

Adaptive beliefs and the volatility of asset prices

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Abstract I present a simple model of an evolutionary financial market with heterogeneous agents, based on the concept of adaptive belief systems introduced by Brock and Hommes [8]. Agents choose between different forecast rules based on past performance, resulting in an evolutionary dynamics across predictor choice coupled to the equilibrium dynamics. The model generates endogenous price fluctuations with similar statistical properties as those observed in real return data, such as fat tails and volatility clustering. These similarities are demonstrated for data from the British, German, and Austrian stock market.

Key words heterogeneous expectations – bounded rationality – evolutionary learning – adaptive dynamics – endogenous price fluctuations in financial markets – bifurcation and chaos

1 Introduction

In the past decades, the degree to which financial markets are efficient has been a matter of heavy debate among academics as well as practitioners. The efficient market hypothesis (EMH) still seems to be one of the dominating paradigms in finance. A second paradigm in economic theory and closely related to the EMH, is the concept of rational expectations equilibria (REE), introduced by Muth [46]. It is assumed that investors have complete knowledge of the fundamental structure of the economy. They take immediately all available information into consideration, optimize expected utilities according to a model which is common knowledge and are able to make arbitrarily difficult logical inferences. Since all traders are rational and process information quickly, asset prices reflect all publically available information and it is not possible to earn “abnormal” or “excess” profits on av-

erage by trading on public information.¹ The value of a risky asset is completely determined by its *fundamental price*, equal to the present discounted value of the expected stream of future dividends. Changes in asset prices would be completely random, solely driven by random economic “news” about changes in fundamentals.

This view was already questioned by Keynes [35], who took the position that investors’ sentiment and mass psychology (“*animal spirits*”) play a significant role in financial markets. Keynes argued that stock prices are not governed by an objective view of “fundamentals”, but by “what average opinion expects average opinion to be.” Arthur [2] writes: “Not knowing how other investors arrive at their expectations, I cannot form mine in a well-defined way. And so I make assumptions – subjective ones – about how other investors form expectations and behave.” He talks about a *problem complexity boundary* beyond which the requirements of perfect rationality fail and the way how human agents arrive at decisions is not well-defined. He suggests to use the term “rational” in a wider sense “that investors are free to use the ultimate in deductive and inductive methods in trying to determine how price dynamics works.”

Also a glance at financial magazines suggests that real financial markets differ from a perfectly rational world. Financial analysts have different expectations about future prices and dividends. They believe that markets are, at least to some degree, forecastable from past prices and that bubbles and crashes are the result of herd behavior rather than adjustment to new information. “Traders, in fact, see the market almost as if it had a personality of its own, a complex psychology that can be read and understood and profited from acquaintance ship with it deepens.” [2]

The concept of rational expectations has certainly importance as a normative tool and provides a useful reference point. It is useful in demonstrating logical equilibrium outcomes and analyzing their consequences. However, if some people deviate, the world that is created may change, so that others should logically expect something different and deviate too. Traditional theory seems not to be appropriate to understand the actual behavior and to explain a lot of pervasive patterns in financial data. For example, stock prices move too much relative to fundamentals (Shiller [52]), the volatility of prices is temporary correlated, and returns distributions exhibit fat tails. In RE models even disagreement among investors due to asymmetric information does not generate trading if all investors are rational and rationality is common knowledge. High trading volumes and the pursuit of active investment strategies seem to be inconsistent with this paradigm.

A comprehensive discussion of stylized facts that are characteristics of financial series is given by Pagan [48] and Brock [7], see also Haugen [33], who discusses “seven mysteries of the stock market” related to the nature of stock volatility. In the following we list some of these facts.

- Stock prices are arguable too volatile, relative to the dividends and underlying cash flows that they are based upon.
- Stock volatility, itself, is too unstable (heteroscedasticity).

¹ For reviews on the EMH see e.g. [21], [22].

- Volatility shows high persistence, that is, high (small) price changes are followed by high (small) price changes (volatility clustering).
- Stock returns exhibit excess kurtosis (fat tails).
- When volatility goes up, stock prices go down, so as to increase the size of the risk premium in the market's aggregate future expected returns (leverage effect).
- Most of the largest changes in price and volatility are unconnected to real-world events.
- When the number of trading hours per week gets smaller (exchange holidays), stock volatility goes down; when the number of trading hours gets larger (cross-listing), stock volatility, and short-term reversal patterns become more pronounced.
- Autocorrelation functions (ACF) of returns, volatility of returns, volume measures, and cross-correlations of these measures have similar shapes across different securities and different indices. Asset returns contain little serial correlation (which is consistent with the weak form of EMH), but there is substantially more correlation between absolute or squared returns.

These facts suggest that price movements are not solely driven by news, but that markets have internal dynamics of their own, which amplify, distort, and create information. Such a view may also be supported by the results of dynamical systems theory since simple nonlinear deterministic models can generate complicated dynamic behavior. Long-run prediction of a chaotic system is impossible due to sensitive dependence on initial conditions. Chaos may not be distinguishable from white noise by linear statistical methods and nonlinear dynamical systems can generate any given autocorrelation structure. Though most empirical studies have rejected the hypothesis that financial data are generated by low dimensional, purely deterministic chaos, the presence of *noisy chaos* cannot be excluded. Thus, at least part of the highly irregular fluctuations in financial data may be explained by a nonlinear law of motion. In fact, investors response to “pseudo signals”, follow advices of market gurus, extrapolate past time series, or imitate successful traders.

Haugen [33] states the hypothesis that volatility consists of three components, the “event-driven volatility” (accurate responses to real-world events), the “error-driven volatility” (time-varying mistakes in pricing, relative to the receipt of new information), and the “price-driven volatility”, which Haugen thinks to be the “by far most important part”. Investors focus on price changes which they as traders co-create, causing price reactions that feed on themselves. Thus, the market reacts in a complicated way to its own price history, creating *price-driven volatility*.

Farmer and Lo [23] argue that patterns in the price tend to disappear as agents evolve profitable strategies to exploit them, but this occurs only over an extended period of time. They view financial markets as evolutionary systems, “in which markets, instruments, institutions, and investors interact and evolve dynamically according to the ‘law’ of economic selection. Under this view, financial agents compete and adapt, but they do not necessarily do so in optimal fashion.”

It seems natural that heterogeneity and evolutionary switching between different trading strategies plays an important role in financial markets.² Some authors take the view that as a new paradigm the *heterogeneous market hypothesis* emerges as an alternative to the EMH, which is closely related to the *hypothesis of bounded rationality* (see e.g. [50]). Financial markets are considered as evolutionary systems, consisting of heterogeneous agents using different bounded rational forecasting rules. The dynamics of these systems are highly nonlinear and can easily lead to market instability and complicated price fluctuations, above and beyond the rational expectations equilibrium (REE-)fundamental price (cf. [34]).

Typically, heterogeneous agents models include “fundamentalists” or “smart money traders” (traders who believe that the price of an asset is completely determined by its economic fundamentals) and “technical traders”, “chartists”, or “noise traders” (believing that asset prices can be predicted by simple technical trading rules based upon patterns in past prices). In the classical economists’ view “irrational” traders lose their money to arbitrageurs and will be driven out of the market. However, De Long et al. [20] show that if arbitrageurs are risk averse and have a finite time horizon, noise traders may on average earn higher profits than smart money traders and survive in the market with positive probability. A survey on the noise trader approach is given by Shleifer and Summers [54]. Frankel and Froot [25] interpret chartists as traders who think short term and fundamentalists as those who think long term. LeBaron [39] shows for a simple agent based financial market that shorter horizon agents increase volatility keeping the larger horizon agents from ever getting a foothold in the market.³ Blume and Easley [5], who look at the evolution of simple strategies find that the link between rationality and (evolutionary) fitness (where the criterion is expected growth rates of wealth share accumulation) is weak.

Other recent approaches deviating from the perfect RE paradigm are psychological approaches (see [18] for a review on the work in behavioral finance) and models that relax the assumption that investors have complete knowledge about the economic structure but maintain the (perfect) rationality assumption.⁴

Agent based models are based on interactions and learning dynamics in groups of traders learning about the relations between prices and market information.

² Goetzmann and Massa [31] examine investors’ trading behavior and characterize heterogeneity in terms of their investment patterns in an S&P 500 index mutual fund.

³ Further examples for heterogeneous agents models include [60], [4], [13], [30], [17], [19], [51], [36], [41], [42].

⁴ Timmermann [56], [57] shows that excess volatility in stock returns can arise under learning processes that converge (slowly) to RE. Routledge [49] investigates an adaptive learning model in a repeated version of the Grossman-Stiglitz model where traders can choose to acquire a costly signal about dividends. He derives conditions under which the learning process converges to a rational expectations equilibrium. In his model the fraction of informed traders is fixed over time. Daniel et al. [16] and Odean [47], for example, analyze how investor overconfidence influence financial markets. A dynamic, rational equilibrium model addressing over- and underreaction to news is Veronesi [59]. Brav and Heaton [6] provide a comparative analysis of behavioral models and approaches relying on rational structural uncertainty.

Price fluctuations are generated by internal dynamics caused by the interaction of diverse trading strategies. Examples of computational based models include the Santa Fe artificial stock market [3], [40] and the stochastic multi-agent models of Lux and Marchesi [43], [44]. LeBaron [38] and Tesfatsion [55] provide overviews on this kind of models. Farmer [24] uses an out-of-equilibrium market mechanism under which different trading strategies evolve. He shows that trend strategies tend to induce positive autocorrelations in the price, whereas value-investing strategies induce negative autocorrelations. If the used strategies are nonlinear the diversity of views results in excess volatility.

Brock and Hommes [8], henceforth BH, have introduced the concept of *Adaptive Belief Systems* (ABS). Agents adapt their beliefs by choosing among a finite number of predictor (expectations or belief) functions which are functions of past observations. Each predictor has a publically available performance or fitness measure attached. According to past performance agents make a (boundedly) rational choice between the predictors. This leads to the so-called *Adaptive Rational Equilibrium Dynamics*, an evolutionary dynamics across predictor choice, coupled to the dynamics of equilibrium prices. BH show that such ABS incorporate a general mechanism leading to local instability of the equilibrium steady state and to chaos as the sensitivity to switch to better prediction strategies increases.

BH [9], [10], [11] have applied this approach to a simple asset pricing model (see also [7] and [34] for comprehensive discussions). Extensions of the model have been studied by Gaunersdorfer [26], Gaunersdorfer and Hommes [27] – henceforth GH –, and Chiarella and He [14], [15]. Under the assumption of homogeneous, rational expectations prices approach the EMH fundamental price. Thus, the model nests the usual rational expectations type of models. Introducing heterogeneous beliefs, however, introduces nonlinearity into the market and rich dynamics emerge, with bifurcation routes to strange attractors. The chaotic price fluctuations are characterized by a switching between phases where prices are close to the fundamental price, phases with temporary upward speculative bubbles, and phases of “pessimism” characterized by declining prices. BH call this “the market is driven by rational animal spirits”.

This paper provides a review of this approach. I describe the model in the next section and present a simple example with two types of traders (fundamentalists and trend followers), where predictor choice is not only based upon evolutionary fitness, but also on market conditions, in section 3. The model captures stylized facts like volatility clustering (with low autocorrelations in returns and significantly positive autocorrelations in absolute returns), excess volatility, and fat tails as observed in real financial data. This is due to the fact that the dynamics is characterized by two phenomena: intermittency (chaotic price fluctuations with phases of almost periodic fluctuations, irregularly interrupted by sudden bursts of erratic fluctuations) and coexistence of attractors (coexistence of a locally fundamental steady state and a “larger” attractor, like a limit cycle). When the system is buffeted with dynamic noise, irregular switching occurs between phases of small price changes and low volatility, when the market is dominated by fundamentalists, and phases of large price fluctuations, when the market is dominated by trend followers. I extend the model of GH [27] by introducing information costs in the

performance measure and present further examples, showing that the results are also valid for more general cases (memory in the performance measure, variation of the parameters in the expectations function of the fundamentalists) than demonstrated in GH [27]. Moreover, I compare the statistical properties of the return series generated by the model with series from the British, German, and Austrian stock market, so as to further document the empirical validity of the model. Section 4 concludes.

2 Adaptive belief systems in a simple asset pricing model

In this section I present a simple asset pricing model in a basic mean variance framework as described in [7] and used in [9], [10]. Consider a financial market where one risky asset and one risk-free asset are traded. The risk-free asset is perfectly elastically supplied at gross return R . The risky asset pays uncertain dividends y_t in future periods t , and therefore has an uncertain return. Let p_t denote the price ex-dividend of this risky asset. The dynamics of wealth of investortype h is given by

$$\tilde{W}_{h,t+1} = RW_{ht} + (\tilde{p}_{t+1} + \tilde{y}_{t+1} - Rp_t)z_{ht},$$

where z_{ht} denotes the numbers of shares of the risky asset purchased at time t . Variables carrying tildes denote random variables. $\mathcal{F}_t = \{p_{t-1}, p_{t-2}, \dots, y_{t-1}, y_{t-2}, \dots\}$ defines a publically available information set consisting of past prices and dividends (\mathcal{F}_t may also contain other economic variables, such as trading volume, interest rates, etc., or other news about economic fundamentals). Let E_{ht} and V_{ht} denote the expectations or *beliefs* of investortype h about expectation $E(\cdot|\mathcal{F}_t)$ and variance $\text{Var}(\cdot|\mathcal{F}_t)$ conditional on this information set. We assume that investors are myopic mean-variance maximizers, so that demand for shares solves

$$\max\{E_{ht}(\tilde{W}_{t+1}) - \frac{a}{2}V_{ht}(\tilde{W}_{t+1})\}, \quad \text{i.e.} \quad z_{ht} = \frac{E_{ht}(\tilde{R}_{t+1})}{aV_{ht}(\tilde{R}_{t+1})}, \quad (1)$$

where $a > 0$ characterizes risk aversion and $\tilde{R}_{t+1} = \tilde{p}_{t+1} + \tilde{y}_{t+1} - Rp_t$ is the excess return per share.

Let z_{st} and n_{ht} denote the supply of shares per investor and the fraction of investors of type h at time t , respectively. Then equilibrium of supply and demand implies

$$\sum_h n_{ht}z_{ht} = z_{st}.$$

Assuming that the number of outstanding shares per investor, z_{st} , is constant over time, we can stick to the simple case $z_{st} \equiv 0$.⁵ Thus, the market equilibrium equation writes as⁶

$$\sum_h n_{ht}z_{ht} = 0. \quad (2)$$

⁵ In the general case one can rewrite the market equilibrium equation as in (2) by introducing risk adjusted dividends $y_{t+1}^\# = y_{t+1} - aV_{ht}(p_{t+1} + y_{t+1})z_{st}$, see [7].

⁶ In contrast to this Walrasian market clearing mechanism, where each agent is viewed as a price taker, Chiarella and He [15] use a market maker scenario.

In order to obtain a benchmark *REE fundamental solution* consider the special case that *all* investors are perfectly rational and their expectations of future prices and dividends are given by the conditional expectations $E(\cdot|\mathcal{F}_t)$. Then the market equilibrium equation (2) reduces to

$$Rp_t = E(\tilde{p}_{t+1} + \tilde{y}_{t+1}|\mathcal{F}_t). \quad (3)$$

That is, today's price of the risky asset must equal the sum of expected tomorrow's price and dividend, discounted by the risk-free rate of return. In such a world where investors are homogeneous and perfectly rational and where it is common knowledge that all investors are rational, the REE or fundamental price is given by the discounted sum of expected future dividends,

$$p_t^* = \sum_{k=1}^{\infty} \frac{E(\tilde{y}_{t+k}|\mathcal{F}_t)}{R^k}.$$

This solution is based on the fact that the fundamental p_t^* must be finite, which means that the long run growth rates of dividends and prices must be smaller than the risk-free growth rate $R - 1$. Note, that also so-called speculative bubble solutions growing at the constant risk-free rate satisfy the market equilibrium equation (3) (for more details see [34]). Along such a rational bubble solution, traders would have perfect foresight. But in a world where all traders are rational they realize that such a rational bubble cannot last forever and therefore, it will never get started. Thus, all traders believe that the value of the risky asset equals its fundamental price forever and deviations are solely driven by unexpected changes in dividends and random news about economic fundamentals. However, in a market where traders have heterogeneous beliefs the situation will be quite different.

We make the following assumptions about the beliefs of the traders.

- A1. Investors have homogeneous beliefs about one step ahead conditional means of earned dividends, $E_{ht}(\tilde{y}_{t+1}) = E(\tilde{y}_{t+1}|\mathcal{F}_t)$, $\forall h, t$.
- A2. Beliefs about the conditional means of next period prices are of the form $E_{ht}(\tilde{p}_{t+1}) =: p_{h,t+1}^e = E(\tilde{p}_{t+1}^*|\mathcal{F}_t) + f_h(x_{t-1}, x_{t-2}, \dots)$, $\forall h, t$, where $x_t = p_t - p_t^*$ and f_h is a deterministic, agent specific function of past deviations from a commonly shared view of the fundamental price.

Realized excess returns over period t to period $t + 1$, can be computed as

$$\tilde{R}_{t+1} = \tilde{p}_{t+1} + \tilde{y}_{t+1} - Rp_t = \tilde{x}_{t+1} - Rx_t + \tilde{\delta}_{t+1},$$

where $\tilde{\delta}_{t+1} = \tilde{p}_{t+1}^* + \tilde{y}_{t+1} - E(p_{t+1}^* + \tilde{y}_{t+1}|\mathcal{F}_t)$ is a martingale difference sequence w.r.t. \mathcal{F}_t , representing intrinsic uncertainty about economic fundamentals. Thus, realized excess returns can be decomposed in an EMH-term $\tilde{\delta}_t$ and an endogenous dynamic term explained by the theory presented here.⁷ This term is nonzero if the price deviates from the fundamental value and thus allows for the possibility of excess volatility.

⁷ This extra term is rather like "endogenous uncertainty" as in Kurz [37], who presents a theory of rational belief equilibria.

We focus on the simple case of an iid dividend process with constant mean $E(\tilde{y}_{t+1}|\mathcal{F}_t) = \bar{y}$ and finite variance, so that the fundamental price is given by $p^* = \bar{y}/(R - 1)$. In that case $\tilde{\delta}_t = \tilde{y}_t - \bar{y}$. This implies

$$E_{ht}(\tilde{R}_{t+1}) = p_{h,t+1}^e - p^* - R(p_t - p^*)$$

and

$$\tilde{R}_{t+1} - E_{ht}(\tilde{R}_{t+1}) = \tilde{p}_{t+1} - p_{h,t+1}^e + \tilde{\delta}_{t+1}. \quad (4)$$

Assuming that $\text{Cov}(\tilde{x}_{t+1} - Rx_t, \tilde{\delta}_{t+1}) \equiv 0$ and $V_t(\tilde{\delta}_{t+1}) =: \sigma_\delta^2$ is constant, we obtain $V_t(\tilde{R}_{t+1}) = V_t(\tilde{x}_{t+1} - Rx_t) + \sigma_\delta^2$.⁸ Gaunersdorfer [26] studies an example where investors have time varying homogeneous beliefs about these conditional variances. The beliefs about $V_t(x_{t+1} - Rx_t)$ are modeled as exponentially moving averages. The qualitative characteristics of the model are quite similar to the case where the beliefs about conditional variances are assumed to be constant. In particular, the local behavior of the steady states coincides in the cases with time varying beliefs and with constant ones. Also the bifurcation routes to chaos are similar. Chiarella and He [14], [15] introduce heterogeneity in the beliefs about variances. They study how the dynamics of asset pricing is affected by different risk attitudes of various investors, as characterized by the different risk aversion coefficients. They show that different attitudes to risk have an influence on the price dynamics. In particular, when the fundamentalists are more risk averse than the trend traders the market becomes more unstable, which is in line with De Long et al. [20].

In what follows, we assume constant beliefs about conditional variances,

$$V_t(\tilde{R}_{t+1}) \equiv \sigma^2. \quad (5)$$

In order to complete the model we have to specify the way the fractions n_{ht} are formed over time. These fractions are defined as discrete choice ‘‘Gibbs’’ probabilities (see, for example [45] or [1])

$$\begin{aligned} n_{ht} &= \text{prob}(\text{choose predictor } h \text{ at time } t) = \exp(\beta U_{h,t-1})/Z_t \\ Z_t &= \sum_h \exp(\beta U_{h,t-1}), \end{aligned} \quad (6)$$

where $U_{h,t-1}$ is a fitness function or performance measure. The parameter β is called *intensity of choice*. It measures how fast the mass of investors switches between the different predictors and thus how sensitive traders are to differences in the fitness. If $\beta = 0$ traders choose predictors randomly (and independently of their performance) with probability $1/H$, where H is the number of different investor types. In the other extreme case, $\beta = \infty$, all traders choose the predictor which performed best in the recent past. For finite, positive β agents are boundedly rational in the sense that fractions of the chosen predictors are ranked according to their fitness. BH [8], [9] and Gaunersdorfer [26] show that this parameter plays a

⁸ In the terminology of Haugen [33] one could interpret $V_t(\tilde{x}_{t+1} - Rx_t)$ as price driven volatility.

crucial role in the bifurcation route to chaotic price fluctuations. In particular, for large values of β the system is close to having a homoclinic orbit, which is a key feature of chaotic systems.

To define the fitness function remember that investors are myopic mean-variance maximizers. We therefore define the performance measure as utilities derived from realized profits or past realized risk-adjusted profits (see [26]). Let

$$\rho_{ht} := E_{ht}(R_{t+1}).$$

Utilities of realized profits in period t are given by

$$\pi_{ht} := \pi(R_{t+1}, \rho_{ht}) := R_{t+1}z(\rho_{ht}) - \frac{a}{2}\sigma^2 z^2(\rho_{ht}), \quad (7)$$

where

$$z(\rho_{ht}) = \arg \max \pi_{ht} = \frac{\rho_{ht}}{a\sigma^2} = z_{ht}.$$

Note, that maximizing expected utility of profits is equivalent to maximizing expected utility of wealth, thus $z(\rho_{ht})$ coincides with (1). Since (6) does not change if the same term is subtracted off the exponents (i.e. fractions are independent of utility levels), we may consider

$$\begin{aligned} \pi_{ht} - \pi_t &= \pi(R_{t+1}, \rho_{ht}) - \pi(R_{t+1}, R_{t+1}) \\ &= -\frac{1}{2a\sigma^2}(E_{ht}R_{t+1} - R_{t+1})^2 \\ &= -\frac{1}{2a\sigma^2}(p_{t+1} - p_{h,t+1}^e + \delta_{t+1})^2 \end{aligned}$$

(see equation (4)), where $\pi_t = \pi(R_{t+1}, R_{t+1})$ are utilities of profits if investors had perfect foresight.

Since we will mainly focus on the deterministic skeleton, that is, the nonlinear asset price dynamics with $\delta_{t+1} \equiv 0$ and constant dividends \bar{y} , we define the performance measure as

$$U_{ht} = \pi_{h,t-1} - \pi_{t-1} + \eta U_{h,t-1} = -\frac{1}{2a\sigma^2}(p_t - p_{ht}^e)^2 + \eta U_{h,t-1}, \quad (8)$$

where the parameter η , $0 \leq \eta \leq 1$, represents ‘‘memory strength’’. The performance measure is thus determined by squared prediction errors. In the special case $\eta = 0$, fitness is given by utilities of realized profits in the most recent past, for positive η it is an exponentially moving average of past utilities of profits. Note, that the performance measure (and hence the fractions) in period t depend upon observable prices at the end of the previous (beginning of this) period.

Another possibility for the performance measure is to take past realized profits ((7) without risk-adjustment term) as in BH [9], [10], [11], with $\pi_{ht} = R_{t+1}z(\rho_{ht})$. For $\eta = 1$ the fitness function coincides with accumulated wealth. In this non-risk adjusted case the fitness function is however inconsistent with the traders being myopic mean-variance maximizers of wealth. BH (1999) show for this non-risk adjusted case that increasing memory strength η has a similar effect concerning the bifurcation route to complicated price dynamics as increasing the intensity of

choice β . They conjecture, however, that the situation is different for the risk adjusted case, where more memory acts as *stabilizing* force (if costs for all trading strategies are zero).

3 A simple example: fundamentalists versus trend followers

Consider a simple example with two types of traders where the beliefs about future prices are given by linear prediction rules

$$\begin{aligned} E_{1t}[p_{t+1}] &\equiv p_{1,t+1}^e = p^* + v(p_{t-1} - p^*), & 0 \leq v \leq 1, \\ E_{2t}[p_{t+1}] &\equiv p_{2,t+1}^e = p_{t-1} + g(p_{t-1} - p_{t-2}), & g \geq 0. \end{aligned}$$

The first trader type are so-called fundamentalists who believe that prices will move toward the fundamental value in the next period (if $v < 1$). The case $v = 1$ corresponds to traders taking today's price as best forecast for tomorrow's. These traders believe that markets are efficient and prices follow a random walk. We call them EMH-believers. The second trader type are trend followers extrapolating the latest observed price change. Thus, the predictor function for future prices of this trader type incorporates two time lags, in contrast to BH [9], [10], [11] who only consider prediction functions with one lag. Chiarella and He [14], [15] study the impact of learning schemes with different lag lengths in the formation of expectations.

The fractions are updated according to past performance as described in the previous section, conditioned upon the deviations of actual prices from the fundamental price (cf. the Santa Fe artificial stock market [3], [40]). The fundamental predictor may be costly since it needs some effort to understand how the market works and to believe that it will price according to the fundamental. The first, evolutionary part of the updating of fractions is determined by risk adjusted realized profits as described in the previous section,

$$\begin{aligned} \hat{n}_{1t} &= \exp(\beta(U_{1,t-1} - C)) / Z_t \\ \hat{n}_{2t} &= \exp(\beta U_{2,t-1}) / Z_t \\ Z_t &= \sum_{h=1}^2 \exp(\beta(U_{h,t-1} - C_h)), \end{aligned} \tag{9}$$

where $U_{h,t-1}$ is defined by (8) and $C_1 = C \geq 0$ are the costs to obtain the fundamental predictor (representing traders effort for information gathering), $C_2 = 0$.⁹ The second part of updating conditions on deviations from the fundamental. If prices are too high or too low technical traders might get nervous and do not believe that a price trend which continued for a while will go on any longer and a

⁹ GH [28] use past realized profits as performance measure. It turns out that the dynamics of the model and the time series properties are very similar to the case where risk adjusted realized profits are used as fitness measure.

correction to the fundamental is about to occur. That is, traders believe that *temporary* speculative bubbles may arise, but these bubbles will not last forever. The fractions of the two trader types are now modeled as

$$n_{2t} = \hat{n}_{2t} \exp[-(p_{t-1} - p^*)^2/\alpha], \quad \alpha > 0 \quad (10)$$

$$n_{1t} = 1 - n_{2t}. \quad (11)$$

According to (10) fractions are almost completely determined by evolutionary fitness as long as prices do not deviate too much from the fundamental value. But when the correction term $\exp[-(p_{t-1} - p^*)^2/\alpha]$ becomes too small most chartist switch to the fundamental predictor. Thus, technical traders are conditioning their charts upon information about fundamentals.

BH [9], [10], [11] only take the evolutionary part for the updating of the fractions. Gaunersdorfer [26] introduces a so-called stabilizing force in the performance measure of the fundamentalists which has a similar effect as the correction term in (10).

Setting $\hat{U}_{ht} := U_{h,t-1}$, we obtain the following dynamical system for the price dynamics,

$$\begin{aligned} p_t &= \frac{1}{R}(p_{1,t+1}^e + n_{2t}(p_{2,t+1}^e - p_{1,t+1}^e) + \bar{y}) \\ \hat{U}_{ht} &= -\frac{1}{2\alpha\sigma^2}(p_{t-1} - p_{h,t-1}^e)^2 + \eta\hat{U}_{h,t-1}, \quad h = 1, 2. \end{aligned} \quad (12)$$

Since the fractions n_{2t} are nonlinear functions of previous prices, this defines a *nonlinear* six dimensional system in $(p_1, p_2, p_3, p_4, u_1, u_2)$, where $p_i(t-1) = p_{t-i}$ and $u_h(t-1) = \hat{U}_{h,t-1} = U_{h,t-2}$. Expectations about future prices, which are determined by past prices, feed back on the actual price.

3.1 Dynamics of the Asset Pricing Model

In the following I briefly discuss the dynamics of system (12). A comprehensive analysis for the case without information costs, $C = 0$, is provided by Gaunersdorfer, Hommes, and Wagener [29], henceforth GHW.

Lemma 1 summarizes the stability analysis of the fundamental steady state for the general case of positive information costs.

Lemma 1 (Existence and stability of steady states)

Let $\eta < 1$. System (12) has a unique steady state $(p^*, p^*, p^*, p^*, 0, 0)$ which is locally stable if

$$-\frac{1}{2}(R + 1 + (v + R)e^{-\beta C}) < g < R(1 + e^{-\beta C}).$$

(1) For parameters satisfying

$$2g + R + 1 + (v + R)e^{-\beta C} = 0, \quad (13)$$

the steady state undergoes a period doubling (flip) bifurcation.

(2) For parameters satisfying

$$g - R(1 + e^{-\beta C}) = 0, \quad (14)$$

the steady state undergoes a Hopf (Neimark-Sacker) bifurcation.

Proof. The characteristic polynomial of the Jacobian at the steady state is given by

$$p(\lambda) = \lambda^2(\eta - \lambda)^2 \left(\lambda^2 - \frac{1 + g + ve^{-\beta C}}{R(1 + e^{-\beta C})} \lambda + \frac{g}{R(1 + e^{-\beta C})} \right). \quad (15)$$

Thus, the eigenvalues of the Jacobian are 0, η (both of multiplicity 2), lying inside the unit circle, and the roots of the last bracket. For the rest of the proof proceed in analogy to the proofs of Lemmas 1 and 2 in [29]. \square

Lemma 1 shows that the local dynamics around the steady state is the same without memory ($\eta = 0$) and with memory ($0 < \eta < 1$).

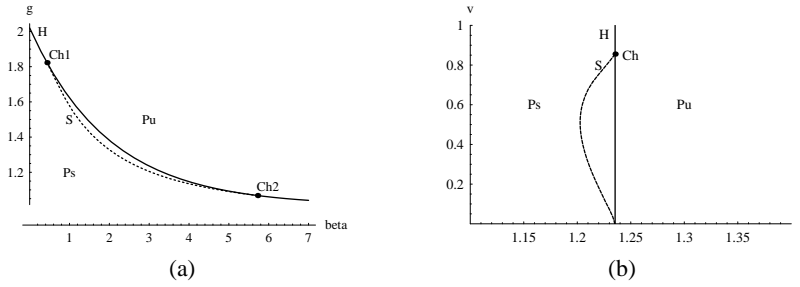


Fig. 1 Bifurcation diagrams for parameter values $C = 0.5$, $R = 1.01$, $\alpha = 10$, $\eta = 0$, $a\sigma^2 = 1$, $\bar{y} = 1$, $p^* = 100$ in the (a) β - g -plane for $v = 0.6$, (b) v - g -plane for $\beta = 3$. $H : g = R(1 + e^{-\beta C})$ is the Hopf bifurcation curve. Ch are points where a Chenciner bifurcation occurs. These points are origin points of a saddle-node bifurcation curve S (dotted curve) of invariant circles. In region $P_s = \{0 < g < R(1 + e^{-\beta C})\}$ the steady state is locally stable, in region $P_u = \{g > R(1 + e^{-\beta C})\}$ it is unstable. As $C \rightarrow 0$, $Ch2$ in diagram (a) moves to infinity. As C is increased, the two Chenciner points approach each other and disappear after collision (i.e. for C large enough the Hopf bifurcation is always supercritical).

Figure 1 shows bifurcation diagrams of the steady state in the β - g and in the g - v -plane, respectively. Ch denote points where the Hopf bifurcation is degenerate and a so-called *Chenciner bifurcation* occurs. At these points the Hopf bifurcation changes from supercritical to subcritical (or vice versa). These points are origin points of a curve S of saddle-node bifurcations of invariant circles.¹⁰ In the region $P_s := \{g < R(1 + e^{-\beta C})\}$ the steady state is stable. When crossing the Hopf bifurcation curve H to the region $P_u := \{g > R(1 + e^{-\beta C})\}$ at parameter value

¹⁰ In the three dimensional parameter space $\{(\beta, g, v)\}$ the Hopf bifurcation manifold H and the set of saddle-node bifurcations of invariant circles S define two dimensional surfaces.

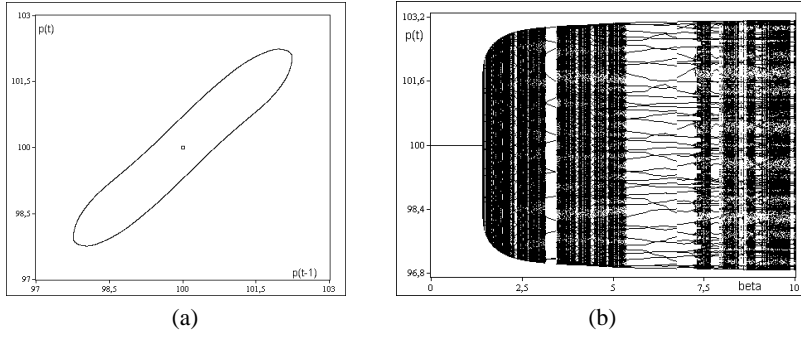


Fig. 2 (a) Phase space projection on the p_{t-1} - p_t -plane for parameter values $\beta = 1.5$, $v = 0.6$, $g = 1.45$, $C = 0.5$, $R = 1.01$, $\alpha = 10$, $a\sigma^2 = 1$, $\eta = 0$, $\bar{y} = 1$, $p^* = 100$ (“volatility clustering region” between S and H): coexisting stable steady state (marked as a square) and stable limit cycle. (b) One-parameter bifurcation diagram p_t versus β for $v = 0.6$, $g = 1.45$, $C = 0.5$, $R = 1.01$, $\alpha = 10$, $a\sigma^2 = 1$, $\eta = 0$, $\bar{y} = 1$, $p^* = 100$.

$\beta < \beta_{Ch1}$ or $\beta > \beta_{Ch2}$ in Figure 1(a), or $v > v_{Ch}$ in Figure 1(b), the steady state becomes unstable by a supercritical Hopf bifurcation and a stable invariant circle is created. Crossing H from region P_u into the region between H and S , a subcritical Hopf bifurcation occurs and an unstable invariant circle arises from the unstable steady state. The steady state becomes stable, whereas the “outer” stable invariant circle still exists. Thus, in the region between the two curves H and S two attractors coexist: the stable steady state and a stable limit cycle (see figure 2(a)). GHW [29] call this the “volatility clustering region”. “Between” these two attractors lies an unstable invariant circle. The stable and the unstable invariant circles collide and disappear at the bifurcation curve S . The emergence of the limit cycle is clearly seen in the bifurcation diagram in figure 2(b). Note the big jump in the amplitude of the attractor at $\beta \approx 1.4$, when parameters cross the saddle-node curve of invariant circles S (cf. also figure 1(b)).¹¹ In the diagram we see windows where the dynamics are phase locked and periodic cycles exist.

GHW [29] show examples for $C = 0$ where the limit circle in the “volatility clustering region” may undergo homoclinic bifurcations and turn into a strange attractor. In such a case a stable steady state and a chaotic attractor “around” it coexist. Figure 3 indicates parameter regions where the largest Lyapunov exponents of orbits on the attractors are positive.¹² In these regions the system has a chaotic attractor. In order to account for coexisting attractors, for every pair of param-

¹¹ The exact value, where the saddle-node bifurcation of invariant circles occurs, can be picked out from figure 1(b). Of course, it is not possible to conclude the exact value from diagram 2(b). Because of coexistence of attractors, depending on initial conditions, a trajectory may either converge to the steady state or to the limit cycle. In diagram 2(b) 1000 points were plotted for every β -value after cutting off the first 10000 points. For every β -value the same initial value was chosen.

¹² I am indebted to Florian Wagener for providing these plots.

ter values the Lyapunov exponents have been computed for ten different initial conditions.

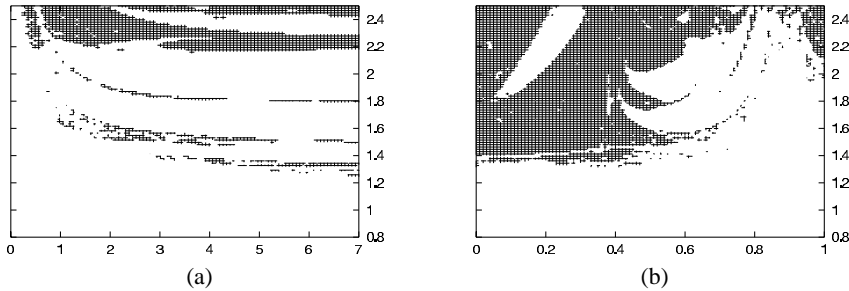


Fig. 3 Regions of positive Lyapunov exponents (a) β - g -plane for $v = 0.6$, (b) in the v - g -plane for $\beta = 3$, with other parameters as in Figure 1 (horizontal axis: (a) β , (b) v ; vertical axis: g)

If system (12) is buffeted with dynamic noise, the price dynamics is characterized by fairly regular phases with prices close to the fundamental steady state, suddenly interrupted by large price changes caused by excursions to the limit cycle triggered by technical trading. Such a phenomenon is known as intermittency. The price dynamics of the chaotic attractor in figure 4 also shows these characteristics: phases of low price fluctuations close to the fundamental interchange with large fluctuations when prices deviate from the fundamental, suggesting some form of volatility clustering.

The geometric shape of the chaotic attractor in figure 4 gives some insight into the structure of the dynamics. Consider a situation where prices are close to the fundamental steady state p^* . For $g > R(1 + e^{-\beta C})$ the steady state is a saddle point (and hence unstable), thus prices will move away. Since the fundamentalists expect that prices return to the fundamental value their performance is bad and the fractions of the trend followers will increase. If prices deviate too much from the fundamental, the factor $\exp[-(p(t) - p^*)^2/\alpha]$ starts to play a role and the fraction of the trend followers n_2 decreases and the fundamentalists take over. In the case when the fundamentalists dominate ($n_1 \approx 1$, $n_2 \approx 0$), the price dynamics is close to

$$p(t) - p^* = \frac{v}{R}(p(t-1) - p^*),$$

which defines a line in the p_{t-1} - p_t -plane, and prices will move along this line towards the fundamental value (this line is clearly seen in figure 4(a)). For more details see [29].

The presented deterministic model is in fact too simple to capture the dynamics of a financial market. In the following section a dynamic noise term is added to (12), representing model approximation error. The analysis given above is however important to understand time series properties of the noisy model.

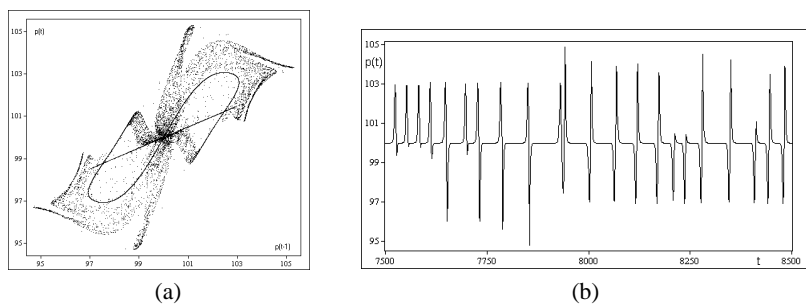


Fig. 4 (a) Phase space projection of a chaotic attractor on the p_{t-1} - p_t -plane and (b) corresponding price series showing intermittent chaos; $\beta = 10$, $v = 0.5$, $g = 2.1$, $C = 0.2$, $R = 1.01$, $\alpha = 10$, $a\sigma^2 = 1$, $\eta = 0.9$, $\bar{y} = 1$, $p^* = 100$

3.2 Time series properties

Figure 5 shows the (continuously compounded) return series $r_t = \log(p_t/p_{t-1})$ for the FTSE, the DAX (German traded index), the ATX (Austrian traded index), and the Austrian stock OMV (Austrian oil company).¹³ Table 1 reports the descriptive statistics of these series.

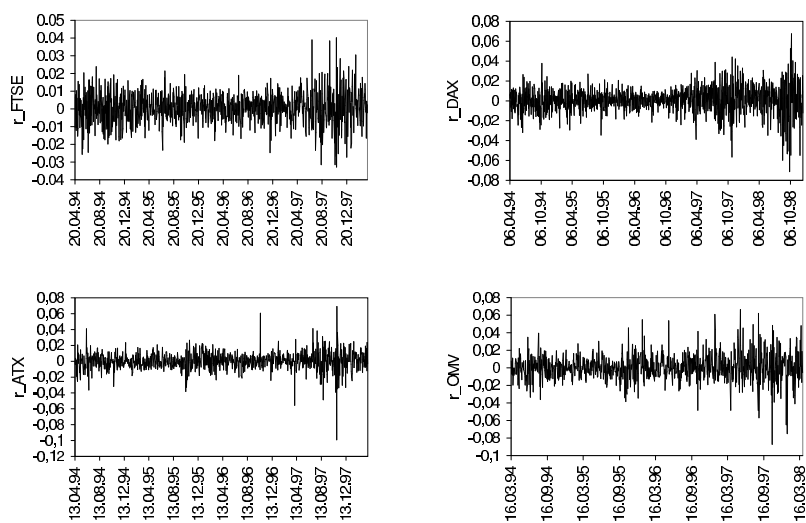


Fig. 5 Returns series of the FTSE (04/20/94–03/13/98), the DAX (04/06/94–03/31/98), the ATX (04/13/94–03/31/98), and the OMV (03/16/94–03/31/98), 1000 observations each

¹³ I am indebted to Robert Tompkins for providing these data. Tompkins [58] provides an analysis of returns and volatility of the Austrian data.

The means and medians are close to zero for all series. All series exhibit excess kurtosis and the hypothesis of normal distribution is clearly rejected by the Jarque-Bera test statistics.¹⁴ The skewness statistics significantly deviates from a normal distribution at the 1% level for the DAX and the ATX, but is not significant for the FTSE and the OMV. As pointed out for example by Tompkins [58] and other authors, the skewness statistics is not significant nor of same sign for all markets. Nevertheless, some authors examine the skewness in addition to excess kurtosis. In a recent paper Harvey and Siddique [32] argue that skewness may be important in investment decisions because of induced asymmetries in realized returns. They present an asset pricing model where skewness is priced and suggest to analyze portfolios in a conditional mean-variance-skewness framework instead of a conditional mean-variance framework.

Table 1 Descriptive statistics of returns

Series	r_FTSE	r_DAX	r_ATX	r_OMV
Mean	0.000656	0.000795	0.000269	0.000527
Median	0.000925	0.001238	8.32E-05	-8.89E-05
Maximum	0.040191	0.044072	0.069210	0.066413
Minimum	-0.032857	-0.056536	-0.099152	-0.087169
Std. Dev.	0.008998	0.010976	0.011182	0.014929
Skewness	-0.059386	-0.200216**	-0.597217**	-0.043749
Kurtosis	4.310033**	4.867463**	12.60597**	6.640624**
Jarque-Bera	72.09555	151.9901	3904.221	552.5750

** null hypothesis of normality rejected at the 1% level

Figure 6 shows autocorrelation plots for the first 36 lags of the returns series and the absolute returns series. For the indices all low order autocorrelations are insignificant. OMV has significant, but small autocorrelation at the first and third lag. In contrast, the autocorrelations of the absolute returns (which are a measure for the volatility) are clearly positive significant at low order lags. That is, all series exhibit volatility clustering.

Our aim is to capture as many as possible of these observed facts by our model. BH [9] calibrate their model to ten years of monthly IBM prices and returns. The autocorrelation structure of their noisy chaotic price and return series are similar to those of the real data. However, squared returns do not have significant autocorrelations.

GH [27] compare the time series properties of the return paths generated by the model without costs ($C = 0$) with 40 years daily S&P 500 data. They show that the model is able to generate clustered volatility and to capture some of the

¹⁴ The Jarque-Bera statistics measures the difference of the skewness and kurtosis of the series with those from the normal distribution. Under the null hypothesis of normality the Jarque-Bera statistic is Chi-squared distributed with two degrees of freedom. The critical value at the one percent level is 9.21.

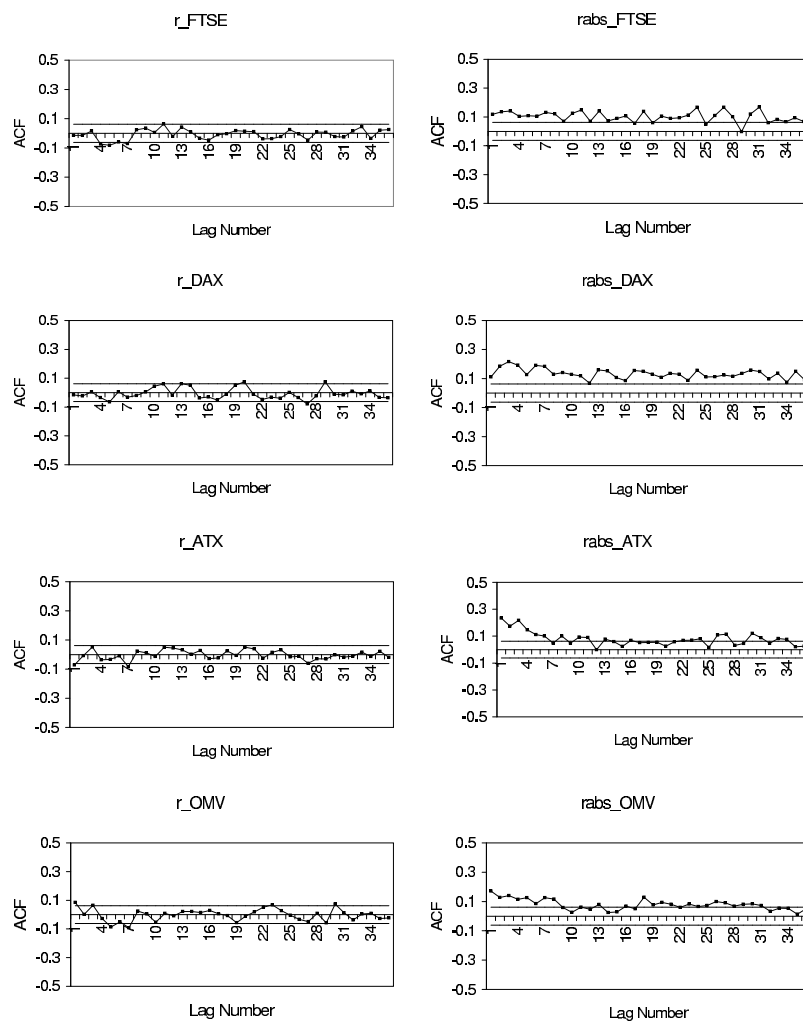


Fig. 6 Autocorrelations of returns and absolute returns of the FTSE, the DAX, the ATX, and the OMV

descriptive statistical properties. Here I present some further examples, including an example with positive costs C for the fundamental predictor.

Returns are defined as relative price changes

$$r_t = \frac{p_t - p_{t-1}}{p_{t-1}}.$$

The price series are generated by system (12) with a dynamic noise term added to the first equation of the system,¹⁵ i.e.

$$p_t = \frac{1}{R}(p_{1,t+1}^e + n_{2t}(p_{2,t+1}^e - p_{1,t+1}^e) + \bar{y}) + \varepsilon_t$$

$$\hat{U}_{ht} = -\frac{1}{2a\sigma^2}(p_{t-1} - p_{h,t-1}^e)^2 + \eta\hat{U}_{h,t-1}, \quad h = 1, 2,$$

where ε_t are iid normally distributed random variables. This noise term can be interpreted as unexpected news about fundamentals or noise created by “noise traders”, agents whose trading is not modeled, but exogenously given.

The return series for the following examples are shown in Figure 7. I have only introduced positive costs for the case $v < 1$. In the case $v = 1$ the first trader type takes the last observed price as predictor for tomorrow’s price. It seems unreasonable to assume positive information costs for using such a predictor. In brackets the dynamical behavior of the system without noise is indicated.

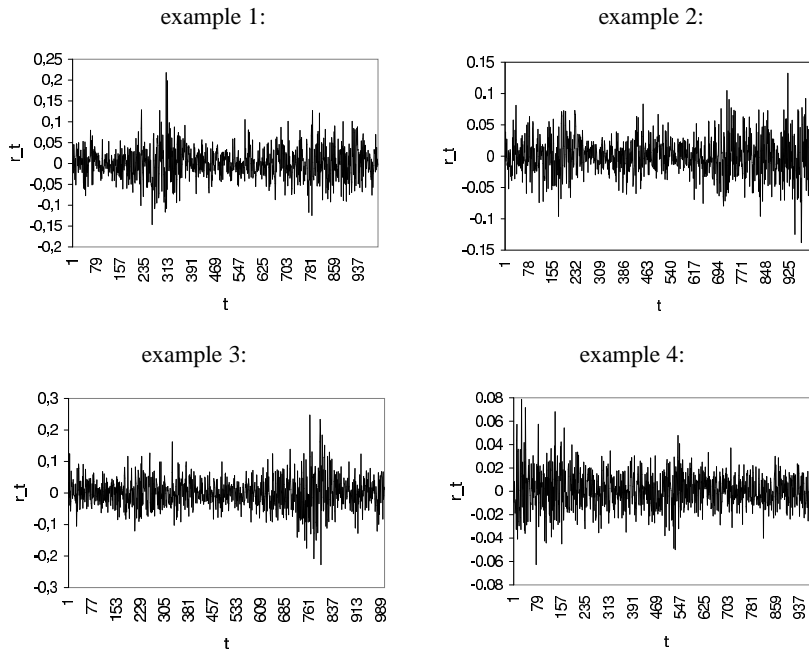


Fig. 7 Return series of examples 1–4

¹⁵ We also ran simulations with the stochastic term δ_t , defined in equation (2), included in the performance measure. However, since δ_t has to be small in comparison to \bar{y} , this hardly affects the results.

Example 1 (coexisting stable steady state and stable limit circle):

$$\beta = 5, R = 1.01, v = 1, g = 2, C = 0, \eta = 0, \bar{y} = 1, \alpha = 10, a\sigma^2 = 1, \varepsilon \sim N(0, 4)$$

Example 2 (stable steady state):

$$\beta = 2, R = 1.01, v = 1, g = 1.5, C = 0, \eta = 0, \bar{y} = 1, \alpha = 10, a\sigma^2 = 1, \varepsilon \sim N(0, 3)$$

Example 3 (stable limit circle, unstable steady state):

$$\beta = 5, R = 1.01, v = 0.9, g = 1.5, C = 0.5, \eta = 0.2, \bar{y} = 1, \alpha = 10, a\sigma^2 = 1, \varepsilon \sim N(0, 5)$$

Example 4 (stable steady state):

$$\beta = 5, R = 1.003, v = 1, g = 1.5, C = 0, \eta = 0.2, \bar{y} = 1, \alpha = 10, a\sigma^2 = 1, \varepsilon \sim N(0, 6)$$

Table 2 shows the descriptive statistics for these examples, the autocorrelation functions of the returns and absolute returns are plotted in Figure 8. The series clearly have excess kurtosis, comparable in size to those of the FTSE and the DAX. The autocorrelations of the returns are for all examples small or not significant, whereas the low order autocorrelations of the absolute returns are all significant and positive. Thus, the model is able to capture the phenomenon of volatility clustering. In examples 1–3 the range of the returns is larger than for the real data. If R is decreased the range becomes smaller as demonstrated by example 4. In fact, for daily data a smaller value of R seems economically more reasonable.¹⁶

Table 2 Descriptive statistics for examples 1–4

Series	Example 1	Example 2	Example 3	Example 4
Mean	0.000732	-0.000739	0.001296	0.000206
Median	0.000732	-0.001567	0.001972	-0.000527
Maximum	0.217724	0.132335	0.247417	0.078711
Minimum	-0.146266	-0.138097	-0.227208	-0.062582
Std. Dev.	0.038897	0.031364	0.050282	0.015815
Skewness	0.258151**	-0.022407	0.168476 *	0.243409**
Kurtosis	4.882617**	4.076606**	5.028881**	4.517690**
Jarque-Bera	158.7839	48.37866	176.2455	105.8490

* null hypothesis of normality rejected at the 5% level

** null hypothesis of normality rejected at the 1% level

Our simulations demonstrate that volatility clustering may even occur for parameters where the deterministic skeleton of the model has a globally stable steady state. Note that for parameter values close to the “volatility clustering region”, the system is *close* to having a limit cycle and orbits spiral *slowly* towards the steady state and appear almost (quasi)periodic. In particular, the transient phases in and

¹⁶ I chose most examples with $R = 1.01$ for numeric reasons. Since prices increase when R is decreased the exponents in equation (9) become rather larger numbers resulting sometimes in overflows in the simulations.

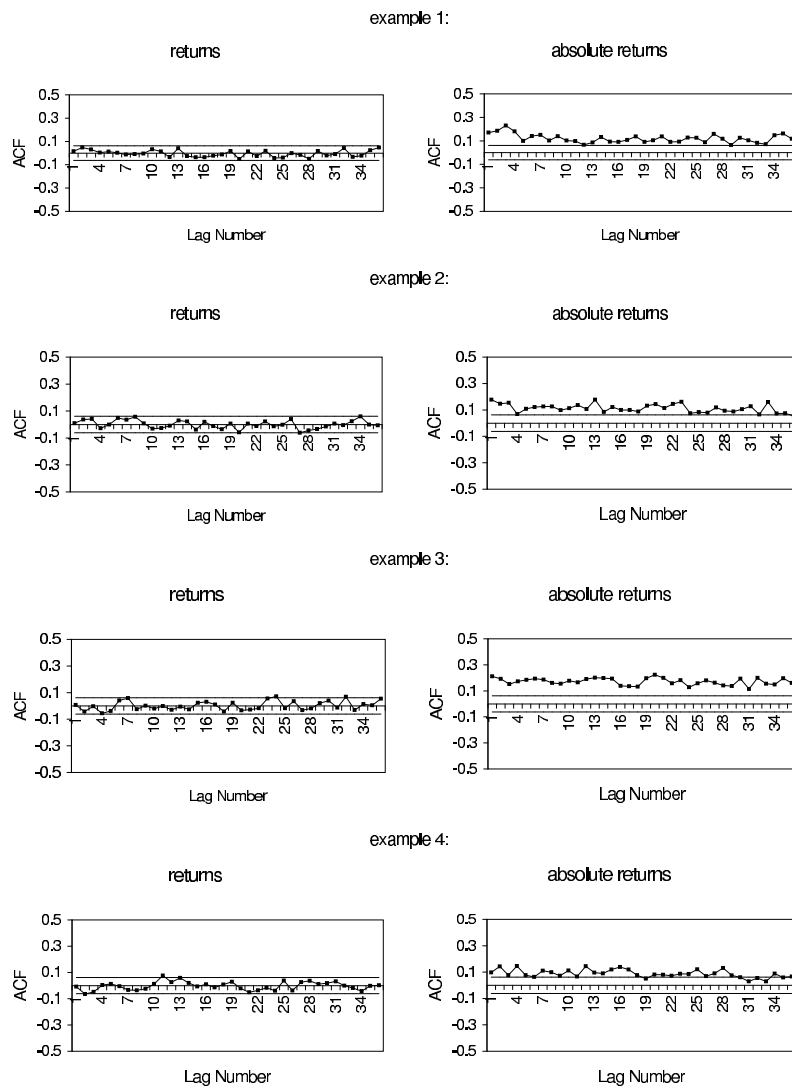


Fig. 8 Autocorrelations of returns and absolute returns of examples 1–4

out of the volatility clustering region are very similar. When the system is buffeted with noise, it is not possible to distinguish if parameters are chosen in or out of the region where a limit cycle exists. In fact, the bifurcations of the deterministic system (cf. Figure 2) cannot be detected in the noisy model. Nevertheless, the analysis of the deterministic system gives important insight into the dynamics of the noisy system.

We obtain the best results when the parameter v is equal or close to 1, i.e. when EMH-believers interact with trend followers. EMH-believers think that prices fol-

low a random walk. Therefore, when this type of traders dominates the market, prices are highly persistent and price changes are small, only driven by random news. Thus, returns are close to zero and volatility is small. As prices move towards the fundamental value, trend followers perform better than EMH-believers and their fraction gradually increases. When they start dominating the market, big price changes occur and volatility becomes large. However, as prices move too far away from the fundamental value, technical traders stop chasing trends, the fraction of EMH-believers increase and the story repeats. Though our system is stationary (due to the fact that we assumed a simple iid dividend process implying a constant fundamental value), it is *close* to having a *unit root*. Note that for $v = R = 1$ and $C = 0$ the characteristic polynomial (15) has an eigenvalue 1.¹⁷ This is illustrated by the results of unit roots test, see table 3. For the examples with $v = 1$ the null hypothesis of a unit root (i.e. non-stationarity) is not rejected for a sample size of 1000 observations, whereas it is clearly rejected if $v < 1$. Nevertheless, example 3 demonstrates that the results are robust with respect to (small) changes of the parameter v . If the sample size is increased the test statistics decreases and the null hypothesis is rejected also in the case $v = 1$ for longer time series. The price series are highly persistent for $v = 1$, the autocorrelation of the first is close to one (see table 3) and is only decreasing very slowly at higher lags, as it is the case in real price series.

Table 3 Unit root test statistics and autocorrelation of first lag on price series

	Example 1	Example 2	Example 3	Example 4
Augmented Dickey-Fuller	-2.386540	-2.684564	-6.986619	-1.866630
Philips Perron	-2.295858	-2.721507	-7.719384	-1.866888
ACF (lag 1)	0.991	0.980	0.895	0.996

MacKinnon critical values for rejection of hypothesis of a unit root:
1%: -3.4396, 5%: -2.8648, 10%: -2.5685

Finally, I estimate GARCH(1,1) models on the return series. The GARCH(1,1) model is specified as

$$\begin{aligned} r_t &= c_1 + \delta_t, & \delta_t &\sim N(0, \sigma_t) \\ \sigma_t^2 &= c_2 + \rho_1 \delta_{t-1}^2 + \rho_2 \sigma_{t-1}^2. \end{aligned}$$

The model can be rewritten in terms of errors

$$\delta_t^2 = c_2 + (\rho_1 + \rho_2) \delta_{t-1}^2 + \nu_t - \rho_2 \nu_{t-1},$$

where $\nu_t = \delta_t^2 - \sigma_t^2$ is the error in squared returns. Thus, the squared errors follow a heteroskedastic ARMA(1,1) process. The autoregressive root which governs the persistence of volatility shocks is the sum of ρ_1 (ARCH term) and ρ_2 (GARCH

¹⁷ Note that the Jacobian of the linear difference equation $y_t = \alpha_0 + \sum_{k=1}^L \alpha_k y_{t-k}$ has an eigenvalue 1 if and only if the time series $y_t = \alpha_0 + \sum_{k=1}^L \alpha_k y_{t-k} + \varepsilon_t$ has a unit root equal to 1.

term). In real return data this root is usually very close to unity, which means that shocks die out rather slowly. This fact can also be observed in the series of the FTSE, the DAX, and the OMV (see table 4). The GARCH terms and the ARCH terms are clearly significant and the GARCH term is much larger than the ARCH term. This is also the fact for the coefficients of the GARCH and ARCH terms of the simulated series, in particular, they resemble those of the FTSE, the DAX, and the OMV.¹⁸

Table 4 GARCH(1,1) estimations for return series, with (standard errors) and [*p*-values]

Series	c_1	c_2	ρ_1	ρ_2	$\rho_1 + \rho_2$
FTSE	0.000657 (0.000256) [0.0103]	7.15E-07 (4.02E-07) [0.0755]	0.039852 (0.010714) [0.0002]	0.950747 (0.014192) [0.0000]	0.990599
DAX	0.000801 (0.000307) [0.0090]	1.78E-06 (7.17E-07) [0.0131]	0.061592 (0.013120) [0.0000]	0.923463 (0.017482) [0.0000]	0.985055
ATX	0.000392 (0.000311) [0.2083]	2.09E-05 (3.97E-06) [0.0000]	0.143104 (0.018830) [0.0000]	0.682189 (0.052898) [0.0000]	0.8252994
OMV	0.000324 (0.000421) [0.4424]	2.40E-06 (8.36E-07) [0.0041]	0.046694 (0.007515) [0.0000]	0.944774 (0.008817) [0.0000]	0.991468
Example 1	0.000834 (0.001028) [0.4169]	3.07E-05 (1.38E-05) [0.0259]	0.080059 (0.017367) [0.0000]	0.900093 (0.022698) [0.0000]	0.980152
Example 2	-3.02E-05 (0.000832) [0.9710]	1.42E-05 (8.21E-06) [0.0829]	0.099011 (0.023513) [0.0000]	0.890440 (0.024322) [0.0000]	0.989451
Example 3	0.000997 (0.001327) [0.4523]	3.54E-05 (1.77E-05) [0.0457]	0.061999 (0.015077) [0.0000]	0.922654 (0.019353) [0.0000]	0.984653
Example 4	0.000695 (0.000453) [0.1246]	1.90E-06 (1.11E-06) [0.0869]	0.039280 (0.008329) [0.0000]	0.952265 (0.010159) [0.0000]	0.991545

4 Conclusions

I have presented a class of models where financial markets are considered as non-linear adaptive evolutionary systems. Endogenous price fluctuations are caused by the interaction of different types of traders, who choose predictors for future prices

¹⁸ A residual test shows that the squared residuals from the estimated GARCH(1,1) model have almost no significant autocorrelations.

according to their performance in the past, conditioned upon the state of the market. In our model only expectations or forecasting rules are changing while everything else stays constant. These expectations feed back into the market equilibrium equation, generating new prices and thus affect the expectations of investors. The resulting volatility of realized market prices is much higher than the RE volatility (“expectation driven excess volatility”).

I have shown for a simple example with only two types of traders (fundamentalists or EMH-believers and trend followers) and a commonly shared view about a constant fundamental, that the model is able to generate some of the stylized facts observed in real data, such as excess kurtosis and volatility clustering. Excess volatility is created by the trading process itself. Deviations from the fundamental value triggered by noise may be amplified by technical trading. In real markets, where the “true” fundamental value is not exactly known, good or bad news about economic fundamentals may be reinforced by evolutionary forces, leading to over- and undervaluation of risky assets.

Volatility clustering, with significant low order autocorrelations in absolute returns, arises in our model due to intermittency and coexistence of attractors, a stable steady state and a “larger” attractor – like a (quasi)periodic orbit or a chaotic attractor – due to a Chenciner bifurcation. We obtain the strongest form of volatility clustering when the system is close to having a unit root and fundamentalists only adapt slowly into the direction of the fundamental value. It should be pointed out that the models of BH [9] and Gaunersdorfer [26] were not able to produce the phenomenon of volatility clustering. In fact, it is not true – as often stated as critics against bounded rationality models – that one “can get everything” by deviating from the concept of RE and introducing new parameters in a model.

Compared to stochastic time series models such a structural nonlinear economical model as presented here has the advantage that it gives insight into the economic mechanism that causes the observed patterns in financial data. Our model is still tractable, generating fluctuations in asset returns similar to those observed in real data. It may also help to understand the more complicated computational oriented, multi-agents models based on computer simulations of asset trading.¹⁹ Intermittency and coexistence of attractors are generic phenomena and occur naturally in nonlinear dynamical systems. Moreover, they are robust with respect to and sometimes reinforced by dynamic noise. Such phenomena may also play a role in the more complicated systems.

An interesting question is, if statistical tests are able to detect the structure in the data, which are generated by a low dimensional dynamical system buffeted with dynamic noise. This dynamic noise destroys predictability in returns, but preserves structure in volatility, measured by absolute returns. In particular, it would be interesting how statistical tests behave in the case of coexisting attractors.

¹⁹ BH [11] and Brock, Hommes and Wagener [12] study so-called Large Type Limits (LTL) systems, deterministic approximations of markets with many types of traders, where initial beliefs are drawn from some random distribution. This type of model may serve as a bridge between simple models with only a few types of traders and the computational oriented multi-agent models.

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