Exercises for T2, Summer term 2019, Sheet 9

1) General uncertainty principle

The variance $(\Delta_\omega A)^2$ of some observable A with respect to a state ω is defined by

$$(\Delta_\omega A)^2 = \omega((A - \omega(A))^2)$$

Given two hermitian operators $A, B \in \mathrm{L}(\mathcal{H})$, show that for an arbitrary state ω the inequality

$$\Delta_\omega A \Delta_\omega B \geq |\omega(\frac{i}{2}[A, B])|.$$

holds. For this one can use the (non-hermitian) operator

$$C = \frac{A - \omega(A)}{\Delta_\omega A} + i\frac{B - \omega(B)}{\Delta_\omega B}$$

and the functional properties (a)-(c) of a general state ω as discussed in Chapter 4.2 of the lecture notes.

2) Density operators in a two dimensional Hilbert space

Let $\{|a_1\rangle, |a_2\rangle\}$ be an orthonormal basis of a complex-values Hilbert space and $A = |a_1\rangle\langle a_1| - |a_2\rangle\langle a_2|$ an observable.

(a) An ensemble of systems, on which A shall be measured, is in the pure state $|b\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle + |a_2\rangle)$. Write down the density operator ρ_1 for this state in abstract bra/het notation as a function of the basis states. Write down the density operator in matrix/coordinate representation for the convension that $|a_1\rangle \simeq \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$, $|a_2\rangle \simeq \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right)$.

(b) An ensemble of systems, on which A shall be measured, is in the mixed state where 50% of the systems is in the state $|a_1\rangle$ and 50% in the state $|a_2\rangle$. Write down the density operator ρ_2 for this state in abstract bra/het notation as a function of the basis states. Write down the density operator in matrix/coordinate representation for the convension above.

(c) Take one of the systems from the ensemble from (b). Does it make sense to ask in which of the pure states $|a_1\rangle$ or $|a_2\rangle$ this system is? In which konkret state is it? (This is a question of understanding. Think about it and discuss it in the exercise class and with your fellow students.)

(d) Write down A in matrix notation. Determine the average measurement value for measurements of A on both ensembles. What is the probability to obtain the value 1 for both ensembles. Write down the state of the system after a measurement of the value 1.

3) Mixed state

(a) Show that a state which is given by the density operator ρ is a mixed state if $\rho^2 \neq \rho$ holds.

(b) Show that a state which is given by the density operator ρ is a mixed (pure) state if $\mathrm{Tr}[\rho^2] < 1$ ($\mathrm{Tr}[\rho^2] = 1$) holds.

4) Harmonic oscillator in thermal equilibrium

Given a harmonic osciallator with angular frequency ω which is in thermal equilibrium with an external heatbath of absolute temperature T. The density matrix then has the form:

$$\rho = \frac{\exp(-\mathbf{H}/kT)}{\mathrm{Tr}[\exp(-\mathbf{H}/kT)]} ,$$

where $\mathbf{H}$ is the Hamilton operator and k the Boltzmann constant.

(a) Calculate the spectral representation of the mixed state ρ in Bra-Ket notation as a function of the temperature, where $|\phi_n\rangle$ is the normalized eigenstate with occupation number n. Note that the sum of the geometric series is very helpful for this calculation.

(b) Calculate the average occupation number $\langle N \rangle$ and the average energy $\langle H \rangle$ as a function of temperature T. ($\langle N \rangle = (\exp(\hbar\omega/kT) - 1)^{-1}$)

(c) Calcuate the average occupation number for visible light ($\lambda = 550\mathrm{nm}$) at room temperature ($T = 295\mathrm{K}$) and at the surface of the sun ($T = 5500\mathrm{K}$).

5) Commutators und exponentials of operators

Given are two non-commutating linear operators A and B which have the properties $[[A, B], A] = [[A, B], B] = 0$.

(a) Show that $[A, B^n] = nB^{n-1}[A, B]$ für $n \in \mathbb{N}_0$

(b) Show that $e^A B e^{-A} = B + [A, B]$, wobei $e^A \equiv \sum_{i=0}^{\infty} \frac{1}{i!} A^i$.

(c)* Show the (weak form of the) Baker-Campbell-Hausdorff formula:

$$e^{A+B} = e^A e^B e^{-[A,B]/2} .$$

Hint: Consider first the operator function $f(\lambda) \equiv e^{\lambda A} e^{\lambda B} e^{\lambda(A+B)}$ for a real parameter λ and show the relation $\frac{df(\lambda)}{d\lambda} = \lambda[A, B]f(\lambda)$. You can then integrate this differential equation. This exercise contains a number of non-trivial manipulations! It is not mandatory to solve it, but you are strongly recommended to try. ($\rightarrow$ Tutorium)