Exercises for T2, Summer term 2019, Sheet 12

1) Transformation law for density matrices under spatial rotations

Let $U(\theta\,\vec{e_z})$ be the spin-1/2 rotation matrix that actively rotates a state by an angle θ around the z-axis.

(a) Show:

$$U(\theta\,\vec{e_z})(\vec{n}\cdot\vec{\sigma})U(\theta\,\vec{e_z})^\dagger = \vec{n}'\cdot\vec{\sigma}, \qquad \vec{n}\in\mathbb{R}^3,$$

where $\vec{n}' = R(\theta\,\vec{e_z})\vec{n}$ and $R(\vec{\alpha})$ is the spatial rotation matrix that rotates vectors by an angle $|\vec{\alpha}|$ around the axis $\vec{\alpha}/|\vec{\alpha}|$.

(b) Use the result from (a) to determine the transformation law for the density matrix

$$\rho = \frac{1}{2}(\mathbb{1}_2 + \vec{n}\cdot\vec{\sigma}), \qquad |\vec{n}| = 1.$$

under a rotation given by $U(\theta\,\vec{e_z})$, and argue why $U(\theta\,\vec{e_z})$ indeed has the physical interpretation described above.

2) Orbital angular momentum

Show that the components of the orbital angular momentum operator

$$L_k = \varepsilon_{klm}X_lP_m$$

fulfill the following commutation relations:

$$[L_k, L_l] = i\hbar\varepsilon_{klm}L_m, \quad [L_k, X_l] = i\hbar\varepsilon_{klm}X_m, \quad [L_k, P_l] = i\hbar\varepsilon_{klm}P_m.$$

Hint: Use the commutation relations $[X_k, P_l] = i\hbar\delta_{kl}$ and $[X_k, X_l] = [P_k, P_l] = 0$, without specifying any particular representation.

3) Translation

Show that the following relation is fulfilled for a finite translation:

$$\exp(-i\vec{a}\,\vec{\mathbf{P}}/\hbar)\,f(\vec{\mathbf{X}})\,\exp(i\vec{a}\,\vec{\mathbf{P}}/\hbar) \;=\; f(\vec{\mathbf{X}} - \vec{a}\,\mathbb{1}),$$

where $\vec{\mathbf{X}}$ and $\vec{\mathbf{P}}$ are the abstract position and momentum operators. You can consider the function $f(\vec{x})$ being defined as a Taylor series in $\vec{x}$.