Extending characterizations of truthful mechanisms from subdomains to domains.

Angelina Vidali *

Theory and Applications of Algorithms Research Group
University of Vienna, Austria

Abstract. The already extended literature in Combinatorial Auctions, Public Projects and Scheduling demands a more systematic classification of the domains and a clear comparison of the results known. Connecting characterization results for different settings and providing a characterization proof using another characterization result as a black box without having to repeat a tediously similar proof is not only elegant and desirable, but also greatly enhances our intuition and provides a classification of different results and a unified and deeper understanding. We consider whether one can extend a characterization of a subdomain to a domain in a black-box manner. We show that this is possible for n-player stable mechanisms if the only truthful mechanisms for the subdomain are the affine maximizers. We further show that if the characterization of the subdomain involves a combination of affine maximizers and threshold mechanisms, the threshold mechanisms for the subdomain cannot be extended to truthful mechanisms for the union of the subdomain with a (very slight) affine transformation of it. We also show that for every truthful mechanism in a domain there exists a corresponding truthful mechanism for any affine transformation of the domain. We finally plug in as a black box to our theorems the characterization of additive 2-player combinatorial auctions that are decisive and allocate all items (which essentially is the domain for scheduling unrelated machines) and show that the 2-player truthful mechanisms of any domain, which is strictly a superdomain of it are only the affine maximizers. This gives a unique characterization proof of the decisive 2-player mechanisms for: Combinatorial Public Projects, Unrestricted domains, as well as for Submodular and Subadditive Combinatorial Auctions that allocate all items.

1 Introduction

Our results and motivation. Suppose that you are in a conference committee and need to choose only one of the following two papers for acceptance: Both give a characterization for n-player stable (/2-player that allocate all items) Combinatorial Auctions, the first for the case when

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the players have additive valuations and the second for the case when the players have sub-modular valuations. Which one will you accept? Is one of the results stronger, in the sense that the two papers can be merged and the weaker result can be derived as a corollary?

In this paper we address and only partially answer the following questions: Which domains have the same characterization? Can we classify different domains in terms of how difficult it is to characterize them or how rich are the mechanisms allowed? Does a characterization for a "more difficult" domain automatically imply a proof for domains that are lower in this hierarchy? Can we establish a bijection between the mechanisms involved in the characterization of different domains? This work gives some explanations we would have liked to find, back when we started working on characterization results and wondered what do the results about other slightly different domains tell us about the domain we were primarily interested in.

Roberts [11](1979) gives an elegant proof, which shows that the only truthful mechanisms for the Unrestricted domain are the affine maximizers. He also gets the Gibbard-Sattherwhaite Theorem (1973) for voting systems as a corollary. For more "restricted" multi-parameter domains, there exist truthful mechanisms other than affine maximizers (see e.g. [15, 10, 4]). An important question, posed in [18, 14], is to determine how much we need to restrict the domain in order to admit truthful mechanisms different than the affine maximizers. Here we show that for the case of 2 players, this transition domain is the additive combinatorial auctions domain: We show that if we slightly enrich the possible valuations, the Threshold mechanisms involved in the characterization [4] seize to be truthful and the only truthful mechanisms that remain are the affine maximizers.

A crucial observation is: the more "unrestricted" the domain of valuations, the fewer the possible truthful mechanisms. An intuitive explanation for this is that in larger domains there are more inputs that need to satisfy the conditions for truthfulness. On the other hand, *this intuition may be misleading*: Given that a sub-domain admits as truthful mechanisms only the affine maximizers does not immediately imply that the domain also admits the same mechanisms; there may be other mechanisms which when restricted to sub-domain are exactly the affine maximizers. In particular, we do not know whether this is possible for non-stable mechanisms. We also do not know if this is possible for domains where the possible truthful mechanisms are richer than combinations of affine maximizers and threshold mechanisms.

Related work. The starting point of characterization attempts goes back to Robert's [11] result. Many papers tried to extend this very elegant proof [16, 10, 19], while others tried different proof techniques [15, 4, 8, 16]. (As the literature in combinatorial auctions is vast we refer the reader to [18] Chapter 11 and the references within and mention here only some recent results.) Computational complexity impossibility results for maximal in range mechanisms (mechanisms obtained by restricting the possible allocations among which an affine maximizer chooses its allocation) where shown in [2, 7]. Dobzinski [6] shows that every universally truthful randomized mechanism for combinatorial auctions with submodular valuations that provides an approximation ratio of $m^{\frac{1}{2}-\epsilon}$ must use exponentially many value queries. Krysta and Ventre show that if verification is introduced sub-modular combinatorial auctions become tractable [13]. Nisan and Ronen introduced the truthful scheduling unrelated machines problem [17, 4, 12, 10, 20]. A characterization of decisive truthful 2-player mechanisms in terms of affine minimizers and threshold mechanisms was given in [4] and it also implies a characterization for additive combinatorial auctions, which we will use here as a black box. We will alternatively use as a black box the characterization of n-player stable (/2-player that allocate all items) subadditive combinatorial auctions by Dobzinski and Sundararajan [10].

1.1 Definitions and preliminaries

A mechanism is *decisive* when (for fixed values of the other players) a player can enforce any outcome (allocation), by declaring very high or very low values. A mechanism is called *stable* if the following holds: *For fixed valuations v_{-i}, the allocation a_i of player i determines uniquely the allocation a_{-i} of the other players.* (In other words: Fix v_{-i}, then for all v_i for which player i has allocation a_i the allocation a_{-i} is the same.) *Stability can be assumed without loss of generality for n-player unrestricted domains, combinatorial public projects and 2-player auctions where all items are allocated.* It is too restrictive for combinatorial auctions with $n \geq 3$ players (see [15] Example 4), however all known characterization results [11, 15, 19, 10, 4, 16, 10] heavily rely on stability, or characterize domains where stability can be assumed essentially without loss of generality. Stability is implied by S-MON or IIA (see [15, 1, 10] for a discussion on these conditions and proofs).

We will denote any finite domain of valuations D as a set of matrices [3]. We have one matrix for each valuation function $v = (v_1, \ldots, v_n) : A \to \mathbb{R}^n$ that belongs to the domain. This matrix has one column for each alternative $a \in A$ and one row for each player. The valuation v_i of player i is a vector of numbers with one coordinate for each possible alternative. Let V_i set of all possible such vectors for player i. (The domain is the set of all possible inputs of the mechanism.) Under this notation the domain of unrestricted valuations [11] contains all possible matrices with real values. We will say that S_i *is a subdomain of* V_i if $S_i \subseteq V_i$. We will say that D *is a subdomain of* D' if $D \subseteq D'$.

2 Our results

Derivation of the characterization of a domain from the characterization of one of its sub-domains. Suppose we know which mechanisms are truthful for a given domain, does this tell us which mechanisms are truthful for any super-domain of it? The first reaction may be: we can read the proofs and produce (tediously) similar ones. But then the mechanism for the bigger domain has to satisfy truthfulness for a superset of the input space. Are then the mechanisms for the bigger domain a subset of the mechanisms for the sub-domain? We have to be careful: it is true that if a mechanism is truthful for the bigger domain, then its restriction to the smaller domain is a truthful mechanism for the smaller domain (for which we assumed that we know a characterization). However it then remains to describe the mechanism for the additional inputs we allowed by enlarging the domain.

We need Lemma 1 in order to avoid assuming decisiveness in Theorem 1. It shows that shows that by truthfulness the range of the mechanism for the bigger domain is the same as the range of it's restriction to the subdomain. In other words *if the characterization that you plug in Theorem 1 or 4 as a black box does not require decisiveness then the characterization you obtain for the superdomain does not require decisiveness either*. Lemma 2 is the core argument for proving Theorem 1.

Lemma 1. *Let S_i be the domain of additive valuations, or any super-domain of it, and $S_i \subseteq V_i$. For for fixed v_{-i}, consider a truthful social choice function $f(\cdot, v_{-i}) : S_1 \times \ldots \times V_i \times \ldots \times S_n \to A$, and restrict it to the sub-domain $S_1 \times \ldots \times S_n$. If the range of the restricted function is a set of alternatives A, then A is also the range of $f(\cdot, v_{-i})$.*

Lemma 2. *Start with an affine maximizer M defined for the domain of valuations $S_1 \times \ldots \times S_n$ and then consider the bigger domain $S_1 \times \ldots \times V_i \times \ldots \times S_n$, where $S_i \subseteq V_i$. There is a unique way to extend M (preserving truthfulness) to an n-player stable (/2-player that allocates* *all items) mechanism for the bigger domain, namely an affine maximizer defined by the same λ, γ as M.*

Note that if we did not require the mechanism to be truthful, then there would have been many possibilities to extend the mechanism to a mechanism that would not be an affine maximizer for the whole domain.

Theorem 1. *Let V be a sub-domain of the domain of unrestricted valuations and superdomain of the domain of additive valuations. If the only possible truthful n-player stable mechanisms for V are affine maximizers, then the same holds for every super-domain of V.*[1]

Plugging in as a black box the characterization of n-player stable scalable (if you multiply all entries of the input matrix by the same number the allocation remains the same) mechanisms for subadditive combinatorial auctions [10] we get:

Corollary 1. *The only truthful n-player stable (/2-player that allocate all items) mechanisms with at least 3 outcomes for any superdomain of Subadditive Combinatorial Auctions that satisfy scalability are affine maximizers. This proves that the only truthful scalable mechanisms for (a) Unrestricted domains as well as for (b) stable Combinatorial Auctions (with general valuations) are affine maximizers.*

Affine transformations of domains. Note that the next theorem holds for any choice of the domain D, and not only for the domain of additive valuations. It implies that if we characterize all possible mechanisms for a domain of valuations D then the same characterization holds for all domains we get by translating D.

If D is the matrix representation of a domain we denote by $\lambda D + c$ the following affine transformation of D: Multiply the valuations of each player i by a positive constant λ_i and add a matrix of constants c, with one row c^i for each player and one column for each possible allocation. For example the following is an affine transformation of 2-player combinatorial auctions:

$$\begin{pmatrix} c^1_\emptyset & \lambda_1 v_1(\{1\}) + c^1_{\{1\}} & \lambda_1 v_1(\{2\}) + c^1_{\{2\}} & \lambda_1 v_1(\{1,2\}) + c^1_{\{1,2\}} \\ \lambda_2 v_2(\{1,2\}) + c^2_{\{1,2\}} & \lambda_2 v_2(\{2\}) + c^2_{\{2\}} & \lambda_2 v_2(\{1\}) + c^2_{\{1\}} & c^2_\emptyset \end{pmatrix}.$$

Theorem 2. *There is a bijection between the mechanisms for D and the mechanisms of $\lambda D + c$. That is the mechanism with the same allocation*

[1] The proof of Theorem 1 for the 2-player case, goes along exactly the same lines as the proof of Lemma 3.1 [5] by Dobzinski. (The statement of that Lemma involves a different setting, with which we don't deal with in this paper, that of two-player multi-unit auctions.)

and payments $p' = \lambda \cdot p + c$ is also truthful for $\lambda D + c$. This holds for any number of players n.

Threshold mechanisms and their payments. The characterization in [4] reveals the class of threshold mechanisms, which are truthful, very simple in their description, and not (necessarily) affine maximizers. The immediate question is whether there exist other domains for which threshold mechanisms are truthful. We describe here the truthful threshold mechanisms for the translated domain $\lambda D + c$. A threshold mechanism for the additive combinatorial auctions (/scheduling) domain is one for which there are threshold functions h_{ij} such that the mechanism allocates item j to player i if and only if $v_i(\{j\}) \geq h_{ij}(v_{-i})$.

Theorem 3. *If D is the domain of additive valuations and a_i is the set of items allocated to player i, then a mechanism for the domain $\lambda D + c$ is a threshold mechanism if and only if it satisfies $p_i(a_i, v_{-i}) - c^i_{a_i} = \sum_{j \in a_i} \left(p_i(\{j\}, v_{-i}) - c^i_{\{j\}}\right)$.*

How to vanish threshold mechanisms. Here we show how starting from the additive domain and slightly enriching the domain of possible valuations we obtain a domain that does not admit any truthful threshold mechanisms. This shows that truthful threshold mechanisms are specific for the domain of additive valuations and its affine transformations and that they cannot be generalized for richer domains.

Let S_i be the set of all valuation functions v_i that are additive. We define the set of valuation functions $S_i + \delta$ as follows: $S_i + \delta$ contains all valuation functions v'_i with $v'_i(a_i) = \sum_{j \in a_i}^{m} v_i(\{j\}) + (|a_i| - 1) \cdot \delta$ where $\delta \neq 0$ is some tiny constant. That is $v_i \in S_i$ and $v'_i \in S_i + \delta$ agree only on the valuation for getting singletons and the emptyset and differ by $\delta \times$ (size of the bundle -1) for bigger bundles. There exist many choices of valuations for which our proofs hold. Of course if you would like to obtain the characterization, say, of sub-modular auctions, you should mind to make a choice of valuations that are submodular.

We start with two domains, that differ slightly in the valuations one of the players. Each one separately admits truthful threshold mechanisms, but their union does not:

Lemma 3. *Consider a truthful mechanism for the domain $\left(S_1 \cup (S_1 + \delta)\right) \times S_2 \times \ldots \times S_2$. If it is a threshold mechanism when restricted to $S_1 \times S_2 \times \ldots \times S_n$, then it is non-threshold when restricted to $(S_1 + \delta) \times S_2 \times \ldots \times S_2$. Consequently for the domain $\left(S_1 \cup (S_1 + \delta)\right) \times S_2 \times \ldots \times S_2$ threshold mechanisms are non-truthful.*

Theorem 4. *If the only truthful mechanisms for the domain* $S_1 \times S_2 \times \ldots \times S_2$ *are either affine maximizers or threshold mechanisms, then the only truthful stable mechanisms, for the domain* $\big(S_1 \cup (S_1 + \delta)\big) \times S_2 \times \ldots \times S_2$*, or any super-domain of it, are affine maximizers.*

Applying our tools for the known characterization. The machinery we just developed opts for a characterization of stable truthful mechanisms for additive combinatorial auctions/scheduling for n players, but this is an important open problem. We only have one [4] for 2-player mechanisms, that are decisive and allocate all items.

The characterization in [4] is only for additive valuations, applying Theorem 2 it also applies to any affine transformation of the domain of additive valuations. We can now state our main Theorem:

Theorem 5. *The only possible truthful mechanisms, for* $S_1 \cup (S_1 + \delta) \times S_2$ *or any super-domain of it, that have at least 3 outcomes, are decisive and allocate all items are the affine maximizers. Consequently the only truthful 2-player mechanisms that are decisive and have at least 3 outcomes for: (a) Combinatorial Auctions with Submodular or Subadditive or Superadditive valuations that allocate all items, as well as for (b) the Unrestricted domain and Combinatorial Public Projects, are the affine maximizers.*

3 Conclusion and future directions

Submodular combinatorial auctions is an important domain [6, 9, 18], whose characterization (assuming decisiveness and that all items are allocated) we obtain in this work for the first time almost for free. Although we characterize at once the very rich class of superdomains of additive combinatorial auctions, the most important aspect of our work is not in characterizing new domains, but in classifying them in terms of which domain's characterization we can use as a black box in order to obtain the characterization of all of it's super-domains and obtaining unified proofs and a unified understanding. An important reason why we used this specific characterization [4] as a black box is that it is the only one that involves truthful mechanisms that are not affine maximizers. We enrich the domain very slightly and these mechanisms seize to be truthful, thus the domain of additive combinatorial auctions is the transition domain [18, 14] where the affine maximizers are not any more the only truthful mechanisms.

Of course the big open question still remains to obtain characterizations of domains that admit non-stable mechanisms. However the approach of classifying domains in the way we propose provides a more

thorough understanding of the existing techniques and results and adds rigor to an intuition that was on the same time helpful and misleading.
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