Universität Wien
Prof. B. Szendrői

Commutative algebra and algebraic geometry
— Exercise Sheet 3 with extended hints —

WS 2023

All rings R are commutative and unital. The symbol k denotes an algebraically closed field. There are some hints at the end of the problem sheet.

Exercise 1. Describe carefully all Zariski closed subsets of the affine plane $\mathbb{A}^2_k$. Be sure to prove all statements you use. Which of these subsets are irreducible?

Exercise 2. Deduce from the result of the previous exercise that the Zariski topology on $\mathbb{A}^2_k$ does not agree with the product topology of $\mathbb{A}^1_k \times \mathbb{A}^1_k$ with the Zariski topology on each factor.

Exercise 3. Describe the irreducible components of $V(J_i) \subset \mathbb{A}^3_k$, for each of the following ideals $J_i \lhd k[x, y, z]$.

(1) $J_1 = (xy, xz)$.
(2) $J_2 = (y^2 - x^4, x^2 - 2x^3 - x^2y + 2xy + x^2 - y^2)$.
(3) $J_3 = (xy + yz + xz, xyz)$.
(4) $J_4 = ((x - z)(x - y)(x - 2z), x^2 - y^2z)$.

Is each of the ideals J_i reduced? If not, find an element of $\sqrt{J_i} \setminus J_i$.

Exercise 4. A *chain of prime ideals* in a ring R *of length* m is a nontrivial chain

$$J_0 \subsetneq J_1 \subsetneq \ldots \subsetneq J_m$$

of ideals, with each $J_i \lhd_p R$ prime. (Note the indexing.)

(1) Find a chain of prime ideals of length n for the ring $R = k[x_1, \ldots, x_n]$.
(2) For $n \leq 2$, show that any chain of prime ideals of $R = k[x_1, \ldots, x_n]$ has length at most n.

Exercise 5. Let $R = k[x, y, z]/(xy, xz)$ be the ring corresponding to the ideal J_1 of Exercise 3. Find maximal (non-extendable) chains of prime ideals of R of different lengths.

Hint for Exercise 1. Any closed set is a finite union of irreducibles. Consider a closed irreducible subset $X = V(I)$ for a radical, prime ideal I. Take an element $f \in I$ of smallest degree, then by prime-ness of I, f must also be irreducible. If $I = (f)$, then $X = V_f$, a hypersurface. Otherwise, there is a $g \in I \setminus (f)$, a polynomial that is not a multiple of f, so relatively prime to f. Then, by Gauss' lemma in the ring $k[x][y]$, we can find a linear combination $af + bg \in I$ which is purely a polynomial in x. By irreducibility, this polynomial must be linear. Similarly, I contains a linear polynomial in y. Hence I is the (maximal) ideal of a point in the plane. So non-trivial irreducible closed subsets of the affine plane are hypersurfaces defined by principal ideals, and points.

Hint for Exercise 2. In the Zariski topology of $\mathbb{A}^1_k$, the only nontrivial closed subsets are points. So in the product topology of Zariski $\mathbb{A}^1_k \times \mathbb{A}^1_k$, the only closed subsets are unions of vertical and horizontal lines, and finite point sets. This clearly does not agree with the list of closed subsets of $\mathbb{A}^2_k$, for example the diagonal line $V(y - x)$ is not of this form.

Hint for Exercise 3. Instead of thinking about the ideals J_i directly, we can think of each $V(J_i)$ as the solution set to a set of equations, and solve the equations.

(1) $V(J_1) = V(x) \cup V(y, z)$, the union of the (y, z)-plane and the x-axis.
(2) $V(J_2) = V(x, y) \cup V(x - 1, y - 1) \cup V(x + 1, y - 1) \cup V(2x - 1, 4y + 1)$, the union of four lines.
(3) $V(J_3) = V(x, y) \cup V(y, z) \cup V(z, x)$, the union of the three coordinate axes.
(4) $V(J_4) = V(z, x) \cup V(z - 1, x - y) \cup (y, x) \cup (x - 2z, y^2 - 4z) \cup (x - z, y^2 - z)$, a union of three lines and two planar parabolas.

Hint for Exercise 4. (1) We can take $(0) \subset (x_1) \subset (x_1, x_2) \subset \ldots \subset (x_1, x_2, \ldots, x_n)$, corresponding to the chain of irreducible varieties given by n-space containing hyperplane containing etc etc containing line containing point. (2) For the affine line, this is easy as $k[x]$ is a PID, and its its nontrivial prime ideals correspond to points. For the affine plane, this was basically explained in the argument of Exercise 1 above: the only possible chains of irreducibles in the plane are plane containing irreducible hypersurface containing point.

Hint for Exercise 5. (Chains of) prime ideals of R/J_1 correspond to (chains of) prime ideals of R containing J_1. Thus at the "bottom" of every relevant chain lies a minimal prime of R, corresponding to an irreducible component of $V(J_1)$. One component $V(y, z)$ is a line, so the only possible chain is $(y, z) \subset (x - \alpha, y, z)$, corresponding to ($x$-axis) containing the point $(\alpha, 0, 0)$. The other component $V(x)$ is a plane, so there are length-two chains possible, for example $(x) \subset (x, y - \beta) \subset (x, y - \beta, z - \gamma)$ corresponding to (y, z)-plane containing line containing point $(0, \beta, \gamma)$.