Exercises for T2, Summer term 2019, Sheet 8

1) Adjoint 0perator

On the space $L^2(\mathbb{R})$ of wave function $\psi(x)$ the operator A is defined by

$$(A\psi)(x) \equiv \frac{1}{\sqrt{\lambda}}\,\psi\left(\frac{x}{\lambda}\right), \quad \text{mit} \quad \lambda > 0\,.$$

(a) Show that A is a linear operator.

(b) Determine the Norm of $A\psi$, given that ψ normalized.

(c) Determine how $A^\dagger$ acts on a wave function $\psi(x)$. ($(A^\dagger\psi)(x) =?$) how do $A^\dagger A$ and $AA^\dagger$ on a wave function $\psi(x)$?

2) Spatial translation

The translation operator T_a acts on a wave function $\psi(x)$ (in one spatial dimension) as

$$(T_a\psi)(x) = \tilde{\psi}_a(x) = \psi(x-a), \quad a \in \mathbb{R}\,. \tag{1}$$

Show that T_a has the explicit form $T_a = \exp(-a\frac{d}{dx})$. Also write this expression as a function of the momentum operator P and formulate Eq. (1) in abstract form for Ket states. (A computation is not needed here!)

3) Distributions

Calculate the first and second derivative of the following functions in the distributional sense ($\theta(x)$ is the Heaviside step function).

- $\theta(x)$
- $\theta(-x)$
- $|x| = -x\,\theta(-x) + x\,\theta(x)$
- $e^{-a|x|} = e^{ax}\,\theta(-x) + e^{-ax}\,\theta(x),\ a > 0.$

4) Particle in the delta-potential

Let the wave function of a particle with one degree of freedom be given by

$$\psi(x) = \mathcal{N}\exp(-a|x|), \quad a > 0.$$

(a) Convince yourself with the help of results from exercise (3) that the wave function $\psi(x)$ is, for a suitable choice of the parameter a, an energy eigenfunction of the Hamilton operator

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \lambda\,\delta(x), \quad (\lambda > 0)\,.$$

Which condition must the wave function satisfy at $x = 0$? What is the necessary choice for a and what is the result for the energy eigenvalue E?

(b) Determine the probability current for $|x| > 0$ and argue that the wave function has to be interpreted as a bound state. What is the interpretation of the corresponding energy eigenvalue?

(c) Argue why there cannot be any further bound states.

5) Simple one-dimensional model for α-decay

The potential an α-particle (consisting of 2 protons and 2 neutrons and having a mass of 3727 MeV in natural units $\hbar = c = 1$) feels in a heavy nuclus with atomic number Z (= total number of protons in the nucleus) has the form

$$V(x) = \begin{cases} 0 & , \ -R \leq x \leq R \\ \dfrac{2(Z-2)\alpha}{|x|} & , \ |x| > R \end{cases} ,$$

where $\alpha = 1/137$ is the fine structure constant and R is the range of the strong attractive nuclear force. Thus R is the effective nucleon radius. A good number for R is 10 fm $= 10 \times 10^{-15}$m. One can now assume that the α-particle carries out free oscillations with kinetic energy E_α in the region $|x| \leq R$ bouncing back and forth between the high potential walls at $|x| = R$ due to the strong attractive nuclear force. So the α-particle is almost all the time reflected. There is, however, a small tunnel probability that the α-particle penetrates outside the nucleus. Due to this effect the heavy nucleus has a finite lifetime. If the α-particle manages to tunnel out of the nucleus potential, then its kinetic energy E_α is the same it had inside the nucleus.

(a) Determine the tunnel probability p_α using the approximation formula $t^{\text{tunnel}}_{\text{approx}}$ discussed in class as a function of E_α, m_α, Z, α und R (see Chap. 3, page 17).

(b) The oscillation time τ_0 of the α-particle inside the nucleus can be determined from the diameter $2R$ of the nucleus and the velocity of the α-particle. Determine the expression for τ_0. Determine numerical value for τ_0 for Radium ($Z = 88$, $E_\alpha = 4.8$ MeV) and Thorium ($Z = 90$, $E_\alpha = 4.0$ MeV) in units of seconds. It is actually useful to first use natural units (see exercise 1, sheet 3) and then convert to seconds at the end.

(c) If the α-particle were completely free (i.e. zero potential), the effective life time would be $\tau_0/2$ because this would be the time the α-particle would be located inside the nucleus. However, due to the very small tunneling probability the actual lifetime is larger by a factor p_α^{-1}, so tha $\tau_\alpha = \tau_0/2 \times p_\alpha^{-1}$. Determine the lifetime τ_α in years which you get from this model for Radium und Thorium. Using natural units is most useful here. Compare to the lifetime numbers quoted in the lecture notes.

Hinweis:

$$\int_R^b dx \sqrt{\frac{b}{x} - 1} = b \left[\arccos\left(\sqrt{\frac{R}{b}} \right) - \sqrt{\frac{R}{b}\left(1 - \frac{R}{b}\right)} \right]$$