Exercises for T2, Summer term 2019, Sheet 11

1) Spin measurements

Initially, let there be a spin-1/2 system in the pure spin state $\chi_{z,+}$, i.e. the spin is pointing in the z-direction.

(a) For which spin operator is $\chi_{z,+}$ an eigenstate and what is the corresponding eigenvalue?

(b) What are the probabilities to get the values $\pm\hbar/2$ if a measurement of the spin in x-direction is carried out? What are the corresponding probabilities for a measurement of the spin in y-direction? What are the respective expectation values?

(d) Suppose you get the value $-\hbar/2$ for a measurement of the spin in x-direction. In which state is the system after the measurement? On this spin state make another measurement of the spin in $\vec{n}$ direction. The vector $\vec{n}$ is a unit vector which is rotated with respect to the x direction by the angle ϕ around the z axis. What is the probability to get the values $\hbar/2$ for this measurement?

(d) Determine the expectation value for that spin measurement in the $\vec{n}$ direction.

2) Magnetic moment im thermodynamic equilibrium

Suppose a spin-1/2 system in an external magnetic field $\vec{B} = B\vec{e}_3$. The operator of the magnetic moment is $\vec{\mathbf{M}} = \gamma\,\vec{\mathbf{S}}$ and the Hamilton operator is $\mathbf{H} = -\vec{\mathbf{M}} \cdot \vec{B}$. If the spin is in contact with a heat bath of temperature T, then the corresponding equilibrium state is described by the density matrix

$$\rho = \mathcal{N} \exp(-\beta \mathbf{H}), \quad \beta = 1/kT$$

Determine the normalization factor $\mathcal{N}$. Calculate the expectation value and the mean square deviations of $\mathbf{M}_i$ $(i = 1, 2, 3)$ and $\mathbf{H}$. Sketch the expectation value of $\mathbf{M}_3$ as function of temperature T.

3) Spin precession in a time-independent magnetic field

Let there be a spin-1/2 system with the Hamiltonian

$$\mathbf{H} = -\vec{\mathbf{M}} \cdot \vec{B}\,, \qquad \vec{\mathbf{M}} = \gamma\,\vec{\mathbf{S}}\,, \qquad \vec{B} = B\,\vec{e}_z\,.$$

(a) Determine the Heisenberg equations for the Heisenberg spin operators $\vec{\mathbf{S}}_{\mathbf{H}}(t)$ and solve them with the initial condition that the Heisenberg spin operators at $t = 0$ coincide with the corresponding Schrödinger spin operators, i.e. $\vec{\mathbf{S}}_{\mathbf{H}}(0) = \vec{\mathbf{S}}_{\mathbf{S}}$.

(b) Solve the problem in the Schrödinger picture for the two-component spin wave function

$$\begin{pmatrix} a_+(t) \\ a_-(t) \end{pmatrix}$$

(c) Illustrate the equivalence of the two solutions by calculating the time evolution of the expectation values for a spin state that initially points in x-direction at $t = 0$, in both the Heisenberg and the Schrödinger picture.