Exercises for T2, Summer term 2019, Sheet 6

1) Matrix elements und wave functions

Let X ($\vec{X}$) und P ($\vec{P}$) be the location and momentum operators in one dimension (three dimensions). Let the state $|\psi\rangle$ have the configuration space wave function $\psi(\vec{x}) \equiv \langle\vec{x}|\psi\rangle$ and the momentum space wave function $\tilde{\psi}(\vec{p}) \equiv \langle\vec{p}|\psi\rangle$.

(a) Determine the following matrix elements in one dimension:
$\langle x|X|p\rangle,\ \langle p|X|x\rangle,\ \langle x|X|x'\rangle,\ \langle p|X|p'\rangle$

(b) Determine the following matrix elements in three dimensions:
$\langle\vec{x}|\vec{P}|\vec{x}'\rangle,\ \langle\vec{p}|\vec{P}|\vec{x}'\rangle$

(c) Determine the following matrix elements in three dimensions (m real and positive):
$\langle\vec{x}|\frac{1}{|\vec{X}|}e^{-m|\vec{X}|}|\vec{x}'\rangle,\ \langle\vec{x}|\frac{1}{|\vec{X}|}e^{-m|\vec{X}|}|\psi\rangle,\ \langle\vec{p}|\frac{1}{|\vec{X}|}e^{-m|\vec{X}|}|\psi\rangle$

2) Real valued wavefunction and wavefunction with spatial phase

(a) Consider the **special case** of a **real valued** wavefunction $\varphi(x)$ in one dimension which vanishes for $x \to \pm\infty$. Show that in this case the expectation value of the momentum operator is zero regardless of the other properties of the wave function.

(b) Consider the wave function of a particle with one degree of freedom of the form

$$\psi(x) = \varphi(x)e^{ip_0x/\hbar}$$

with p_0 being real and a **real valued** function φ. Calculate the expectation value of the momentum operator. What is the physical meaning of p_0 and the phase of the wavefunction?

3) Gaussian wave packet I

Consider a wave function of a particle in one spatial dimension, given by

$$\psi(x) = \mathcal{N}\exp(-x^2/4\sigma^2)\,, \qquad (\sigma \in \mathbb{R}^+)\,.$$

Try to be efficient and concise and recycle the formulae you already obtained on exercise sheet 2 and the previous exercises. The point is not to do very long computations.

(a) Determine the normalization constant $\mathcal{N}$ and the expectation value for a measurement of the position.

(b) Is the square of the position operator X^2 a hermitian operator? Determine the expectation value for a measurement of X^2 and calculate the expected standard deviation Δx, that one will get in the limit of infinitely many position measurements (on identical copies, each of them in the state $\psi(x)$).

(c) Determine the expectation values for measurements of P and P^2 and the standard deviation ΔP for momentum measurements. Check the uncertainty principle. What are the physical properties of the particle described by this wavefunction? Discuss the physical situation of having a very small or very large value of σ.

4) Gaussian wave packet II

The general form of a wavefunction of a particle in one dimension with a minimal value of the product of position and momentum uncertainty ($\Delta X \Delta P = \hbar/2$) is

$$\psi(x) = \mathcal{N} \exp\Big(- \frac{(x-x_0)^2}{4\sigma^2} \Big) e^{ip_0 x/\hbar} \,, \qquad (\sigma \in \mathbb{R}^+; x_0, p_0 \in \mathbb{R}) \,.$$

Again, try to be efficient and concise and recycle the formulae from the previous exercises. The point is not to do very long computations.

(a) Determine the normalization constant $\mathcal{N}$. Argue why a computation is not needed any more.

(b) Determine the expectation values for measurements of X, X^2, P und P^2. Determine the position uncertainty ΔX and momentum uncertainty ΔP

(c) What are the physical properties of the particle described by this wavefunction? Discuss the physical situation of having a very small or very large value of σ.

5) Gaussian wave packet III

(a) Determine the momentum space wavefunction $\tilde{\psi}(p)$ associated to $\psi(x)$ from exercise (4), by solving the differential equation

$$\Big(\frac{X - x_0}{\sigma} + i \, \frac{P - p_0}{\hbar/2\sigma} \Big) \, \tilde{\psi}(p) = 0$$

directly in momentum space representation. For this you have to account for the momentum space representations of the momentum operator P and the position opeartor X. This method was actually discussed in the handwritten scriptum in Chap. 2.9.

(b) Determine the momentum space wavefunction $\tilde{\psi}(p)$ directly from the form of $\psi(x)$ through a transformation from configuration to momentum space representation.

6) Commuting operators

Suppose A, B and C are linear operators which fulfill $[A, C] = [B, C] = 0$. Does this also mean that $[A, B] = 0$ holds? If not, you may just quote a counter example.