Universität Wien
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Group Theory
— Exercises —

SS 2024

Exercise 1. Let K be a field.

(1) Find a subgroup of $GL_n(K)$ isomorphic to the product of n copies of the multiplicative group $(K^\times, \cdot)$ of K.
(2) Show that the set of matrices
$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$
with $a \in K$ forms an abelian subgroup of $GL_2(K)$ isomorphic to the additive group $(K, +)$ of K.
(3) Show that the set of matrices
$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$
with $a, b, c \in K$ forms a non-abelian subgroup of $GL_3(K)$. Find an explicit formula for all powers A^n, with $n \in \mathbb{Z}$.
(4) Describe this group when $K = \mathbb{F}_2$ is the field with two elements.

Exercise 2. Show that $\mathrm{Aut}(\mathrm{S}_3) \cong \mathrm{Aut}(\mathrm{C}_2 \times \mathrm{C}_2) \cong \mathrm{S}_3$.

Exercise 3. Find out (without proof) the number n of different groups of order 1536. Show that n is not the sum of three integer cubes, but find integers x, y, z with $1536 = x^3 + y^3 + z^3$.

Exercise 4. Determine the number of elements in the group $GL_n(\mathbb{F}_q)$.

Exercise 5. Write down the subgroup lattice (list of all subgroups, with containment relations indicated) of the following groups: the cyclic group C_n; the group $C_p \times C_p$ for p a prime; and the symmetric group S_4.

Exercise 6. Show that every subgroup of the quaternion group Q_8 is normal. Is this true for the dihedral group D_4?

Exercise 7. Show that the groups $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{R}/\mathbb{Q}$ cannot be isomorphic.

Exercise 8.

(1) Show that for all $n \geq 1$, the symmetric group S_n has a subgroup of index n.
(2) **Extra, optional** Show that for all $n \geq 5$, if $H < S_n$ is a subgroup of index $m < n$, then $m = 2$.

Exercise 9. Show that every finite group can be embedded into a finite simple group.

Exercise 10. Determine the order of the group $G = SL_2(\mathbb{F}_3)$. Prove that this group is not isomorphic to a symmetric group S_n.

Exercise 11. Let $\mathrm{Sym}(\mathbb{N})$ be the group of permutations of a countable set $\mathbb{N}$. Let $S_\infty \subset \mathrm{Sym}(\mathbb{N})$ be the subset of all permutations which fix all but finitely many elements in $\mathbb{N}$.

(1) Prove that S_∞ is a non-trivial normal subgroup of $\mathrm{Sym}(\mathbb{N})$.
(2) Prove that the group S_∞ is an increasing union of finite symmetric groups. Deduce that there exists a non-trivial homomorphism $\varphi\colon S_\infty \to C_2$.
(3) Prove that the group $A_\infty = \ker \varphi$ is an increasing union of finite alternating groups. Deduce that A_∞ is simple.

Exercise 12 - optional. Let G be an infinite group with the property that every nontrivial subgroup H of G is isomorphic to a cyclic group of order p for a fixed prime number p. Prove that G is simple.

It is a highly nontrivial fact that such groups G exist at all, but they do for sufficiently large p, and are called *Tarski monsters*.

Exercise 13. Consider the following group actions (X, G). In each case, describe the set of orbits, and the stabilizer of a chosen point from each orbit.

(1) Let $G_1 = S_n$ act on the set $X_1 = \{1, \ldots, n\}$ by permutations.
(2) Let $G_2 \cong (\mathbb{R}, +)$ act on the plane $X_2 = \mathbb{R}^2$ by horizontal translations.
(3) Let X_3 be the (boundary of the) unit square in the plane $\mathbb{R}^2$. Let $G_3 = \mathrm{Sym}(X_3)$ be the set of symmetries of the square.

Exercise 14. Let G be a group, and H a subgroup of G of index 2. Show that H is a normal subgroup in G. (Hint: how many cosets are there?)

Exercise 15. Let $G = D_n$ be the dihedral group, the group of symmetries of the regular n-gon in the plane. Describe the conjugacy classes of D_n, and verify the class equation. (See Example 2.2.4 in the notes for an example. You will need to distinguish two different cases.)

Exercise 16. Show that there is no simple group of order 312.

Exercise 17. Let G be a group of order 30. Show that either G has a normal subgroup N of order 5 and a subgroup H of order 3, or G has a normal subgroup N of order 3 and a subgroup H of order 5. Deduce that G has a normal subgroup K isomorphic to the cyclic group C_{15}.

Exercise 18. Let G be a non-abelian group of order less than 60. Show that G is not simple.

Exercise 19 - extra.

(1) Show that the action of S_5 on its Sylow-5 subgroups defines an injective group homomorphism $\varphi\colon S_5 \to S_6$ with transitive image.
(A subgroup H of S_n is called transitive, if the action of H on $\{1, \dots n\}$ is transitive, i.e. has one orbit.)
(2) Let $H = \mathrm{im}\varphi < S_6$. By using the action of S_6 on the cosets of H, find a non-inner automorphism of S_6, showing that S_6 is not complete.

Exercise 20. Show that the quaternion group Q_8 cannot be written as a semidirect product of two non-trivial subgroups.

Exercise 21. Let G be a group of order 30. As we saw in Exercise 17, G contains a normal subgroup $K = \langle g \rangle$ that is cyclic of order 15. By Cauchy's theorem, G also contains an element h of order 2.

(1) Show that
$$G = \{g^i h^j : 0 \leq i \leq 14, 0 \leq j \leq 1\}.$$
(2) Let $\psi \in \mathrm{Aut(K)}$ satisfy $\psi^2 = id_K$. Show that $\psi : g \mapsto g^i$, where $i \in \{1, 4, 11, 14\}$.
(3) Deduce that there are exactly four groups of order 30, up to isomorphism.

Exercise 22. Classify all groups that arise as semidirect products of $\mathbb{Z}$ by $\mathbb{Z}$. Show that there is one abelian group and one non-abelian group.

Exercise 23. Let
$$D_\infty = \langle r, s \mid s^2 = e, srs^{-1} = r^{-1} \rangle$$
be the *infinite dihedral group*, satisfying the same relations as the dihedral group D_n, except that the order of r is assumed infinite. Show that D_∞ has a normal subgroup isomorphic to the infinite cyclic group $C_\infty \cong \mathbb{Z}$, and a complementary subgroup isomorphic to C_2. Deduce that D_∞ is a semidirect product.

Exercise 24. Let
$$G = \langle x, y \mid x^2 = y^2 = e \rangle.$$

(1) Show that G is infinite.
(2) Let $z = xy$. Show that every element of G can be uniquely written as z^k or yz^k, where k is an integer.
(3) Deduce that G can be written as a semi-direct product, identifying it with the infinite dihedral group D_∞ from the previous Exercise.
(4) (Optional) By considering appropriate reflections and translations, show that G may be identified with the isometry group of the integers $\mathbb{Z}$, considered as a subset of the real line with the Euclidean metric.

Exercise 25. Let $G = S_7$ and $x = (1234567)$, $y = (265734)$ in G. Let $H = \langle x, y \rangle$. By experimenting with different combinations of x, y, describe the structure of the group H. Find a composition series for H and determine its quotients.

Exercise 26. The projective special linear group $\mathrm{PSL}_2(\mathbb{F}_p)$ is the group of 2×2 matrices over the finite field $\mathbb{F}_p$ of determinant 1, modulo diagonal matrices.

(1) Determine the order of $\mathrm{PSL}_2(\mathbb{F}_p)$ for a prime $p > 2$.
(2) (Optional) Show that $\mathrm{PSL}_2(\mathbb{F}_p)$ is isomorphic to a subgroup of S_{p+1}.
(3) Assuming the result in (2), show that the group $\mathrm{PSL}_2(\mathbb{F}_3)$ is isomorphic to A_4 and that $\mathrm{PSL}_2(\mathbb{F}_5)$ is isomorphic to A_5.
(4) Is $\mathrm{PSL}_2(\mathbb{F}_7)$ also isomorphic to an alternating group?

Exercise 27. Determine all composition series for the quaternion group Q_8.

Exercise 28. Show that the infinite dihedral group D_∞ from Exercise 23 is solvable. Compute its derived length.

Exercise 29. Let G be a non-abelian group of order pq, for distinct primes p, q. Show that G is solvable. Compute its derived length.

Exercise 30. Determine the values of n for which the symmetric group S_n is solvable.

Exercise 31. Let F be a field, and let $B \subset \mathrm{GL}_n(F)$ be the subgroup of the general linear group of F consisting of upper triangular matrices. Prove that B is a solvable group.

Exercise 32 - optional. Let G be a finite solvable group, all of whose Sylow subgroups are abelian. Prove that $Z(G) \cap G' = 1$.