The Grothendieck class of the moduli space of pointed stable curves of genus 0

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References

Joint work with Matilde Marcolli, Stephanie Chen, Eduardo Nascimento:

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ACM: 'Log concavity of the Grothendieck class of $\overline{\mathcal{M}}_{0,n}$'

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AMN: 'Explicit formulas for the Grothendieck class of $\overline{\mathcal{M}}_{0,n}$'

arXiv:2406.13095
To appear in *Moduli.*

The main result, I

$\overline{\mathcal{M}}_{0,n}$:= moduli space of stable n-pointed genus-0 curves

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Theorem (AMN: —, Marcolli, Nascimento)

For $n \geq 3$

$$[\overline{\mathcal{M}}_{0,n}] = (1-\mathbb{L})^{n-1} \sum_{j\geq 0} \sum_{k\geq 0} s(k+n-1, k+n-1-j)\, S(k+n-1-j, k+1)\, \mathbb{L}^{k+j}$$

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Here $[\cdot]$ = class in the Grothendieck group of varieties; $\mathbb{L} = [\mathbb{A}^1]$

s, S: Stirling numbers of 1st, 2nd kind.

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For $3 \le n \le 11$, $[\overline{\mathcal{M}}_{0,n}] =$

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$$\mathbb{L}^2+5\mathbb{L}+1$$
$$\mathbb{L}^3+16\mathbb{L}^2+16\mathbb{L}+1$$
$$\mathbb{L}^4+42\mathbb{L}^3+127\mathbb{L}^2+42\mathbb{L}+1$$
$$\mathbb{L}^5+99\mathbb{L}^4+715\mathbb{L}^3+715\mathbb{L}^2+99\mathbb{L}+1$$
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$$\mathbb{L}^7+466\mathbb{L}^6+13333\mathbb{L}^5+63173\mathbb{L}^4+63173\mathbb{L}^3+13333\mathbb{L}^2+466\mathbb{L}+1$$
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Coefficient of $\mathbb{L}^\ell$ in $[\overline{\mathcal{M}}_{0,n}]$: $\operatorname{rk} H^{2\ell}(\overline{\mathcal{M}}_{0,n})$

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Corollary (AMN)

For $n \geq 3$ (and over $\mathbb{C}$), $\operatorname{rk} H^{2\ell}(\overline{\mathcal{M}}_{0,n})$ *equals*

$$\sum_{j=0}^{\ell} \sum_{k=0}^{\ell-j} (-1)^{\ell-j-k} \binom{n-1}{\ell-j-k} s(k+n-1, k+n-1-j) S(k+n-1-j, k+1)$$

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Remark: In particular, the formula must equal 0 for $\ell > n-3$; this is not 'combinatorially obvious'.
Also clear from geometry, but not combinatorially obvious: $[\overline{\mathcal{M}}_{0,n}]$ is 'palindromic'.

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'First kind': $s(n, k)$; 'second kind': $S(n, k)$

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Example: $(x)_3 = x(x-1)(x-2) = x^3 - 3x^2 + 2x$, so

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Fact (exercise): $S(n,k) = \sum_{i=0}^{k} \dfrac{(-1)^{k-i}\, i^n}{(k-i)!\,i!}$

Matrix form: $\mathfrak{s} = (s(i,j))_{i,j\geq 1}$, resp., $\mathfrak{S} = (S(i,j))_{i,j\geq 1}$:

$$\mathfrak{s} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & -3 & 1 & 0 & 0 & \cdots \\ -6 & 11 & -6 & 1 & 0 & \cdots \\ 24 & -50 & 35 & -10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathfrak{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 7 & 6 & 1 & 0 & \cdots \\ 1 & 15 & 25 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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Notation:

$$1\!\!1_q = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & q & 0 & \dots \\ 0 & 0 & q^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$ (so $1\!\!1_1 = \mathrm{id}$); and

tr_k = sum of entries in k-th subdiagonal (so tr_0 = ordinary trace).

$$\mathfrak{s} \cdot \mathfrak{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\mathrm{id}\cdot\mathfrak{s}\cdot\mathrm{id}\cdot\mathfrak{S}\cdot\mathrm{id}=\begin{pmatrix}1&0&0&0&\dots\\0&1&0&0&\dots\\0&0&1&0&\dots\\0&0&0&1&\dots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

$$\mathbb{1}_1 \cdot \mathfrak{s} \cdot \mathbb{1}_1 \cdot \mathfrak{S} \cdot \mathbb{1}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$\mathbb{1}_{\mathbb{L}} \cdot \mathfrak{s} \cdot \mathbb{1}_{\mathbb{L}-1} \cdot \mathfrak{S} \cdot \mathbb{1}_{\mathbb{L}} =$

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$$\begin{aligned} \text{E.g., } \mathrm{tr}_1(\mathbb{1}_{\mathbb{L}} \cdot \mathfrak{s} \cdot \mathbb{1}_{\mathbb{L}^{-1}} \cdot \mathfrak{S} \cdot \mathbb{1}_{\mathbb{L}}) &= (1-\mathbb{L}) + (3\mathbb{L}-3\mathbb{L}^2) + (6\mathbb{L}^2-6\mathbb{L}^3) + \cdots \\ &= 1 + 2\mathbb{L} + 3\mathbb{L}^2 + 4\mathbb{L}^3 + \cdots \end{aligned}$$

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E.g., $\mathrm{tr}_1(\mathbb{1}_{\mathbb{L}} \cdot \mathfrak{s} \cdot \mathbb{1}_{\mathbb{L}^{-1}} \cdot \mathfrak{S} \cdot \mathbb{1}_{\mathbb{L}}) = (1-\mathbb{L}) + (3\mathbb{L}-3\mathbb{L}^2) + (6\mathbb{L}^2-6\mathbb{L}^3) + \cdots$

$$= 1 + 2\mathbb{L} + 3\mathbb{L}^2 + 4\mathbb{L}^3 + \cdots$$

so $(1-\mathbb{L})^{3-1} \cdot \mathrm{tr}_{3-2}(\mathbb{1}_{\mathbb{L}} \cdot \mathfrak{s} \cdot \mathbb{1}_{\mathbb{L}^{-1}} \cdot \mathfrak{S} \cdot \mathbb{1}_{\mathbb{L}}) = 1 = [\overline{\mathcal{M}}_{0,3}]$.

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Then:

Theorem (restatement of main result)

$$[\overline{\mathcal{M}}_{0,n}] = (1-\mathbb{L})^{n-1} \cdot \mathrm{tr}_{n-2}(\mathbb{1}_{\mathbb{L}} \cdot \mathfrak{s} \cdot \mathbb{1}_{\mathbb{L}^{-1}} \cdot \mathfrak{S} \cdot \mathbb{1}_{\mathbb{L}}) .$$

Some history

Definition/construction of $\overline{\mathcal{M}}_{0,n}$: Knudsen 1983

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$A^\ell(\overline{\mathcal{M}}_{0,n}) \cong H^{2\ell}(\overline{\mathcal{M}}_{0,n})$, $H^{\text{odd}}(\overline{\mathcal{M}}_{0,n}) = 0$, recursion for $\operatorname{rk} H^{2\ell}$.

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Recursion: With $a^k(n) := \operatorname{rk} H^{2k}(\overline{\mathcal{M}}_n)$, we have

$$a^k(3) = 1 \text{ if } k = 0, \ 0 \text{ otherwise,}$$

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$$a^k(n+1) = a^k(n) + a^{k-1}(n) + \frac{1}{2}\sum_{j=2}^{n-2}\binom{n}{j}\sum_{\ell=0}^{\ell=k-1} a^\ell(j+1)a^{k-1-\ell}(n-j+1)$$

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Other proofs of this recursion or equivalent information:
Fulton-MacPherson '94, Getzler '95, Manin '95.

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Consider the generating function

$$M := 1 + z + \sum_{n\geq 3} P_{\overline{\mathcal{M}}_{0,n}}(t) \frac{z^{n-1}}{(n-1)!}$$

where $P_{\overline{\mathcal{M}}_{0,n}}(t) =$ the Poincaré polynomial.

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Manin (*'Generating functions in algebraic geometry and sums over trees'*): a differential equation and a functional equation determining M.

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Theorem (Manin 1995; Manin-Marcolli 2016)

$$\frac{dM}{dz} = \frac{M}{1+\mathbb{L}(1+z)-\mathbb{L}M}$$

$$M^{\mathbb{L}} = \mathbb{L}^2 M + (1-\mathbb{L})(1+(z+1)\mathbb{L})$$

The main result, II

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Theorem (AMN)

$$M = \sum_{\ell \geq 0} \frac{(\ell+1)^{\ell}}{(\ell+1)!} \left(((1-\mathbb{L})(1+(z+1)\mathbb{L}))^{\frac{1+\ell}{\mathbb{L}}-\ell} \prod_{j=0}^{\ell-1} \left(1 - \frac{j\mathbb{L}}{\ell+1}\right) \right) \mathbb{L}^{\ell}$$

$$= \sum_{\ell \geq 0} \left(\sum_{k \geq 0} \frac{(\ell+1)^{\ell+k}}{(\ell+1)!k!} (z - \mathbb{L} - z\mathbb{L})^{k} \prod_{j=0}^{\ell+k-1} \left(1 - \frac{j\mathbb{L}}{\ell+1}\right) \right) \mathbb{L}^{\ell}.$$

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The second form implies the Stirling expression given earlier for $[\overline{\mathcal{M}}_{0,n}]$.

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The second form implies the Stirling expression given earlier for $[\overline{\mathcal{M}}_{0,n}]$. The first implies the second and is independently useful.

Plan for the rest of the talk

- $\overline{\mathcal{M}}_{0,n}$
- Grothendieck classes, recursion
- Proof — After-the-fact approach
- One consequence and some contextual comments

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E.g., there is a 'universal subbundle' over the Grassmannian:

$$\begin{array}{ccc} \mathcal{S} & \hookrightarrow & G(k,V) x V \\ & \searrow^{\pi} & \downarrow \nu \\ & & G(k,V) \end{array}$$

such that for $W \in G(k, V)$, $\pi^{-1}(W) = W$ as a subspace of $V = \nu^{-1}(W)$.

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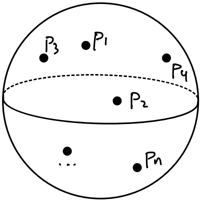
Simplest case: $g = 0$. Smooth genus-0 curves are isomorphic to $\mathbb{P}^1$, so

$$\mathcal{M}_{0,n} = \{C = \mathbb{P}^1 \text{ along with } p_1, \dots, p_n \text{ distinct points on } C\}$$



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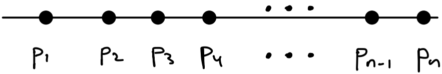

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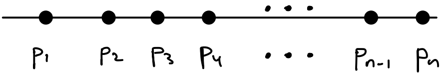

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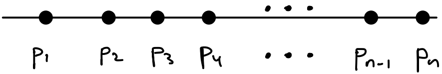

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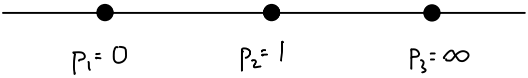

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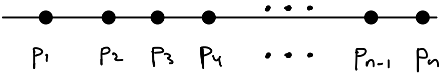

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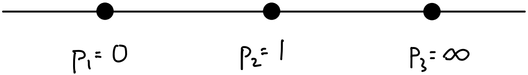

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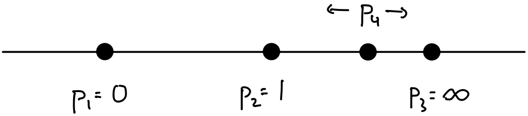

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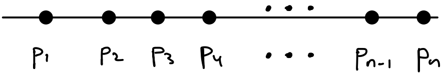

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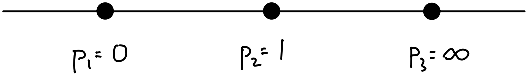

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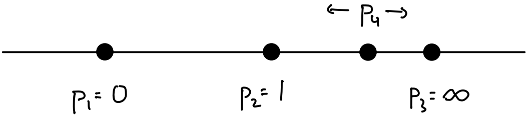

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  (The limits as points come together depend on the choice of which points are identified with 0, 1, ∞.)

Amazing fact (Knudsen 1983):

- For all n there exists a modular compactification $\overline{\mathcal{M}}_{0,n}$.
- Each point of $\overline{\mathcal{M}}_{0,n}$ may be viewed as a configuration of n distinct points on a tree of $\mathbb{P}^1$'s. (More precise statement below.)
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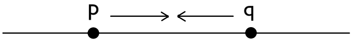

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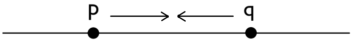
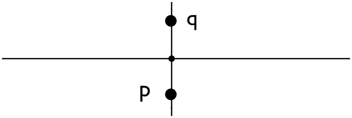
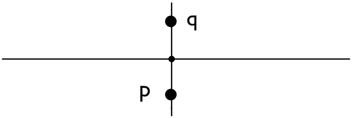
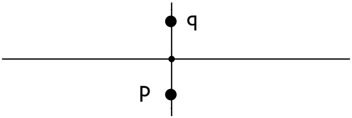


In the limit, the curve 'sprouts' a new component $\mathbb{P}^1$ and p, q end up as distinct points on this component:





Note that the new component now has three 'special' points: p, q, and the intersection with the other component.



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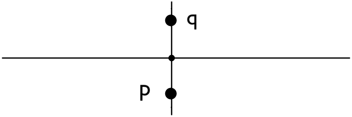
So the limiting p, q cannot move further.



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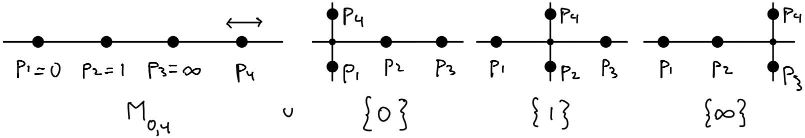

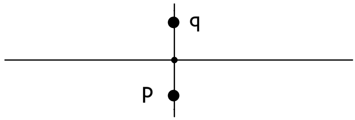

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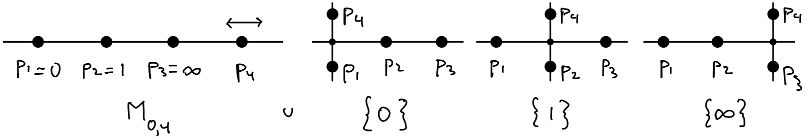

Each component in the limits has exactly 3 special points, so the configurations in the limits are rigid.

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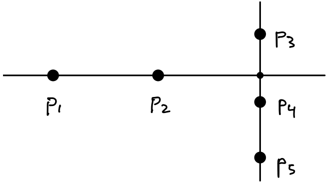

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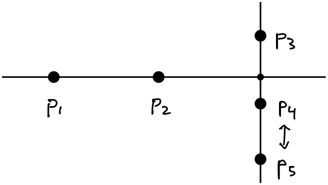

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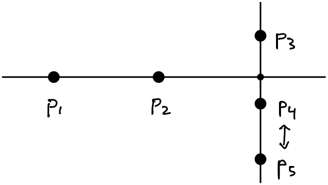

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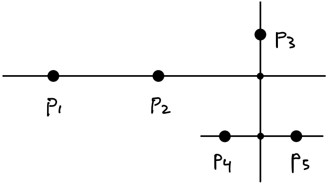

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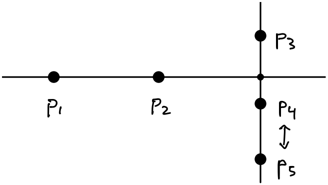

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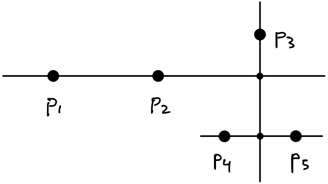

and now each component has exactly 3 special points, so this configuration cannot degenerate further.

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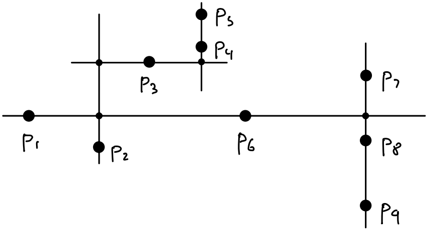

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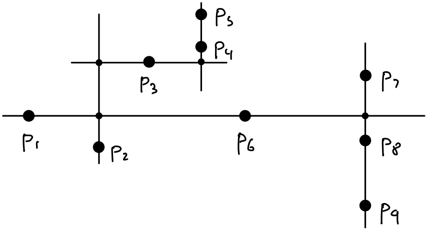

Of course these pictures only capture some combinatorial information.
There is a better way to express the same information.

Dual tree:

- One 'internal' vertex for each component;
- Edges join vertices if components meet;
- n marked points $\to$ n leaves.

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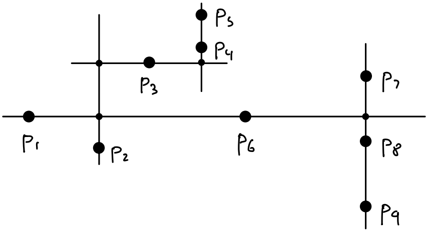

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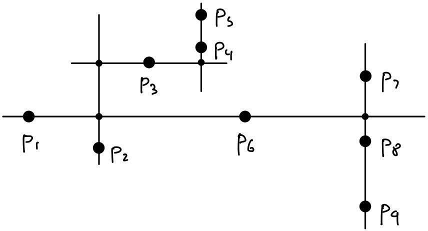

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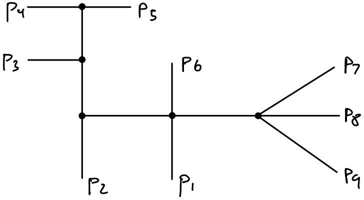

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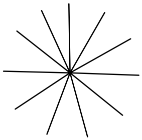

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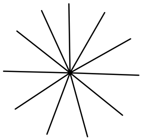

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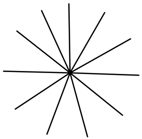

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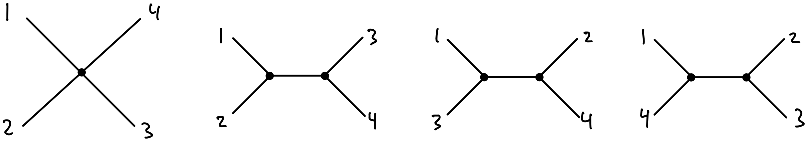

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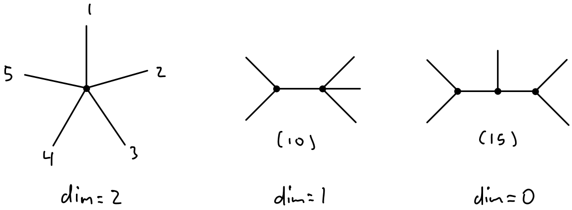

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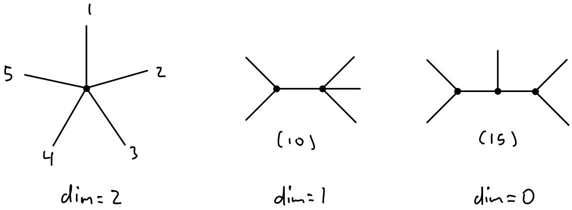

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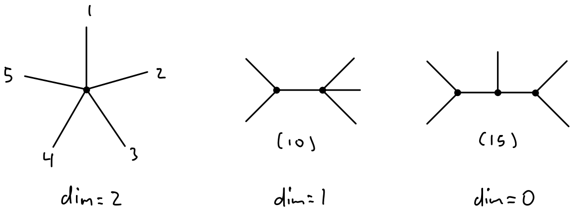

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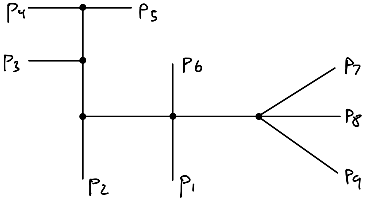

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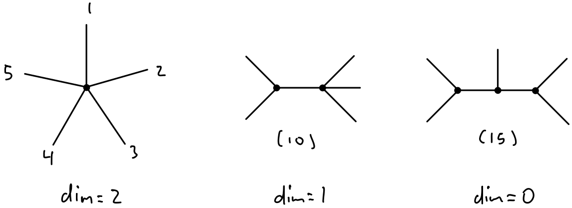

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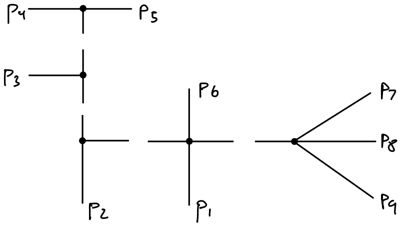

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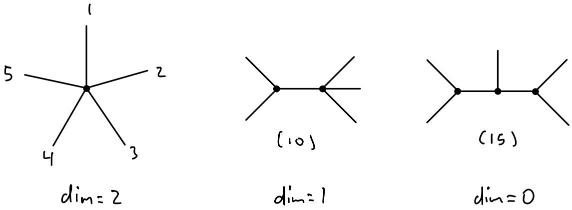

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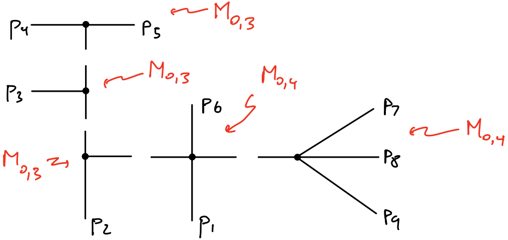

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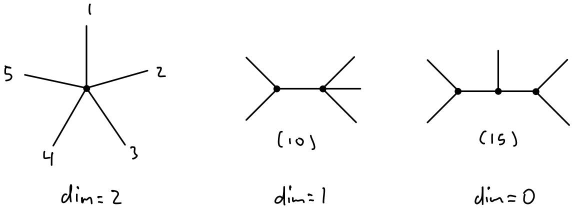

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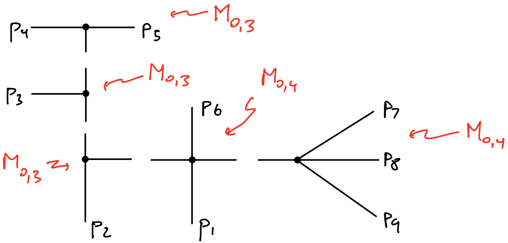

Stratum $\cong \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,4} \times \mathcal{M}_{0,4}$, dim $= 2$.

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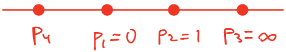
If $p_4 \neq 0, 1, \infty$:



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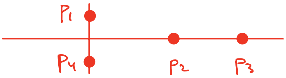

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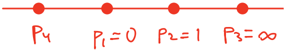

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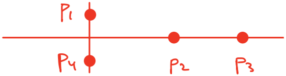
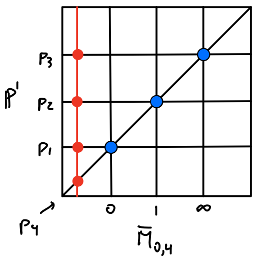


How to construct $\overline{\mathcal{M}}_{0,5}$?

First approximation: try $\overline{\mathcal{M}}_{0,4} \times \mathbb{P}^1$, where second factor parametrizes p_5:



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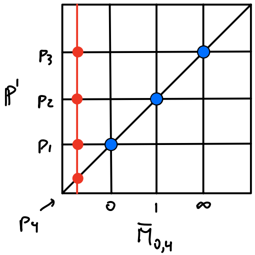

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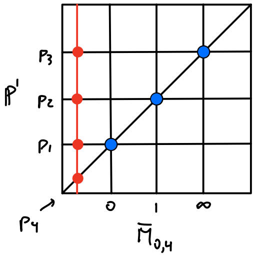

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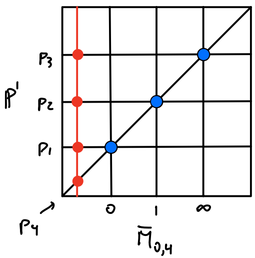

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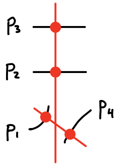
But not so when $p_4 \to p_1, p_2$, or p_3. What to do?
Answer: Blow-up the three blue points!

Each exceptional divisor is $\cong \mathbb{P}^1$. Fiber over 0:



as it should be.

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- In general, $\overline{\mathcal{M}}_{0,n+1}$ = result of sequence of blow-ups of $\overline{\mathcal{M}}_{0,n} \times \mathbb{P}^1$ along smooth subvarieties of codimension 2.
- All centers of blow-up are isomorphic to $\overline{\mathcal{M}}_{0,i+1} \times \overline{\mathcal{M}}_{0,j+1}$ with $i + j = n$.

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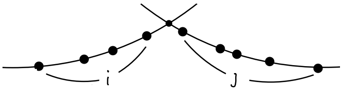

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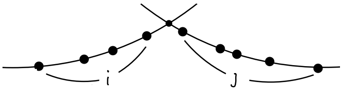

Keel's construction easily implies a recursion formula for the Grothendieck class $[\overline{\mathcal{M}}_{0,n}]$!

Class in the Grothendieck ring

Definition (Grothendieck group of varieties)

$$K(\mathrm{Var}_k) = \frac{\text{Free abelian group on isomorphism classes of varieties}}{\text{`scissor': } [X] = [Y] + [Z] \text{ if } Z \text{ closed in } X,\ Y = X \smallsetminus Z}$$

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Fact:

$$[\overline{\mathcal{M}}_{0,n}] = \sum_{k=0}^{n-3} \mathrm{rk}\, H^{2k}(\overline{\mathcal{M}}_{0,n})\, \mathbb{L}^k\,.$$

Example: $[\overline{\mathcal{M}}_{0,5}] = ?$

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$(H^1 = H^3 = 0)$.

$$1$$

$$\mathbb{L}+1$$

$$\mathbb{L}^2+5\mathbb{L}+1$$

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$$\mathbb{L}^7+466\mathbb{L}^6+13333\mathbb{L}^5+63173\mathbb{L}^4+63173\mathbb{L}^3+13333\mathbb{L}^2+466\mathbb{L}+1$$

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Example: $[\overline{\mathcal{M}}_{0,5}] = ?$
Recall $\overline{\mathcal{M}}_{0,5} \cong B\ell_{3\text{pts}}(\mathbb{P}^1 \times \mathbb{P}^1)$.

$$\begin{aligned}[B\ell_{3\text{pts}}(\mathbb{P}^1 \times \mathbb{P}^1)] &= [\mathbb{P}^1 \times \mathbb{P}^1] - 3[\mathbb{P}^0] + 3[\mathbb{P}^1] \\ &= (\mathbb{L}+1)^2 - 3 + 3(\mathbb{L}+1) \\ &= \mathbb{L}^2 + 5\mathbb{L} + 1\,. \end{aligned}$$

Therefore

$$\operatorname{rk} H^0(\overline{\mathcal{M}}_{0,5}) = 1, \quad \operatorname{rk} H^2(\overline{\mathcal{M}}_{0,5}) = 5, \quad \operatorname{rk} H^4(\overline{\mathcal{M}}_{0,5}) = 1$$

$(H^1 = H^3 = 0)$.
Keel's construction $\to$ can do the same in general. Reason:
If $B \subseteq V$ nonsingular, $d = \operatorname{codim}_B V$, then

$$[B\ell_B V] = [V] + (\mathbb{L} + \cdots + \mathbb{L}^{d-1})[B]\,.$$

By Keel,

$$[\overline{\mathcal{M}}_{0,n}] = [\overline{\mathcal{M}}_{0,n-1}] \cdot (\mathbb{L}+1) + \mathbb{L}\sum_k [B_k]$$

where B_k ranges over products $\overline{\mathcal{M}}_{0,i+1} \times \overline{\mathcal{M}}_{0,j+1}$, with care about numberings.

Bookkeeping $\to$ explicit recursive relation. For $n \geq 4$:

$$[\overline{\mathcal{M}}_{0,n}] = [\overline{\mathcal{M}}_{0,n-1}](\mathbb{L}+1) + \mathbb{L}\sum_{i=3}^{n-2} \binom{n-2}{i-1} [\overline{\mathcal{M}}_{0,i}][\overline{\mathcal{M}}_{0,n+1-i}] \,.$$

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Example:

$$\begin{aligned}[\overline{\mathcal{M}}_{0,5}] &= [\overline{\mathcal{M}}_{0,4}](\mathbb{L}+1) + \mathbb{L} \sum_{i=3}^{3} \binom{3}{i-1} [\overline{\mathcal{M}}_{0,i}][\overline{\mathcal{M}}_{0,6-i}] \\ &= (\mathbb{L}+1)^2 + \mathbb{L}\binom{3}{2} \\ &= \mathbb{L}^2 + 5\mathbb{L} + 1 \,.\end{aligned}$$

Proof of the main result

Recursion $\leftrightarrow$ differential equation for the generating function

$$M := 1 + z + \sum_{n\geq 2} [\overline{\mathcal{M}}_{0,n+1}] \frac{z^n}{n!} \ :$$

$$\frac{dM}{dz} = \frac{M}{1 + \mathbb{L}(1+z) - \mathbb{L}M}, \quad M(0) = 1\,.$$

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(Cf. Manin for a more direct way to prove this.) Our result is that

$$M = \sum_{\ell\geq 0} \frac{(\ell+1)^\ell}{(\ell+1)!} \left(((1-\mathbb{L})(1+(z+1)\mathbb{L}))^{\frac{1+\ell}{\mathbb{L}}-\ell} \prod_{j=0}^{\ell-1} \left(1 - \frac{j\mathbb{L}}{\ell+1}\right) \right) \mathbb{L}^\ell$$

satisfies this differential equation & initial condition.

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But *how* would one come up with the formula to begin with?

After-the-fact reconstruction
(this is not how it went, but how it should have gone):

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Then, you have to come up with the following.

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Let $y = (x-1)x^{-m}$. *Then*

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Then

$$M = Nw = ((1-\mathbb{L})(1+(z+1)\mathbb{L}))^{\frac{1}{\mathbb{L}}}\, w = ((1-\mathbb{L})(1+(z+1)\mathbb{L}))^{\frac{1}{\mathbb{L}}}\, x^m$$

$$\stackrel{\text{lemma}}{=} ((1-\mathbb{L})(1+(z+1)\mathbb{L}))^{\frac{1}{\mathbb{L}}} \sum_{\ell\geq 0} \binom{\frac{1}{\mathbb{L}}(\ell+1)}{\ell} \frac{1}{\ell+1} \mathbb{L}^{2\ell} \left((1-\mathbb{L})(1+(z+1)\mathbb{L})\right)^{\frac{\ell}{\mathbb{L}}-\ell}$$

and this is equivalent to the formula in the claim!

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E.g., study generating functions

$$\alpha_k(z) := \sum_{n\geq 3} H^{2k}(\overline{\mathcal{M}}_{0,n}) \frac{z^{n-1}}{(n-1)!}$$

for individual k, $0 \leq k \leq n-3$.

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Theorem (ACM: —, Chen, Marcolli)

For $0 \leq m \leq k$ *there exist polynomials* $p_m^{(k)}(z) \in \mathbb{Q}[z]$ *such that*

$$\alpha_k(z) = e^z \sum_{m=0}^{k} (-1)^m p_m^{(k)}(z) e^{(k-m)z} .$$

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What do these polynomials look like?

E.g.: $p_0^{(k)} = \dfrac{(k+1)}{(k+1)!}$, $p_1^{(1)} = 1 + z + \dfrac{z^2}{2}$, so

$$\alpha_1(z) = e^z \left(p_0^{(1)} e^z - p_1^{(1)}(z) \right) = e^z \left(e^z - 1 - z - \frac{z^2}{2} \right)$$
$$= \frac{z^3}{3!} + 5\frac{z^4}{4!} + 16\frac{z^5}{5!} + 42\frac{z^6}{6!} + 99\frac{z^7}{7!} + 219\frac{z^8}{8!} + 466\frac{z^9}{9!} + \cdots$$

is the generating function for rk $H^2(\overline{\mathcal{M}}_{0,n})$.

$$1$$

$$1\mathbb{L} + 1$$

$$\mathbb{L}^2 + 5\mathbb{L} + 1$$

$$\mathbb{L}^3 + 16\mathbb{L}^2 + 16\mathbb{L} + 1$$

$$\mathbb{L}^4 + 42\mathbb{L}^3 + 127\mathbb{L}^2 + 42\mathbb{L} + 1$$

$$\mathbb{L}^5 + 99\mathbb{L}^4 + 715\mathbb{L}^3 + 715\mathbb{L}^2 + 99\mathbb{L} + 1$$

$$\mathbb{L}^6 + 219\mathbb{L}^5 + 3292\mathbb{L}^4 + 7723\mathbb{L}^3 + 3292\mathbb{L}^2 + 219\mathbb{L} + 1$$

$$\mathbb{L}^7 + 466\mathbb{L}^6 + 13333\mathbb{L}^5 + 63173\mathbb{L}^4 + 63173\mathbb{L}^3 + 13333\mathbb{L}^2 + 466\mathbb{L} + 1$$

$$\mathbb{L}^8 + 968\mathbb{L}^7 + 49556\mathbb{L}^6 + 429594\mathbb{L}^5 + 861235\mathbb{L}^4 + 429594\mathbb{L}^3 + 49556\mathbb{L}^2 + 968\mathbb{L} + 1$$

$$\dots$$

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A polynomial $a_0 + a_1 t + a_2 t^2 + \cdots + a_d t^d$ with positive coefficients is log-concave if $\forall i = 1, \ldots, d-1$

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Beautiful example (Huh 2012): The chromatic polynomial of a graph is log-concave (w/alternating signs).

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Moral: Better think twice before making conjectures.

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For example, $T(t) = -W(-t)$ is the 'tree function':

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Surprisingly (or maybe *un*surprisingly) the generating function M has a relatively simple expression in the tree function T.

Theorem (ACM)

Let $T = T(e^z\mathbb{L})$, where T is the tree function. Then for $m > 0$ there exist polynomials $F_m(z,\tau) \in \mathbb{Q}[z,\tau]$, of degree $2m$ in z and $< 3m$ in τ, such that

$$M = \frac{T}{\mathbb{L}} \sum_{m\geq 0} \frac{(-1)^m F_m(z,T)}{(1-T)^{2m-1}}\,\mathbb{L}^m\,.$$

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More combinatorial identities involving Stirling numbers. □

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For instance, this gives the asymptotics

$$\sum_{k=0}^{n-3} \frac{\operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n})}{(k+1)^{n-1}} t^{k+1} \sim -W(-t)$$

as $n \to \infty$.

Theorem (ACM)

Let $T = T(e^z\mathbb{L})$, where T is the tree function. Then for $m > 0$ there exist polynomials $F_m(z,\tau) \in \mathbb{Q}[z,\tau]$, of degree $2m$ in z and $< 3m$ in τ, such that

$$M = \frac{T}{\mathbb{L}} \sum_{m\geq 0} \frac{(-1)^m F_m(z,T)}{(1-T)^{2m-1}} \mathbb{L}^m .$$

Proof.

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Why do we care about such things?

Application: Asymptotic log-concavity

Corollary (ACM)

For all $k \geq 0$,

$$\operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n}) \sim \frac{(k+1)^{k+n-1}}{(k+1)!}$$

as $n \to \infty$. In particular, $\forall i \geq 1$

$$\left(\frac{\operatorname{rk} H^{2i}(\overline{\mathcal{M}}_{0,n})}{\binom{n-3}{i}}\right)^2 \geq \frac{\operatorname{rk} H^{2(i-1)}(\overline{\mathcal{M}}_{0,n})}{\binom{n-3}{i-1}} \cdot \frac{\operatorname{rk} H^{2(i+1)}(\overline{\mathcal{M}}_{0,n})}{\binom{n-3}{i+1}}$$

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This says that the 'even-Poincaré' ('Chow'?) polynomial of $\overline{\mathcal{M}}_{0,n}$ is 'asymptotically ultra-log-concave'.

Huh: We should *always* care that a polynomial is log-concave; this never happens by chance.

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Also, asymptotic ultra-log-concavity is the strongest evidence to date for the following.

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The polynomial $\sum_{k=0}^{n-3} \operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n}) t^k$ has only real zeros.

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This conjecture fits a separate context, and indeed after we made it we found that people in combinatorics (e.g., Louis Ferroni) had already considered it.

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Can one prove asymptotic log-concavity for matroidal Chow rings w.r.t. maximal building sets?

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Characteristic polynomials of $\overline{\mathcal{M}}$ *and log-concavity,*
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There is much more to explore!

Thank you for your attention!