Exercises for T2, Summer term 2019, Sheet 7

1) Time evolution of the momentum space free particle wave function

Let a free particle with mass m be described at $t = 0$ by the wave function calculated in exercise (6) of sheet 6. (See also the lecture notes, chap. 2.10.)

(a) Determine the time-dependent momentum space wave function $\tilde{\psi}(p,t)$.

(b) Determine the mean and mean square deviations of X and P at time t.

(c) How would the results change if the particle would be in a spatially constant potential of the form $V(x) = E_0$?

2) Time evolution of the configuration space free particle wave function

Use the result of exercise (1) for $x_0 = 0$, and determine the time-dependent configuration space wave function $\psi(x,t)$. Write down $|\psi(x,t)|^2$ as well. Use the method of quadratic completion. (See also the hand-written lecture notes, Chap. 2.9. for the discussion of the Gaussian wave packet).

Hint: Let $c, d \in \mathbb{C}$ (!) where $\mathrm{Re}\, c > 0$. Then the formula

$$\int_{-\infty}^{+\infty} dx\, e^{-c(x-d)^2} = \sqrt{\frac{\pi}{c}}$$

still holds! You may check this by simply performing some numerical tests (e.g. with Mathematica).

3) Commutators I

Show the following idenities for the commutator $[A, B] = AB - BA$ of two linear operators A and B. (Let C also be a linear operator and α, β two complex numbers.)

(a) $[A, B] = -[B, A]$

(b) $[\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C]$

(c) $[AB, C] = A[B, C] + [A, C]B\,, \quad [A, BC] = B[A, C] + [A, B]C$

(d) $[A, [B, C] + [B, [C, A]] + [C, [A, B]] = 0$ (Jacoby identity)

4) Commutators II

Determine:

In one dimension $[X^2, P^2]$.

In three dimensions $[A, X_1^2]$, where $A \equiv X_1 P_2 - X_2 P_1$.

5) Ladder operators

The ladder operators of a one-dimensional harmonic oscillator $a, a^\dagger$ fullfill the commutation relation $[a, a^\dagger] = \mathbb{1}$.

(a) Show: $\left[a, (a^\dagger)^n\right] = n(a^\dagger)^{n-1}$

(b) Show: $\left[a, f(a^\dagger)\right] = f'(a^\dagger)$, where you can assume that the function $f(x)$ is defined as a power series.
(c) Show that the state $|\psi_z\rangle \equiv c\, e^{za^\dagger}|0\rangle$, where $z, c \in \mathbb{C}$ is valid and $|0\rangle$ is the ground state of the harmonic oscillator, satisfies the eigenvalue equation $a|\psi_z\rangle = z|\psi_z\rangle$ (coherent state).

6) Expectation values and matrix elements for the harmonic oscillator

(a) Calculate the expectation values $\langle X\rangle_n$, $\langle P\rangle_n$, $\langle X^2\rangle_n$, $\langle P^2\rangle_n$ for the energy eigenstates $|n\rangle$ of the harmonic oscillator ($\langle O\rangle_n \equiv \langle n|O|n\rangle$).

(b) Calculate the matrix elements $x_{nm} \equiv \langle n|X|m\rangle$ und $p_{nm} \equiv \langle n|P|m\rangle$.

Use algebraic methods with the ladder operators.