Exercises for T2, Summer term 2019, Sheet 5

1) H-Atom

The wave function of the ground state of an electron in an hydrogen atom has the form

$$\psi(\vec{x}) = \mathcal{N} \exp(-r/a).$$

Here $r = |\vec{x}|$ is the distance to the nucleus, $a = \hbar/m_e \alpha c$ the Bohr radius, m_e the mass of the electron and $\alpha = e^2/\hbar c \simeq 1/137$ the fine structure constant.

(a) How do you have to choose $\mathcal{N}$, such that the wave function is normalized?

(b) Determine the average distance $\langle r \rangle_\psi$ of the electron to the nucleus. What is the numerical value?

Hint: Use spherical coordinates to do the computation. Also note that after having computed the integral $I(u) = \int_0^\infty dr\, e^{-ur}$ you may obtain integrals $\int_0^\infty dr\, r^n\, e^{-ur}$ for $n \in \mathbb{N}$ without doing any additional integration.

2) Two-dimensional Hilbert space and representation of bra- and ket-vectors

Consider a two-dimensional complex valued Hilbert space with an (orthonormal) basis $\{|a_1\rangle, |a_2\rangle\}$ ("a-representation"). Two stated vectors are given by:

$$|b_1\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle + i\,|a_2\rangle) \qquad |b_2\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle - i\,|a_2\rangle)\,.$$

(a) Show that $\{|b_1\rangle, |b_2\rangle\}$ also form an orthonormal basis ("b-representation").

(b) Write down the coordinate representation (i.e. in terms of complex-valued 2-vectors) of the ket-vectors $|a_1\rangle$, $|a_2\rangle$, $|b_1\rangle$, $|b_2\rangle$ and of the respective bra-vectors in the a-representation. (Use the convention discussed in class and note that the upper/first entry shall be related to $|a_1\rangle$.) Also write down the coordinate entries in terms of the abstract scalar products $\langle ..|..\rangle$!

(c) Do the same as in (b) for the b-representation. Note it is useful to first rewrite $|a_1\rangle$, $|a_2\rangle$ as functions of $|b_1\rangle$, $|b_2\rangle$.

(d) Determine the entries (in terms of scalar products $\langle b_i|a_j\rangle$, $i, j = 1, 2$) of the 2×2 matrix $U_{b\leftarrow a}$ which is transforming a state ket-vector given in the a-represention to the b-representation. Use the completeness relation from the lecture! How does the transformation matrix look for the bra-vectors?

3) Measurements and the collaps of the state

Consider a two-dimensional complex valued Hilbert space in a suitable basis. There are two observables A and B which in the representation of this basis have the form

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B(\theta) = \begin{pmatrix} \cos(\theta) & -i\sin(\theta) \\ i\sin(\theta) & -\cos(\theta) \end{pmatrix}, \qquad \left[\, |\phi\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$

where θ is a real number.

(a) Determine the possible measurement values for both observables.

(b) Determine the expectation value and the standard deviation for measurements of observable A on a large number of systems which are all in the state $|\phi\rangle$. Do the same for observable $B(\theta)$. (Do not forget about the normalization!)

(c) Determine the probablity that the value 1 is obtained in the measurement of $B(\pi/4)$ when the system is in the state $|\phi\rangle$. Consider the case that 1 is indeed measured. In which state $|\phi_{B(1)}\rangle$ is the system immediatly after that measurement? What are the answers if you would consider the measurement of observable A instead of observable $B(\pi/4)$?

(d) Consider a system in state $|\phi\rangle$. What is the state $|\phi_A\rangle$ the system has immediately after the measurement of observable A?

4) Abstract linear operator

Suppose some linear operator T which acts on a complex-valued Hilbert space is defined by $T := |u\rangle\langle u|$ (with $|u\rangle \neq 0$).

(a) Is T hermitian?

(b) What feature is needed from $|u\rangle$ so that T is a projection operator?

(c) Suppose B is an arbitrary linear operator which acts on the same Hilbert space. Show that the trace of the operator TB is given by $\langle u|B|u\rangle$. Recall that the trace of an operator does not depend on the used basis.

5) Computation with an observable in a Hilbert space with finite dimensions

Let $\mathcal{H} = \mathbb{C}^3$ be the Hilbert space of complex 3-dimensional vectors with the usual scalar product $\langle\chi|\psi\rangle = \sum_{k=1}^{3} \chi_k^* \psi_k$. Let a particular observable be given by the matrix A

$$A = \begin{pmatrix} 0 & -2i & 0 \\ 2i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad \left[|\phi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]$$

(a) What are the possible values a_1, a_2, a_3 that can occur in a measuerment of this observable?

(b) Determine the orthonormalized eigenvectors that correspond to each eigenvalue. These eigenvectors are unique up to a complex phase which you can choose as you wish.

(c) Determine the probability for the measurement of the values a_1, a_2, a_3 when the system is in the state $|\phi\rangle$.