Exercises for T2, Summer term 2019, Sheet 10

1) Harmonic oscillator in the Heisenberg picture

(a) Write down the Hamilton operator of the one-dimensional harmonic oscillator in the Schrd̈inger and the Heisenberg pictures, i.e. by using location and momentum operators in the Schrd̈inger and the Heisenberg pictures, respectively. Which connection do they have?

(b) Determine the Heisenberg equations for the time-dependence (1st derivative with respect to time) of the location and momentum operators in the Heisenberg picture.

(c) Solve the Heisenberg equations for the Heisenberg location and momentum operators $\mathbf{X_H}(t)$ and $\mathbf{P_H}(t)$, respectively. Express the solutions in terms of the initial conditions $\mathbf{X_H}(0) = \mathbf{X_S}$ and $\mathbf{P_H}(t) = \mathbf{P_S}$. Recall how to solve the problem for the classical equations of motion and argue why you can apply the same method here.

(d) Determine the standard deviations $\Delta\mathbf{X_H}(t)$ und $\Delta\mathbf{P_H}(t)$ in an arbitrary state at time t as a function of those at time $t = 0$.

(e) How does the result of (d) change, if the state at time $t = 0$ minimizes Heisenberg's uncertainty relation? How does it change if one further assumes that $\mathbf{X_H}(0) = \frac{1}{m\omega}\mathbf{P_H}(0)$?

2.1) Pauli matrices

The Pauli matrices are defined as

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show by explicit calculation that the following relations are satified: $(k, l = 1, 2, 3)$

(a) $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i\mathbb{1}_{2x2}$

(b) $[\boldsymbol{\sigma}_k, \boldsymbol{\sigma}_l] = 2i\varepsilon_{klm}\boldsymbol{\sigma}_m$ (Note the sum convention and that $\varepsilon_{123} = 1$)

(c) $\{\boldsymbol{\sigma}_k, \boldsymbol{\sigma}_l\} = \boldsymbol{\sigma}_k\boldsymbol{\sigma}_l + \boldsymbol{\sigma}_l\boldsymbol{\sigma}_k = 2\delta_{kl}\mathbb{1}_{2x2}$

2.2) Pauli matrices

(d) Derive from (b) and (c) that $\boldsymbol{\sigma}_k\boldsymbol{\sigma}_l = \delta_{kl}\mathbb{1}_{2x2} + i\varepsilon_{klm}\boldsymbol{\sigma}_m$

(e) Derive from (d) that: $(\vec{a} \cdot \vec{\boldsymbol{\sigma}})(\vec{b} \cdot \vec{\boldsymbol{\sigma}}) = (\vec{a} \cdot \vec{b})\mathbb{1}_{2x2} + i(\vec{a} \times \vec{b}) \cdot \vec{\boldsymbol{\sigma}}$ for $\vec{a}, \vec{b} \in \mathbb{R}^3$

(f) Each comples 2×2 matrix $\mathbf{A}$ can be written in the form $\mathbf{A} = a_0\mathbb{1}_{2x2} + \sum_{i=1}^{3} a_i\boldsymbol{\sigma}_i$ with complex coefficients $a_{0,1,2,3}$. Derive the additional properties the a's have to satisfy if $\mathbf{A}$ is unitary?