Malfatti's problems

Emil Kostadinov
Mathematics High School "Acad. S. Korolov", 11 grade
Blagoevgrad, Bulgaria

Abstract

The purpose of the present project is to consider the original Malfatti problem of cutting three circles of maximal combined area from a given triangle as well as various related problems of this type. The project is divided into five sections. In Section 1 some useful formulae for the radii of the Malfatti circles of a triangle are given. In Section 2 a trigonometric inequality (Zalgaller's inequality) for the angles of an acute triangle is proved. In particular, it is shown that the Malfatti circles never give a solution of the Malfatti problem. In Section 3 we solve the Malfatti problems for equilateral triangles and squares by using an approach different from the one proposed by Zalgaller and Loss. It is based on the so-called dual Malfatti problems. In Section 4 the possibility for solving Malfatti type problems by means of the greedy algorithm is discussed. In the last section of the project we state some open problems that might be objects of future investigations.

1 Introduction

In 1803 the Italian mathematician Gianfrancesco Malfatti [3] posed the following problem:

Given a right triangular prism of any sort of material, such as marble, how shall three circular cylinders of the same height as the prism and of the greatest possible volume be related to one another in the prism and leave over the least possible amount of material?

It is clear that this problem is equivalent to the plane problem of cutting three circles from a given triangle so that the sum of their areas is maximized. As noted in [1], Malfatti, and many others who considered the problem, assumed that the solution would be the three circles that are tangent to each other, while each circle is tangent to two

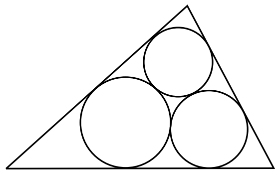
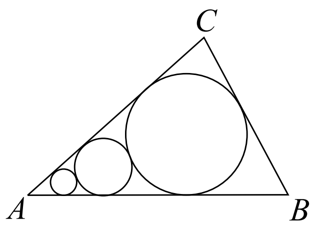


Fig. 1

sides of the triangle (Fig. 1). These circles have became known in the literature as the Malfatti circles, and we refer the reader to [2] and [5] for some historical remarks on the derivation of their radii.

In 1929, H. Lob and H. W. Richmond [2] gave a very simple counterexample to the Malfatti conjecture. Namely, they noted that in the case of an equilateral triangle the sum of the areas of the incircle and the two circles tangent to it and inscribed in two angles of the triangle is greater than the sum of the areas of the Malfatti circles. Moreover, M. Goldberg [1] proved in 1967 that the Malfatti circles never give a solution of the Malfatti problem.

To the best of the author's knowledge the Malfatti problem was first solved by V. Zalgaller and G. Loss [6] in 1991. They proved that for any triangle ABC with angles $\angle A \le \angle B \le \angle C$ the solution of the Malfatti problem is given by the three circles k_1, k_2 and k_3, where k_1 is the incircle of the triangle, k_2 is the circle inscribed in $\angle A$ and externally tangent to k_1, and k_3 is either the circle inscribed in $\angle A$ and externally tangent to k_2 (Fig. 2) or the circle inscribed in $\angle B$ and externally tangent to k_1 (Fig. 3), depending on whether $\sin \frac{A}{2} \ge \operatorname{tg} \frac{B}{4}$.



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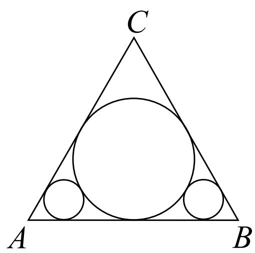
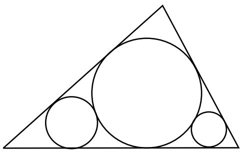


Fig. 3

The proof of the above result given by Zalgaller and Loss is very long (more than 25 pages) and at some key points it uses computer computations. In short the proof goes as follows. The first step is to show that it is enough to consider the following 14 configurations of the three circles: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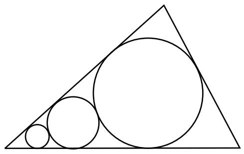
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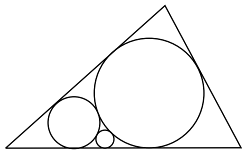
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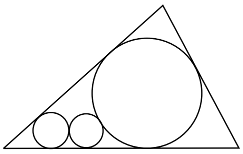
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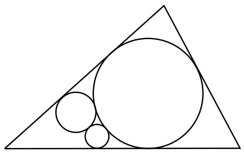
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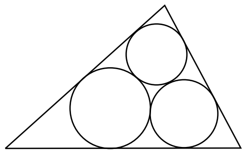
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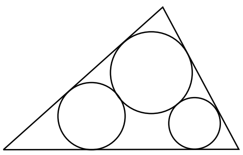
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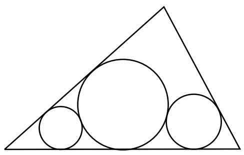
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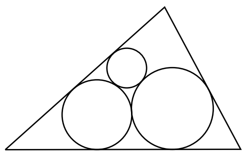
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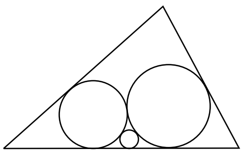
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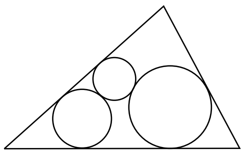
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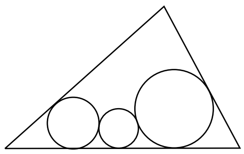
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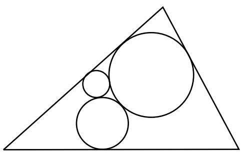
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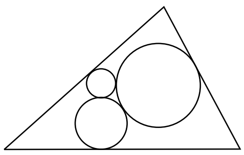
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Then configurations 3 – 14 are excluded by using case-by-case arguments showing that the respective three circles have less combined area than those of configurations 1 or 2.

The purpose of this project is two-fold. Firstly, we discuss in details one of the main difficulties in the Zalgaller and Loss solution of the Malfatti problem, namely the exclusion of configuration 6. Note that this is equivalent to prove that the Malfatti circles never give a solution of the Malfatti problem. Secondly, we give new solutions of the Malfatti problems for an equilateral triangle or a square by using the so-called *dual Malfatti problems*. At the end of the project we discuss the possibility of applying the so-called *greedy algorithm* for solving Malfatti's type problems and state some open related problems.

The project is organized as follows. In Section 2 the well-known formulae [2] for the radii of the Malfatti circles of a triangle are derived. In Section 3 a trigonometric inequality for acute-angled triangles, due to Zallgarer [6], is proved. It implies that the Malfatti circles never give a solution of the Malfatti problem. Section 4 is devoted to some problems which in a sense are dual to the Malfatti problems for two circles. More precisely, we find explicit formulae for the side-lengths of the smallest equilateral triangle and square containing two non-intersecting circles of given radii. These formulae are used to solve the Malfatti problem for an equilateral triangle or a square. In Section 5 we consider some problems of Malfatti's type and show that they cannot be solved by using the so-called *greedy algorithm*. In the last section of the project we state some open problems that might be objects of future investigations.

2 Malfatti's circles

It is well-known that for any triangle there exist three circles such that each of them is tangent to the other two and to two sides of the triangle. These circles are uniquely determined and are known in the literature as *Malfatti's circles.*

In this section we shall derive some useful formulae for the radii of the Malfatti circles of a triangle.

Theorem 2.1 [2]. Let *ABC be a triangle with angles α, β, γ and inradius r and let r_A, r_B and r_C be the radii of the Malfatti circles inscribed respectively in the angles A, B and C. Then*

$$
\begin{aligned}
r_A &= \frac{r}{2}\,\frac{\left(1+\operatorname{tg}\dfrac{\beta}{4}\right)\left(1+\operatorname{tg}\dfrac{\gamma}{4}\right)}{\left(1+\operatorname{tg}\dfrac{\alpha}{4}\right)} \\
r_B &= \frac{r}{2}\,\frac{\left(1+\operatorname{tg}\dfrac{\alpha}{4}\right)\left(1+\operatorname{tg}\dfrac{\gamma}{4}\right)}{\left(1+\operatorname{tg}\dfrac{\beta}{4}\right)} \qquad (1)\\
r_C &= \frac{r}{2}\,\frac{\left(1+\operatorname{tg}\dfrac{\alpha}{4}\right)\left(1+\operatorname{tg}\dfrac{\beta}{4}\right)}{\left(1+\operatorname{tg}\dfrac{\gamma}{4}\right)}.
\end{aligned}
$$

Proof. Let O_A and O_B be the centres of the Malfatti circles inscribed in angles A and B, and let L and M be their tangent points with the side AB (Fig. 4). Then

$$
\begin{aligned}
&AL = r_A \operatorname{ctg}\frac{\alpha}{2}, \quad MB = r_B \operatorname{ctg}\frac{\beta}{2},\\
&LM = \sqrt{(r_A + r_B)^2 - (r_A - r_B)^2} = 2\sqrt{r_A r_B}.
\end{aligned}
$$

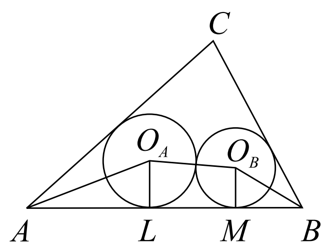

Fig. 4

On the other hand $AB = r\left(\operatorname{ctg}\dfrac{\alpha}{2} + \operatorname{ctg}\dfrac{\beta}{2}\right)$ and we get the identity

$$
r_A \operatorname{ctg}\frac{\alpha}{2} + r_B \operatorname{ctg}\frac{\beta}{2} + 2\sqrt{r_A r_B} = r\left(\operatorname{ctg}\frac{\alpha}{2} + ctg\frac{\beta}{2}\right).
$$

The same reasoning shows that the radii $r_A,\ r_B$ and r_C of the Malfatti circles are solutions of the following system:

$$\begin{aligned} r_A \operatorname{ctg}\frac{\alpha}{2} + r_B \operatorname{ctg}\frac{\beta}{2} + 2\sqrt{r_A r_B} &= r\left(\operatorname{ctg}\frac{\alpha}{2} + \operatorname{ctg}\frac{\beta}{2}\right) \\ r_B \operatorname{ctg}\frac{\beta}{2} + r_C \operatorname{ctg}\frac{\gamma}{2} + 2\sqrt{r_B r_C} &= r\left(\operatorname{ctg}\frac{\beta}{2} + \operatorname{ctg}\frac{\gamma}{2}\right) \\ r_C \operatorname{ctg}\frac{\gamma}{2} + r_A \operatorname{ctg}\frac{\alpha}{2} + 2\sqrt{r_C r_A} &= r\left(\operatorname{ctg}\frac{\gamma}{2} + \operatorname{ctg}\frac{\alpha}{2}\right). \end{aligned} \tag{2}$$

Now we shall check that r_A, r_B and r_C given by formulae (1) satisfy the above system. The uniqueness of the solutions of (2) is proved in [2].

Set $\operatorname{tg}\frac{\alpha}{4} = x$, $\operatorname{tg}\frac{\beta}{4} = y$ and $\operatorname{tg}\frac{\gamma}{4} = z$. Then

$$\begin{aligned} \operatorname{ctg}\frac{\alpha}{2} &= \frac{1-\operatorname{tg}^2\frac{\alpha}{4}}{2\operatorname{tg}\frac{\alpha}{4}} = \frac{1-x^2}{2x} \\ \operatorname{ctg}\frac{\beta}{2} &= \frac{1-\operatorname{tg}^2\frac{\beta}{4}}{2\operatorname{tg}\frac{\beta}{4}} = \frac{1-y^2}{2y} \\ \operatorname{ctg}\frac{\gamma}{2} &= \frac{1-\operatorname{tg}^2\frac{\gamma}{4}}{2\operatorname{tg}\frac{\gamma}{4}} = \frac{1-z^2}{2z}. \end{aligned}$$

Hence plugging the expressions for r_A, r_B and r_C from (1) in the first identity of (2) we see that we have to prove the identity

$$\frac{(1+y)(1+z)(1-x^2)}{2(1+x)x} + \frac{(1+x)(1+z)(1-y^2)}{2(1+y)y} + 1 + z = \frac{1-x^2}{2x} + \frac{1-y^2}{2y}.$$

Simple algebraic manipulations show that the above identity is equivalent to the identity

$$x + y + z + xy + yz + zx - xyz = 1. \tag{3}$$

On the other hand $\operatorname{tg}\left(\frac{\alpha}{4} + \frac{\beta}{4} + \frac{\gamma}{4}\right) = \operatorname{tg}\frac{\pi}{4} = 1$ and using the formula

$$\begin{aligned} \operatorname{tg}\left(\frac{\alpha}{4} + \frac{\beta}{4} + \frac{\gamma}{4}\right) &= \frac{\operatorname{tg}\frac{\alpha}{4} + \operatorname{tg}\left(\frac{\beta}{4} + \frac{\gamma}{4}\right)}{1 - \operatorname{tg}\frac{\alpha}{4}\operatorname{tg}\left(\frac{\beta}{4} + \frac{\gamma}{4}\right)} \\ &= \frac{\operatorname{tg}\frac{\alpha}{4} + \operatorname{tg}\frac{\beta}{4} + \operatorname{tg}\frac{\gamma}{4} - \operatorname{tg}\frac{\alpha}{4}\operatorname{tg}\frac{\beta}{4}\operatorname{tg}\frac{\gamma}{4}}{1 - \operatorname{tg}\frac{\alpha}{4}\operatorname{tg}\frac{\beta}{4} - \operatorname{tg}\frac{\beta}{4}\operatorname{tg}\frac{\gamma}{4} - \operatorname{tg}\frac{\gamma}{4}\operatorname{tg}\frac{\alpha}{4}} \end{aligned}$$

we get

$$\operatorname{tg}\frac{\alpha}{4} + \operatorname{tg}\frac{\beta}{4} + \operatorname{tg}\frac{\gamma}{4} + \operatorname{tg}\frac{\alpha}{4}\operatorname{tg}\frac{\beta}{4} + \operatorname{tg}\frac{\beta}{4}\operatorname{tg}\frac{\gamma}{4} + \operatorname{tg}\frac{\alpha}{4}\operatorname{tg}\frac{\gamma}{4} - \operatorname{tg}\frac{\alpha}{4}\operatorname{tg}\frac{\beta}{4}\operatorname{tg}\frac{\gamma}{4} = 1.$$

Hence the identity (3) holds true and the Theorem is proved.

3 An inequality for acute triangles

In this section we shall prove an inequality of Zalgaller [6] for the angles of an acute triangle which implies that the Malfatti circles never give a solution of the Malfatti problem. In the exposition below we follow [6] and [7].

Theorem 3.1. *Let α, β and γ be the angles of an acute triangle. Then*

$$\frac{\left(1+\operatorname{tg}\frac{\alpha}{2}\right)^4 + \left(1+\operatorname{tg}\frac{\beta}{2}\right)^4 + \left(1+\operatorname{tg}\frac{\gamma}{2}\right)^4}{\left(1+\operatorname{tg}\frac{\alpha}{2}\right)^2\left(1+\operatorname{tg}\frac{\beta}{2}\right)^2\left(1+\operatorname{tg}\frac{\gamma}{2}\right)^2} \le C + \frac{2}{3}\left(\operatorname{tg}^4\frac{\alpha}{2} + \operatorname{tg}^4\frac{\beta}{2} + \operatorname{tg}^4\frac{\gamma}{2}\right), \tag{4}$$

where

$$C = \frac{9}{(\sqrt{3}+1)^2} - \frac{2}{9} \approx 0,98355.$$

The identity is achieved only for equilateral triangles.

Proof. We first prove two lemmas.

Lemma 1. *Let*

$$f(x) = \frac{4}{3}x^4 + \frac{(1-x^2)^4}{24x^4} - \frac{8x^2}{(1+2x-x^2)^2} - \frac{(1+2x-x^2)^2}{4x^2(1+x)^4}. \tag{5}$$

Then the minimum of the function $f(x)$ in the interval $x \in [0,4\ ,\ 1]$ is attained at the point $x = \frac{1}{\sqrt{3}}$ and is equal to $f\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{9} - \frac{9}{(\sqrt{3}+1)^2} = -C$.

Proof of Lemma 1. Set

$$A(x) = \frac{4}{3}x^4 + \frac{(1-x^2)^4}{24x^4}, \qquad B(x) = -\frac{8x^2}{(1+2x-x^2)^2} - \frac{(1+2x-x^2)^2}{4x^2(1+x)^4}.$$

Direct computations show that the derivatives of $A(x)$ and $B(x)$ are given by

$$A'(x) = \frac{(3x^2-1)(11x^6+3x^4+x^2+1)}{6x^5}$$

and

$$\begin{aligned} B'(x) &= -\frac{3x^2-1}{2x^3(1+x)^5(1+2x-x^2)^3}\cdot(11x^9+49x^8+140x^7 \\ &\quad +148x^6+194x^5+214x^4+140x^3+52x^2+11x+1). \end{aligned}$$

Hence

$$f'(x) = A'(x)+B'(x) = (3x^2-1)\frac{\varphi(x)}{\phi(x)}, \tag{6}$$

where

$$\begin{aligned} \varphi(x) &= -11x^{17}+11x^{16}+118x^{15}+14x^{14}-474x^{13}-546x^{12}+301x^{11} \\ &\quad +1091x^{10}+892x^9+424x^8-132x^7-408x^6-234x^5+22x^4 \\ &\quad +89x^3+47x^2+11x+1 \end{aligned}$$

and

$$\phi(x) = 6x^5(1+x)^5(1+2x-x^2)^3.$$

It is obvious that $\phi(x) > 0$ for $x \in [0,4\ ,\ 1]$. On the other hand the function $\varphi(x)$ can be represented as a sum of positive and non-negative functions in the given interval as follows:

$$\begin{aligned} \varphi(x) &= (1-x^3)^6(22x^4+68x^3+39x^2)+(1-x^8)^2+11x(1-x^7)^2 \\ &\quad +8x^2(1-x^6)^2+21x^3(1-x^5)^2+220x^{16}(x-0,4)+320x^9(x-0,4) \\ &\quad +x^{16}(408x^2-65)+x^8(441x^2-70)+x^8(141x^3-9) \\ &\quad +x^{19}(132-39x-68x^2-22x^3)+3x^{17} \\ &\quad +x^{11}(940+814x-55x^2-579x^3-913x^4-167x^5). \end{aligned}$$

Hence $\varphi(x) > 0$ and it follows from (6) that the function $f(x)$ decreases in the interval $\left[0,4\ ,\ \frac{1}{\sqrt{3}}\right]$ and increases in the interval $\left[\frac{1}{\sqrt{3}}\ ,\ 1\right]$. This shows that the minimum of $f(x)$ in the interval $[0,4\ ,\ 1]$ is attained at the point $x = \frac{1}{\sqrt{3}}$ and the lemma is proved.

Lemma 2. *Theorem 3.1 is true for isosceles triangles.*

Proof of Lemma 2. Let the angles of an isosceles triangle be α, α and $2\left(\frac{\pi}{2}-\alpha\right)$, where $\frac{\pi}{4}<\alpha<\frac{\pi}{2}$. Set $\operatorname{tg}\frac{\alpha}{2}=x$. Then the identity $\operatorname{tg}\frac{\pi}{8}=\dfrac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}=\dfrac{1}{\sqrt{2}+1}=\sqrt{2}-1$ shows that $\sqrt{2}-1<x<1$. It is easy to check that in this case the inequality (4) has the form

$$\frac{8x^2}{(1+2x-x^2)^2}+\frac{(1+2x-x^2)^2}{4x^2(1+x)^4}\leq C+\frac{2}{3}\left[2x^4+\frac{(1-x^2)^4}{16x^4}\right]$$

and Lemma 2 follows from Lemma 1.

Now we are ready to prove Theorem 3.1. Denote by p, R and r the semi-perimeter, the circumradius and the inradius of a triangle, respectively. Without loss of generality we may assume that

$$4R+r=1. \tag{7}$$

Under this condition we have the following well-known identities:

$$\begin{aligned}
\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}&=\frac{1}{p}\\
\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}+\operatorname{tg}\frac{\gamma}{2}\operatorname{tg}\frac{\alpha}{2}&=1\\
\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}&=\frac{r}{p}.
\end{aligned}$$

Hence we may express the following symmetric functions of $\operatorname{tg}\frac{\alpha}{2}$, $\operatorname{tg}\frac{\beta}{2}$ and $\operatorname{tg}\frac{\gamma}{2}$ by means of p and r:

$$\begin{aligned}
\operatorname{tg}^4\frac{\alpha}{2}+\operatorname{tg}^4\frac{\beta}{2}+\operatorname{tg}^4\frac{\gamma}{2}&=\frac{1}{p^4}(4rp^2+2p^4-4p^2+1),\\
(1+\operatorname{tg}\frac{\alpha}{2})^4+\left(1+\operatorname{tg}\frac{\beta}{2}\right)^4+\left(1+\operatorname{tg}\frac{\gamma}{2}\right)^4&=\\
\frac{1}{p^4}(4rp^2(3p+1)-7p^4-8p^3+2p^2+4p+1),&\\
\left(1+\operatorname{tg}\frac{\alpha}{2}\right)\left(1+\operatorname{tg}\frac{\beta}{2}\right)\left(1+\operatorname{tg}\frac{\gamma}{2}\right)&=\frac{1}{p}(r+2p+1).
\end{aligned}$$

The Euler's inequality $R \geq 2r$ and (7) imply that $0 < r \leq \dfrac{1}{9}$. On the other hand by Blundon's inequality [4] we have

$$\begin{aligned} &2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2 - 2Rr} \\ &\leq\ p^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr}. \end{aligned}$$

Hence for fixed r and $R = \dfrac{(1-r)}{4}$ the semi-perimeter p ranges from

$$p_{\min}(r) = \sqrt{\frac{1}{8} + \frac{9r}{4} - \frac{27r^2}{8} - \frac{1-9r}{8}\sqrt{1-10r+9r^2}}$$

to

$$p_{\max}(r) = \sqrt{\frac{1}{8} + \frac{9r}{4} - \frac{27r^2}{8} + \frac{1-9r}{8}\sqrt{1-10r+9r^2}}$$

and $p = p_{\max}(r)$ is attained for isosceles triangles only. Note that $p < \pi R = \dfrac{\pi(1-r)}{4} < \dfrac{\pi}{4} < 1$. It is also well-known [4] that $p > 2R + r = \dfrac{1+r}{2}$ for acute triangles. The admissible values of r and p are presented by the region KLM on Fig. 5, where $KL:\ p = p_{\max}(r)$, $ML:\ p = p_{\min}(r)$, $KM:\ p = \dfrac{1+r}{2}$.

Consider the function

$$\begin{aligned} \phi(r,p) &= \frac{2}{3}\left(\mathrm{tg}^4\frac{\alpha}{2} + tg^4\frac{\beta}{2} + \mathrm{tg}^4\frac{\gamma}{2}\right) \\ &\quad - \frac{\left(1+\mathrm{tg}\frac{\alpha}{2}\right)^4 + \left(1+\mathrm{tg}\frac{\beta}{2}\right)^4 + \left(1+\mathrm{tg}\frac{\gamma}{2}\right)^4}{\left(1+\mathrm{tg}\frac{\alpha}{2}\right)^2\left(1+\mathrm{tg}\frac{\beta}{2}\right)^2\left(1+\mathrm{tg}\frac{\gamma}{2}\right)^2} \\ &= \frac{2}{3}\left(4rp^2 + 2p^4 - 4p^2 + 1\right) - \frac{4rp^2(3p+1) - 7p^4 - 8p^3 + 2p^2 + 4p + 1}{(r+2p+1)^2p^2}. \end{aligned}$$

Direct computations show that the partial derivative of $\phi(r,p)$ with respect to r is given by

$$\begin{aligned} \frac{\partial \phi}{\partial r} &= \frac{2}{3p^2(r+2p+1)^3}\big(4p^4r^3 + 12p^4(2p+1)r^2 \\ &\quad +\ 6p^2(8p^4 + 8p^3 + 2p^2 + 3p + 1)r \\ &\quad +\ (32p^7 + 48p^6 + 24p^5 - 53p^4 - 54p^3 + 12p + 3)\big) > 0. \end{aligned}$$

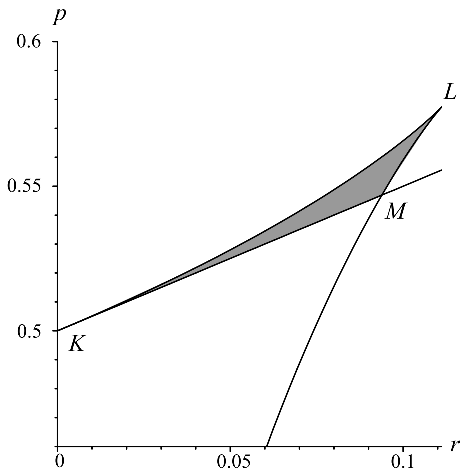

Fig. 5

Hence the minimum of the function $\phi(r,p)$ is attained for $p = p_{\max}(r)$ (see Fig. 5), i. e. for isosceles triangles only.

Now we shall use Theorem 3.1 to prove that the Malfatti circles never give a solution of the Malfatti problem. Set $\tilde{\alpha} = \dfrac{\pi}{4} - \dfrac{\alpha}{4}$, $\tilde{\beta} = \dfrac{\pi}{4} - \dfrac{\beta}{4}$, $\tilde{\gamma} = \dfrac{\pi}{4} - \dfrac{\gamma}{4}$. Then

$$\left(1 + \mathrm{tg}\,\frac{\alpha}{4}\right)^2 = 2\frac{1+\sin\dfrac{\alpha}{2}}{1+\cos\dfrac{\alpha}{2}} = 2\,\frac{1+\cos 2\tilde{\alpha}}{1+\sin 2\tilde{\alpha}} = \frac{4}{(1+\mathrm{tg}\,\tilde{\alpha})^2}$$

and we get from Theorem 2.1 that

$$r_A^2 = r\frac{(1+\mathrm{tg}\,\tilde{\alpha})^2}{(1+\mathrm{tg}\,\tilde{\beta})^2(1+\mathrm{tg}\,\tilde{\gamma})^2}.$$

Consider the three circles that are inscribed in the angles of the triangle and are tangent to the incircle. It is easy to check that their radii are given by

$$r_\alpha = r\,\frac{1-\sin\dfrac{\alpha}{2}}{1+\sin\dfrac{\alpha}{2}},\quad r_\beta = r\,\frac{1-\sin\dfrac{\beta}{2}}{1+\sin\dfrac{\beta}{2}},\quad r_\gamma = r\,\frac{1-\sin\dfrac{\gamma}{2}}{1+\sin\dfrac{\gamma}{2}}.$$

Note also that $r_\alpha = r\,tg^2\tilde{\alpha}$ etc. Hence Theorem 3.1 implies the inequality

$$r_A^2 + r_B^2 + r_C^2 < r^2 + \frac{2}{3}\left(r_\alpha^2 + r_\beta^2 + r_\gamma^2\right) \tag{8}$$

since it is equivalent to the inequality

$$\frac{(1+\mathrm{tg}\,\tilde{\alpha})^4+(1+\mathrm{tg}\,\tilde{\beta})^4+(1+\mathrm{tg}\,\tilde{\gamma})^4}{(1+\mathrm{tg}\,\tilde{\alpha})^2\,(1+\mathrm{tg}\,\tilde{\beta})^2\,(1+\mathrm{tg}\,\tilde{\gamma})^2}<1+\frac{2}{3}\left(\mathrm{tg}^4\tilde{\alpha}+\mathrm{tg}^4\tilde{\beta}+\mathrm{tg}^4\tilde{\gamma}\right).$$

The inequality (8) shows that the incircle and the two largest circles inscribed in the angles of a triangle and tangent to the incircle have greater combined area than the Malfatti circles.

4 The Malfatti problems for equilateral triangles and squares

In this section we shall give simple proofs of the Malfatti problems for equilateral triangle and square. To do this we shall use the so-called *dual Malfatti problems.*

4.1 Dual Malfatti problems for equilateral triangles and squares

We first consider the dual Malfatti problem for equilateral triangle and two circles.

Problem 4.1. *Given two positive numbers a and b find the side-length of the smallest equilateral triangle containing two nonintersecting circles of radii a and b respectively.*

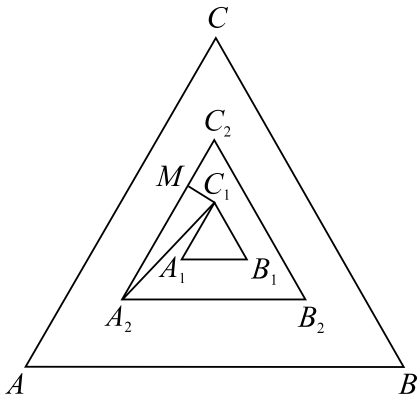

Fig. 6

Solution. We shall assume that $a \geq b$. Let ABC be an equilateral triangle of side-length x which contains two nonintersecting circles $k_1(O_1, a)$ and $k_2(O_2, b)$. Then the inradius of ABC is not less than a and therefore we have the inequality

$$x \geq 2\sqrt{3}\,a. \tag{9}$$

Further note that the center O_1 of the circle k_1 lies inside the equilateral triangle $A_1B_1C_1$ whose sides are at distance a apart from the sides of triangle ABC (Fig. 6).

Analogously O_2 lies inside the equilateral triangle $A_2B_2C_2$ whose sides are at distance b apart from the sides of triangle ABC. Denote by O the center of triangle ABC. Then

$$O_1O_2 \leq A_2C_1. \tag{10}$$

Since the distances from O to AC and A_1C_1 are respectively $\dfrac{x}{2\sqrt{3}}$ and $\dfrac{x}{2\sqrt{3}} - a$ it follows that $\dfrac{A_1C_1}{AC} = \dfrac{\dfrac{x}{2\sqrt{3}} - a}{\dfrac{x}{2\sqrt{3}}}$. Hence $A_1C_1 = x - 2\sqrt{3}\,a$. Analogously $A_2C_2 = x - 2\sqrt{3}\,b$. Let M be the foot of the perpendicular from C_1 to A_2C_2. Then

$$\begin{aligned} A_2M &= A_2C_2 - MC_2 = x - 2\sqrt{3}\,b - \frac{A_2C_2 - A_1C_1}{2} \\ &= x - 2\sqrt{3}\,b - \frac{x}{2} + \sqrt{3}\,b + \frac{x}{2} - a\sqrt{3} = x - (a+b)\sqrt{3}. \end{aligned}$$

Now using the Pythagorean theorem for $\triangle A_2C_1M$ we get

$$(A_2C_2)^2 = (a-b)^2 + (x - \sqrt{3}\,(a+b))^2. \tag{11}$$

On the other hand since k_1 and k_2 are nonintersecting circles we have

$$(O_1O_2)^2 \geq (a+b)^2. \tag{12}$$

Hence (9), (10), (11) and (12) imply the inequality

$$x \geq \sqrt{3}(a+b) + 2\sqrt{ab}. \tag{13}$$

Set $t(a,b) = \max\left\{2\sqrt{3}\,a,\ \sqrt{3}\,(a+b) + 2\sqrt{ab})\right\}$. Then it follows from (9) and (13) that $x \geq t(a,b)$.

Now we shall show that an equilateral triangle ABC of side-length $t(a,b)$ contains two nonintersecting circles k_1 and k_2 of radii a and b. Indeed, if $a \geq 3b$ then $t(a,b) = 2\sqrt{3}\,a$ and in this case k_1 is the incircle of triangle ABC. Let k be the circle inscribed in angle A and which is tangent to k_1. Then it is easy to check that its radius is equal to $r = \dfrac{a}{3}$. Hence we can place a circle k_2 of radius b in the interior of k and it does not intersect the circle k_1. If $b \leq a \leq 3b$ then $t(a,b) = \sqrt{3}\,(a+b) + 2\sqrt{ab}$ and we can take as k_1 and k_2 the circles of radii a and b with centers at the points C_1 and A_2, respectively.

Note that these circles are inscribed in angles C and A of $\triangle ABC$, respectively and they are tangent to each other.

Thus we have shown that the side-length of the smallest equilateral triangle containing two nonintersecting circles of radii a and b is equal to

$$t(a,b) \quad = \quad \begin{cases} \sqrt{3}\,(a+b)+2\sqrt{ab} & \text{if } b \le a \le 3b \\ 2\sqrt{3} & \text{if } a \ge 3b. \end{cases}$$

Now we shall solve the dual Malfatti problem for a square and two circles.

Problem 4.2. *Given two positive numbers a and b find the side-length of the smallest square containing two nonintersecting circles of radii a and b.*

Solution. The solution is similar to that of Problem 4.1. Let $ABCD$ be a square of side-length x containing two nonintersecting circles $k_1(O_1,a)$ and $k_2(O_2,b)$, where $a \ge b$. Since its inradius is not less than a we get

$$x \ge 2a. \tag{14}$$

The center O_1 of k_1 lies inside the square $A_1B_1C_1D_1$ whose sides are at distance a apart from the sides of $ABCD$ (Fig. 7). Analogously the center O_2 of the circle k_2 lies inside the square $A_2B_2C_2D_2$ whose sides are at distance b apart from the sides of $ABCD$. Hence

$$A_2C_1 \ge O_1O_2. \tag{15}$$

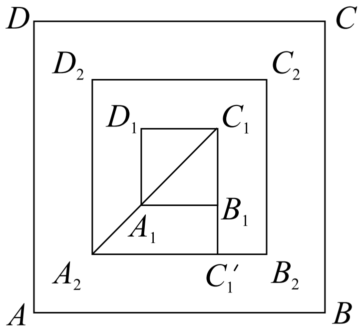

Fig. 7

Now we shall find the length of A_2C_1 in terms of a, b and x. Let $C_1C_1' \perp A_2B_2$ and $C_1' \in A_2B_2$. Then $\triangle\, A_2C_1C_1'$ is an isosceles right triangle with $C_1C_1' = x-a-b$ and therefore

$$A_2C_1^2 = 2(x-a-b)^2 \;\Rightarrow\; A_2C_1 = \sqrt{2}(x-a-b). \tag{16}$$

On the other hand since the circles k_1 and k_2 are nonintersecting we have that

$$O_1O_2 \geq a + b. \tag{17}$$

Hence (14), (15), (16) and (17) imply that $\sqrt{2}\,(x - a - b) \geq a + b$, i. e.

$$x \geq (a + b)\left(1 + \frac{1}{\sqrt{2}}\right). \tag{18}$$

Set $t(a, b) = \max\left\{2a,\ (a + b)\left(1 + \frac{1}{\sqrt{2}}\right)\right\}$. Then it follows from (14) and (18) that $x \geq t(a, b)$.

Now we shall show that a square of side-length $t(a, b)$ contains two nonintersecting circles of radii a and b. If $a \geq b(\sqrt{2} + 1)^2$ then $t(a, b) = 2a$ and k_1 is the incircle of the square. Let k be the circle inscribed in angle A and which is tangent to k_1. Then it easy to check that its radius is $r = (\sqrt{2} - 1)^2 a$. Hence we can place a circle k_2 of radius b in the interior of k and it does not intersect k_1. If $b \leq a \leq b(\sqrt{2} + 1)^2$ then $t(a, b) \geq (a + b)\left(1 + \frac{1}{\sqrt{2}}\right)$ and we can take as k_1 and k_2 the circles of radii a and b and centers at the points A_2 and C_1, respectively.

Thus we have shown that the side-length of the smallest square containing two nonintersecting circles of radii a and b is equal to

$$t(a, b) = \begin{cases} 2a & \text{if } a \geq b(\sqrt{2} + 1)^2 \\ (a + b)\left(1 + \frac{1}{\sqrt{2}}\right) & \text{if } b \leq a \leq b(\sqrt{2} + 1)^2. \end{cases}$$

4.2 The Malfatti problems for equilateral triangles and squares

In this subsection we shall solve the Malfatti problems for equilateral triangles and squares by using Problems 4.1 and 4.2. We first consider the case of an equilateral triangle.

Malfatti problem 1. *Three nonintersecting circles lie in the interior of an equilateral triangle. Prove that the sum of their areas is maximal when one of them is the incircle of the triangle and the other two are inscribed in its angles and are tangent to the incircle.*

Proof. We shall assume that the side-length of the triangle is 1. Then the radius of its incircle is equal to $\frac{1}{2\sqrt{3}}$ and the radii of the circles inscribed in its angles and tangent to the incircle are equal to $\frac{1}{6\sqrt{3}}$. Let us consider three arbitrary nonintersecting circles lying in the interior of the triangle and denote their radii by a, b and c, where $a \geq b \geq c$. We have to prove that

$$a^2 + b^2 + c^2 \leq \frac{11}{108}.$$

To do this we shall consider two cases.

Case 1. Let $a \geq 3b$. The inequality $a \leq \frac{1}{2\sqrt{3}}$ implies $a^2 + b^2 + c^2 \leq a^2 + 2b^2 \leq a^2 + \frac{2a^2}{9} \leq \frac{11}{108}$ and the above inequality is proved.

Case 2. Let $b \leq a \leq 3b$. It follows from Problem 4.1 that $\sqrt{3}(a+b) + 2\sqrt{ab} \leq 1$. Set $a = 3x^2 b$, where $x > 0$. Then the above inequalities are equivalent to

$$\frac{1}{\sqrt{3}} \leq x \leq 1 \text{ and } b \leq \frac{1}{\sqrt{3}(3x^2 + 2x + 1)}.$$

Hence

$$a^2 + b^2 + c^2 \leq a^2 + 2b^2 = (9x^2 + 2)b^2 \leq \frac{9x^4 + 2}{3(3x^2 + 2x + 1)^2}$$

and it is enough to prove that

$$\frac{9x^4 + 2}{(3x^2 + 2x + 1)^2} \leq \frac{11}{36} \text{ for } x \in \left[\frac{1}{\sqrt{3}}, 1\right].$$

It is easy to check that this inequality is equivalent to $225x^4 - 132x^3 - 110x^2 - 44x + 61 \leq 0$ in the given interval for x. We can rewrite the latter inequality in the form $(225x^3 + 93x^2 - 17x - 61)(x - 1) \leq 0$. Hence it is fulfilled for $x \in \left[\frac{1}{\sqrt{3}}, 1\right]$ since $(x - 1) \leq 0$ and $225x^3 + 93x^2 - 17x - 61 = 51x(x^2 - \frac{1}{3}) + 174x^3 + 93x^2 - 61 \geq 174x^3 + 93x^2 - 61 \geq \frac{174}{3\sqrt{3}} + \frac{93}{3} - 61 > 0$. This completes the proof.

Next we consider the case of a square.

Malfatti problem 2. *Three nonintersecting circles lie in the interior of a square. Prove that the sum of their areas is maximal when one of them is the incircle of the square and the other two are inscribed in its angles and are tangent to the incircle.*

Proof. We shall assume that the side-length of the square is 1. Then the radius of its incircle is equal to $\frac{1}{2}$ and the radii of the circles inscribed in the angles of the square and tangent to the incircle are equal to $\frac{(\sqrt{2}-1)^2}{2}$. Let us consider arbitrary three nonintersecting circles lying in the interior of the square and denote their radii by a, b and c, where $a \ge b \ge c$. Then we have to prove that $a^2+b^2+c^2 \le \frac{35-24\sqrt{2}}{4}$. To do this we shall consider two cases.

Case1. Let $a > b(\sqrt{2}+1)^2$. Then $a^2+b^2+c^2 \le a^2+2b^2 = a^2\left(1+\frac{2}{(\sqrt{2}+1)^4}\right)$. Since $a \le \frac{1}{2}$ we get $a^2\left(1+\frac{2}{(\sqrt{2}+1)^4}\right) \le \frac{1}{4}+\frac{1}{2(\sqrt{2}+1)^4} = \frac{35-24\sqrt{2}}{4}$.

Case 2. Let $b \le a \le b(\sqrt{2}+1)^2$. Then it follows from Problem 4.2 that

$$(a+b)\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right) \le 1.$$

Set $a = tb$, where $t > 0$. Then the above inequalities are equivalent to

$$1 \le t \le (\sqrt{2}+1)^2 \text{ and } b \le \frac{\sqrt{2}}{(\sqrt{2}+1)(t+1)}.$$

Hence

$$a^2+b^2+c^2 \le a^2+2b^2 = (t^2+2)b^2 \le \frac{2(t^2+2)}{(\sqrt{2}+1)^2(t+1)^2}$$

and it is enough to prove that

$$\frac{2(t^2+2)}{(\sqrt{2}+1)^2(t+1)^2} \le \frac{35-24\sqrt{2}}{4} \text{ for } t \in \left[1, (\sqrt{2}+1)^2\right].$$

Simple algebraic manipulations show that the former inequality is equivalent to

$$(2\sqrt{2}-1)t^2 + (4\sqrt{2}-18)t + 7 + 2\sqrt{2} \le 0. \tag{19}$$

Since this inequality is quadratic and $2\sqrt{2}-1 > 0$, it is enough to check that it is fulfilled for $t_1 = 1$ and $t_2 = (\sqrt{2}+1)^2$. Set $f(x) = (2\sqrt{2}-1)x^2 + (4\sqrt{2}-18)x + 7 + 2\sqrt{2}$. Then $f(1) = 8\sqrt{2} - 12 < 0$ and $f((\sqrt{2}+1)^2) = (2\sqrt{2}-1)(3+2\sqrt{2})^2 + (4\sqrt{2}-18)(\sqrt{2}+1)^2 + 7 + 2\sqrt{2} = 0$. Hence the inequality (19) is true for $t \in \left[1, (\sqrt{2}+1)^2\right]$ and the equality is attained if and only if $t = (\sqrt{2}+1)^2$.

5 Malfatti type problems and the greedy algorithm

The Theorem of Zalgaller and Loss shows that the solution of the Malfatti problem is given by using the so-called *greedy algorithm*. Namely, at each step we cut the largest possible circle. As we saw in Section 3 the same is also true for the solution of the Malfatti problem for a square. Of course, one can formulate various other Malfatti type problems and it is tempting to conjecture that their solutions are given by using the greedy algorithm. The purpose of this section is to show that this is not true in general.

Let us consider the following problem:

Problem 5.1. *Given a circle cut three nonintersecting triangles so that the sum of their areas is maximal.*

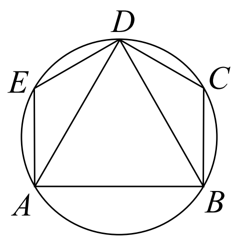

Fig. 8

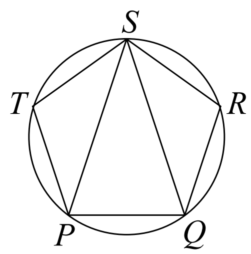

Fig. 9

We shall show that the greedy algorithm does not give a solution of this problem. To do this let us assume the contrary. Then at the first step we have to cut an equilateral triangle ABD, inscribed in the given circle (Fig. 8). This is so, since it is well-known that among all triangles inscribed in a given circle the equilateral triangles have the greatest area. Then according to the greedy algorithm at the next two steps we have to cut two triangles of maximal areas from the segments of the circle cut by the sides of triangle ABD. Consider, for example, the segment cut by the side AD. Then the triangle of maximal area that we can cut from it is ADE, where E is the midpoint of the arc $\overset{\frown}{AD}$. Therefore the solution of the problem given by the greedy algorithm consists of triangles ABD, ADE and BDC, where E and C are the midpoints of the arcs $\overset{\frown}{AD}$ and $\overset{\frown}{BD}$ (Fig. 8).

Consider now the regular pentagon $PQRST$ inscribed in the given circle (Fig. 9). Then, as is well-known, the area of $PQRST$ is greater than the area of the pentagon $ABCDE$ since the latter is not regular. Therefore the sum of the areas of triangles PQS, PST and QSR is greater than the sum of the areas of triangles ABD, ADE and BDC. This shows that the solution of Problem 5.1 is not given by the greedy algorithm.

It is easy to prove that the solution of Problem 5.1 for two triangles is given by any two isosceles right triangles forming a square inscribed in the circle. So, it is natural to conjecture that the solution of the analogous problem for n triangles is given by any n triangles forming a regular $(n + 2)$-gon inscribed in the given circle. It is interesting to note that if the above conjecture is true then the solution of the problem for an odd number of triangles is not given by the greedy algorithm, whereas if the number of triangles is an even integer of the form $3 \cdot 2^{n+1} - 2$, a solution of the problem can be obtained by using the greedy algorithm (Fig. 10).

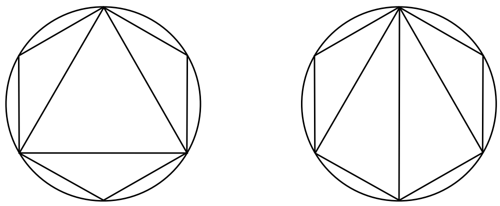

Fig. 10

6 Concluding remarks and questions

As we noted in the Introduction the solution of the Malfatti problem given by Zalgaller and Loss [7] is very difficult and we presented in Section 3 a different approach for solving this problem in the case of an equilateral triangle. This approach was based on the so-called dual Malfatti problem for two circles and we saw that the same idea can be used also for solving the Malfatti problem for a square. One may also consider the respective Malfatti problems for four (or more) circles and try to use the dual Malfatti problems for equilateral triangles and squares and three circles. Unfortunately we do not know how to solve these problems in the general case.

Another interesting question is connected with Zalgaller's inequality [6] which implies that the Malfatti circles never give a solution of the Malfatti problem (see Section 2). Its proof is based on Blundon's inequality and the investigation of a rather complicated function of two variables. So, it is interesting to know whether one can prove Zalgaller's inequality by using only Blundon's inequality.

Finally, let us note that we do not know how to solve Problem 5.1 and its analogs for $n > 2$ circles.

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