that $\forall C' \in \pi(\mathcal{N}_{c \upharpoonright i})\ C \cap j = C' \cap j$ and $\forall D' \in \pi(\mathcal{N}_{d \upharpoonright i})\ D \cap j = D' \cap j$.

Suppose, towards a contradiction, that for all $n \in \mathbb{N}$ and extensions s and t of $c \upharpoonright i$ and $d \upharpoonright i$ in T, there exist $A \in \pi(\mathcal{N}_s \cap [T])$ and $B \in \pi(\mathcal{N}_t \cap [T])$ such that $A \cap B \not\subseteq n$. Then there are extensions a and b of s and t in $[T]$ such that $A = \pi(a)$ and $B = \pi(b)$, in which case

2010 *Mathematics Subject Classification.* Primary 03E15; secondary 28A05.
Key words and phrases. Almost disjoint, analytic, eventually different.

$\forall A' \in \pi([T] \cap \mathcal{N}_{s'}) \forall B' \in \pi([T] \cap \mathcal{N}_{t'})\ A' \cap B' \nsubseteq n$, where $s' = a' \upharpoonright k$ and $t' = b' \upharpoonright k$, for all sufficiently large $k \in \mathbb{N}$.

By recursively applying this observation, we obtain strictly increasing sequences $(s_n)_{n \in \mathbb{N}}$ and $(t_n)_{n \in \mathbb{N}}$ of extensions of $c \upharpoonright i$ and $d \upharpoonright i$ in T such that $\forall A \in \pi([T] \cap \mathcal{N}_{s_n}) \forall B \in \pi([T] \cap \mathcal{N}_{t_n})\ A \cap B \nsubseteq n$. Then the sequences $a = \bigcup_{n \in \mathbb{N}} s_n$ and $b = \bigcup_{n \in \mathbb{N}} t_n$ are in $[T]$, but the sets $\pi(a)$ and $\pi(b)$ are neither equal nor almost disjoint, a contradiction. ⊠

We can now establish the promised result.

Theorem 3 (Mathias, Törnquist)**.** *Suppose that $\mathscr{A} \subseteq [\mathbb{N}]^{\aleph_0}$ is an infinite analytic set of pairwise almost disjoint sets. Then there is a countable set $\mathscr{C} \subseteq [\mathbb{N}]^{\aleph_0}$ with the following properties:*

(1) $\forall A \in \mathscr{A} \exists \mathscr{F} \in [\mathscr{C}]^{<\aleph_0}\ A \subseteq^* \bigcup \mathscr{F}$.
(2) $\forall \mathscr{F} \in [\mathscr{C}]^{<\aleph_0} \exists A \in \mathscr{A}\ A \nsubseteq^* \bigcup \mathscr{F}$.

Thus $\mathscr{A}$ is not a maximal set of pairwise almost disjoint sets.

Proof (essentially Törnquist). Fix a continuous surjection $\pi \colon \mathbb{N}^{\mathbb{N}} \to \mathscr{A}$. We will recursively construct trees T^α on $\mathbb{N}$, $n^\alpha \in \mathbb{N}$, and $s^\alpha, t^\alpha \in \mathbb{N}^{<\mathbb{N}}$, from which we define $\mathscr{C}^\alpha \subseteq [\mathbb{N}]^{\aleph_0}$, $\mathscr{A}^\alpha \subseteq \mathscr{A}$, and $A^\alpha \in [\mathbb{N}]^{\aleph_0}$ by

– $\mathscr{C}^\alpha = \{A^\beta \mid \beta < \alpha\}$.
– $\mathscr{A}^\alpha = \{A \in \mathscr{A} \mid \forall \mathscr{F} \in [\mathscr{C}^\alpha]^{<\aleph_0}\ A \nsubseteq^* \bigcup \mathscr{F}\}$.
– $A^\alpha = \bigcup \pi(\mathcal{N}_{s^\alpha} \cap [T^\alpha])$.

Note that the $\mathscr{C}^\alpha$ are increasing, so the $\mathscr{A}^\alpha$ are decreasing. We will ensure that the following conditions are satisfied:

(a) $\forall A \in \pi(\mathcal{N}_{s^\alpha} \cap [T^\alpha]) \forall B \in \pi(\mathcal{N}_{t^\alpha} \cap [T^\alpha])\ A \cap B \subseteq n^\alpha$.
(b) $\mathscr{A}^\alpha \cap \pi(\mathcal{N}_{s^\alpha} \cap [T^\alpha])$ and $\mathscr{A}^\alpha \cap \pi(\mathcal{N}_{t^\alpha} \cap [T^\alpha])$ are uncountable.

Suppose that $\alpha < \omega_1$ and we have already constructed T^β, n^β, s^β, and t^β, for all $\beta < \alpha$. If $\mathscr{A}^\alpha$ is countable, then the construction terminates. Otherwise, define $T^\alpha = \{t \in \mathbb{N}^{<\mathbb{N}} \mid \mathscr{A}^\alpha \cap \pi(\mathcal{N}_t) \text{ is uncountable}\}$. As $\pi([T^\alpha])$ is uncountable, we can apply Proposition 2 to π and T^α to obtain $n^\alpha \in \mathbb{N}$ and $s^\alpha, t^\alpha \in T^\alpha$ satisfying condition (a). Condition (b) then follows from the countability of $\mathscr{A}_\alpha \cap \pi(\mathbb{N}^{\mathbb{N}} \setminus [T^\alpha])$.

As $s^\alpha \in T^\alpha \setminus T^{\alpha+1}$, the T^α are strictly decreasing, so the construction terminates at some $\alpha < \omega_1$. Then the set $\mathscr{C} = \mathscr{A}^\alpha \cup \mathscr{C}^\alpha$ is countable and satisfies condition (1). To see that it satisfies condition (2), suppose that $\mathscr{F} \subseteq \mathscr{C}$ is finite. If $\mathscr{F} \subseteq \mathscr{A}^\alpha$, then almost disjointness ensures that $A \nsubseteq^* \bigcup \mathscr{F}$ for all $A \in \mathscr{A} \setminus \mathscr{F}$. Otherwise, there is a maximal $\beta < \alpha$ with $A^\beta \in \mathscr{F}$, so $A \nsubseteq^* \bigcup \mathscr{F}$ for all $A \in (\mathscr{A}^\beta \cap \pi(\mathcal{N}_{t^\beta} \cap [T^\beta])) \setminus \mathscr{A}^\alpha$.

The non-maximality of $\mathscr{A}$ now follows from Proposition 1. ⊠

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