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Commutative algebra and algebraic geometry
— Exercise Sheet 2 with extended hints —

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All rings R are commutative and unital. The symbol k denotes a field. There are some hints at the end of the problem sheet.

Exercise 1. Let $I = \langle x, y\rangle \lhd R = k[x, y]$. Consider the R-module homomorphism

$$\varphi\colon R \oplus R \to I$$

which maps the basis vectors $(1, 0)$, $(0, 1)$ of the free module to $x, y \in I$. Find the kernel of φ. Find a short exact sequence of R-modules that includes the map φ. Show that the kernel of φ is a free R-module, but that I is not a free R-module.

Exercise 2. Let $I = \langle x, y, z\rangle \lhd R = k[x, y, z]$. Consider the R-module homomorphism

$$\psi\colon R \oplus R \oplus R \to I$$

which maps the basis vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ of the free module to $x, y, z \in I$. Find the kernel of ψ.

Exercise 3. Let

$$0 \to L \to M \to N \to 0$$

be a short exact sequence of R-modules. Show that if L, N are finite, then so is M. Show that if R is Noetherian, then the implication is reversible: if M is finite, then so are L, N.

Exercise 4. Let R be a k-algebra which is finite dimensional as a k-vector space. Prove that R is both Noetherian and Artinian (satisfies both ACC and DCC for ideals).

Exercise 5. Let $R = k[x_1, \dots, x_n]$. Show that for any non-negative integers $m_1, \dots, m_n$, the ideal $I = \langle x_1^{m_1}, \dots, x_n^{m_n}\rangle \lhd R$ is *of finite codimension*, meaning that the quotient R/I is a finite dimensional k-vector space (and hence an Artinian ring by Exercise 4). Compute $\dim_k R/I$.

Exercise 6. Let R be an Artinian integral domain. Show that R is a field. Deduce that every prime ideal of an Artinian ring is maximal.

Exercise 7. Let $R = k[x, y]/\langle y^2 - x^2 - x^3\rangle$. Show that R is a domain. Find the normalization of R.

Hint for Exercise 1. The map φ takes $(f, g) \in R \oplus R$ to $xf + yg \in R$. We need to find the kernel, so (f, g) such that $xf + yg = 0$. By unique factorisation in R, this must mean that $(f, g) = h \cdot (-y, x)$. So the kernel is isomorphic to a free module under the map $h \mapsto h \cdot (-y, x)$. The ideal I could only be a free module if it was principal, but it cannot be generated by one element.

Hint for Exercise 2. This starts in the same way as Exercise 1, but the answer is a little more involved: you will need more than one generator for the kernel.

Hint for Exercise 3. In one direction, if L, N are finite, consider a set of generators for L, and lift to M a set of generators of N. Show that these generate M. In the other direction, if M is finite, then obviously N is finite, since a finite set of generators for M projects to a generating set of N. The fact that every submodule of a finitely generated module is itself finitely generated is one of the defining properties of a Noetherian ring.

Hint for Exercise 4. Ideals in a k-algebra are in particular linear subspaces, so we can use the notion of k-dimension. For a strictly descending or ascending chain, the dimensions must go down, respectively up, but we can only have finitely many such steps.

Hint for Exercise 5. A finite spanning set, in fact a basis, for the quotient R/I is given by $\prod_i x_i^{a_i}$ with $0 \le a_i < m_i$. In particular, the dimension is $\prod_i m_i$.

Hint for Exercise 6. For any $r \in R$, consider the descending chain $\langle r \rangle \supseteq \langle r^2 \rangle \supseteq \langle r^3 \rangle \dots$. When this stabilizes, deduce a relation $r^n = sr^{n+1}$. Use this to find an inverse for r.

Hint for Exercise 7. Let $t = y/x$; show that the normalization is $R \subset k[t] \subset \mathrm{FoF}(R)$.